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ON MAGIC LABELLINGS OF TYPE (1,1,1) FOR THREE CLASSES OF PLANE GRAPHS

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1. Introduction

The notions of magic and consecutive labelling of plane graphs were defined by Lih Ko-Wei [1]. However, the subject can be traced back to the 13th century when similar notions were investigated by Yang Hui (1275) and later by Chang Chhao (1670), Pao Chhi-Shou (1880) and Li Nien (1935).
Magic labellings of type (1,1,0) for wheels, friendship graphs, prisms and some of the Platonic polyhedra are given in [1].

This paper describes magic labellings of type (1,1,1) for three classes of plane graphs.

2. Necessary notions and definitions

We shall consider non-trivial finite connected planar graphs without loops or multiple edges. If a planar graph is embedded in the plane, then it is called a plane graph. Let G be such a graph with the vertex set $V(G)$, the edge set $E(G)$ and the face set $F(G)$, where $|V(G)|$, $|E(G)|$ and $|F(G)|$ are the number of vertices, edges and faces of G.

A labelling of type (1,1,1) assigns labels from the set $\{1, 2, 3, \ldots, |V(G)| + |E(G)| + |F(G)|\}$ to the vertices, edges and faces of graph G in such a way that each vertex, edge and face receives exactly one label and each number is used exactly once as a label. If we label only vertices or only edges or only faces, we call such a labelling a vertex labelling, an edge labelling or a face labelling, respectively.

The weight of a face under a labelling is the sum of the label of the face itself and the labels of vertices and edges surrounding that face.

A *labelling* is said to be *magic* if for every integer s all s-sided faces have the same weight [1]. We allow different weights for different s.

This notion of magicality is different from the definition given by J. Sedláček in [2]. However, a magic edge labelling of a plane graph, in our sense, is equal to a supermagic labelling of the plane dual graph G^* of G as defined, for instance, in [3, 4, 5].

A *labelling* is said to be *consecutive* if for every integer s the weights of all s-sided faces constitute a set of consecutive integers. Two *labellings* g and g' are said to be *complementary* if for every integer s the sum of the g-weight and the g'-weight of each s-sided face is a constant.

We shall use $\lceil r \rceil$ to denote the least integer greater than or equal to r, and $\lfloor r \rfloor$ to denote the greatest integer smaller than or equal to r, and further we shall use the expressions

$$\alpha = \frac{(-1)^n + 1}{2} \quad \text{and} \quad \beta = \frac{(-1)^{n+1} + 1}{2} \quad \text{to simplify later notations.}$$

3. Results

For $n \geq 2$ let B_n be the Cartesian product $P_n \times P_3$ of a path on n vertices with a path on three vertices, embedded in the plane and labelled as in Fig. 1.

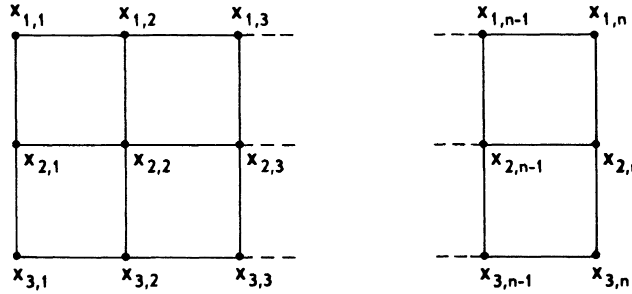

Fig. 1

Define the vertex labelling g_0 as follows.

$$g_0(x_{1,i}) = \begin{cases} \alpha(i+1) + \beta(n+i-1) & \text{if } i \text{ is odd} \\ \alpha(n+i-1) + \beta(i-1) & \text{if } i \text{ is even} \end{cases}$$

$$g_0(x_{2,i}) = \begin{cases} \alpha(3n-i+1) + \beta\dfrac{5n-i+2}{2} & \text{if } i \text{ is odd} \\ \alpha(3n-i+1) + \beta\dfrac{5n+i+1}{2} & \text{if } i \text{ is even} \end{cases}$$

$$g_0(x_{3,i}) = \begin{cases} \alpha i + \beta(n+i) & \text{if } i \text{ is odd} \\ \alpha(n+i) + \beta i & \text{if } i \text{ is even} \end{cases}$$

Lemma 1. *The vertex labelling g_0 of B_n is magic if $n \geq 2$ is even and is consecutive* if *$n \geq 3$ is odd.*

Proof. Under the labelling g_0 the weight for all 4-sided faces is $7n + 2$ (if n is even) and the weights for all 4-sided faces successively assume consecutive values $6n + 3, 6n + 4, \ldots, 8n$ (if n is odd).

Define the edge labelling g_1 as follows.

$$g_1(x_{1,i}x_{1,i+1}) = \alpha(2n - 2i - 1) + \beta(2i - 1) \qquad \text{if } 1 \leq i \leq n-1$$

$$g_1(x_{1,i}x_{2,i}) = \begin{cases} \alpha \dfrac{5n + i - 3}{2} + \beta \dfrac{6n - i - 1}{2} & \text{if } i \text{ is odd} \\ \alpha \dfrac{6n + i - 4}{2} + \beta \dfrac{7n - i - 1}{2} & \text{if } i \text{ is even} \end{cases}$$

$$g_1(x_{2,i}x_{2,i+1}) = 5n - i - 2 \qquad \text{if } 1 \leq \text{i} \leq n-1$$

$$g_1(x_{2,i}x_{3,i}) = \begin{cases} \alpha \dfrac{4n + i - 3}{2} + \beta \dfrac{5n - i - 2}{2} & \text{if } i \text{ is odd} \\ \alpha \dfrac{7n + i - 4}{2} + \beta \dfrac{8n - i - 2}{2} & \text{if } i \text{ is even} \end{cases}$$

$$g_1(x_{3,i}x_{3,i+1}) = \alpha(2n - 2i) + \beta 2i \qquad \text{if } 1 \leq i \leq n-1$$

Lemma 2. *The edge labelling g_1 of B_n is magic if $n \geq 3$ is odd and is consecutive if $n \geq 2$ is even.*

Proof. For the weights of 4-sided faces we have

$$g_1(x_{1,i}x_{1,i+1}) + g_1(x_{1,i}x_{2,i}) + g_1(x_{1,i+1}x_{2,i+1}) + g_1(x_{2,i}x_{2,i+1}) = \\ = \alpha \frac{25n - 4i - 12}{2} + \beta \frac{23n - 9}{2}$$

and

$$g_1(x_{2,i}x_{2,i+1}) + g_1(x_{2,i}x_{3,i}) + g_1(x_{2,i+1}x_{3,i+1}) + g_1(x_{3,i}x_{3,i+1}) = \\ = \alpha \frac{25n - 4i - 10}{2} + \beta \frac{23n - 9}{2} \qquad \text{for } i = 1, 2, \ldots, n-1.$$

It is easy to see that under the labelling g_1 if n is odd, the common weight for all 4-sided faces is $\dfrac{23n - 9}{2}$ and if n is even, the set of weights of 4-sided faces consists of the consecutive integers $\left\{\frac{21n-8}{2}, \frac{21n-6}{2}, \ldots, \frac{25n-14}{2}\right\}$.

Theorem 1. *For $n \geq 2$ the graph B_n has a magic labelling of type* (1,1,1).

Proof. Label the vertices and the edges of B_n by g_0 and $|V(B_n)| + g_1$, respectively. From the previous lemmas it easily follows that in the resulting labelling of type (1,1,0) the weights of 4-sided faces constitute a set of consecutive integers. Hence, if g_2 is the complementary face labelling with values in the set $\{|V(B_n)| + |E(B_n)| + 1, \ldots, |V(B_n)| + |E(B_n)| + |F(B_n)|\}$, then the labellings g_0, $|V(B_n)| + g_1$ and g_2 combine to a magic labelling of type (1,1,1).

Let Q_1, Q_2 and Q_3 be paths on n, $2n$ and n vertices, respectively. Denote the vertices of Q_i by $x_{i,1}, x_{i,2}, \ldots$, in the order they occur on Q_i, $i = 1, 2, 3$. Form the graph C_n from the disjoint union $Q_1 \cup Q_2 \cup Q_3$ by adjoining the edges $x_{1,i}x_{2,2i-1}$ and $x_{2,2i}x_{3,i}$ for $i = 1, 2, \ldots, n$. (Fig. 2).

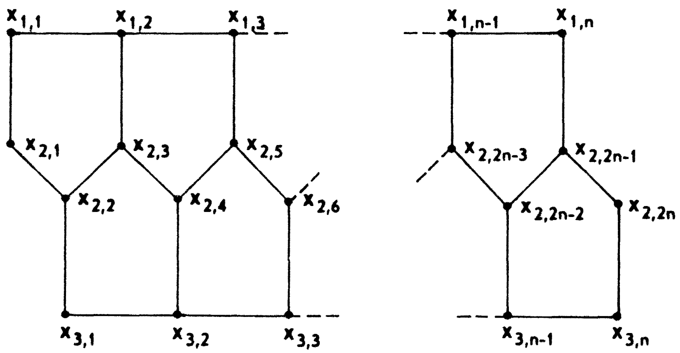

Fig. 2

We construct a vertex labelling g_3 and an edge labelling g_4 of C_n in the following way.

$$g_3(x_{1,i}) = 2i - 1$$

$$g_3(x_{2,2i-1}) = 4n - 2i + 2$$

$$g_3(x_{2,2i}) = 4n - 2i + 1$$

$$g_3(x_{3,i}) = 2i$$

for $i = 1, 2, \ldots, n$ and

$$g_4(x_{1,i}x_{1,i+1}) = 4n - 2i - 1 \qquad \text{if } 1 \le i \le n-1$$

$$g_4(x_{1,i}x_{2,2i-1}) = \begin{cases} 4n + i - 3 & \text{if } i \text{ is odd} \\ 5n + i - 4 & \text{if } i \text{ is even} \end{cases}$$

$$g_4(x_{2,2i-1}x_{2,2i}) = n - i + 1 \qquad \text{if } 1 \le i \le n$$

$$g_4(x_{2,2i}x_{2,2i-1}) = n + i \qquad \text{if } 1 \le i \le n-1$$

$$g_4(x_{2,2i}x_{3,i}) = \begin{cases} 4n + i - 2 & \text{if } i \text{ is odd} \\ 5n + i - 3 & \text{if } i \text{ is even} \end{cases}$$

$$g_4(x_{3,i}x_{3,i+1}) = 4n - 2i - 2 \qquad \text{if } 1 \le i \le n-1$$

Lemma 3. *The vertex labelling g_3 of C_n is consecutive if $n \ge 2$.*

The set of weights of 5-sided faces under the labelling g_3 consists of consecutive integers $\{10n + 4, 10n + 5, \ldots, 12n + 1\}$.

Lemma 4. *The edge labelling g_4 of C_n is magic if $n \ge 2$.*

By direct computation we obtain that the weight for all 5-sided faces is $15n - 6$.

Theorem 2. *For $n \ge 2$ the graph C_n has a magic labelling of type* (1,1,1).

Proof. Label the vertices and the edges of C_n by g_3 and $|V(C_n)| + g_4$, respectively. If g_5 is the complementary face labeling defined analogously as in the previous theorem, then the labellings g_3, $g_4 + |V(C_n)|$ and g_5 combine to a magic labelling of type (1,1,1).

We define D_n to be the graph obtained from the graph C_n by inserting the edges $x_{2,2i-1}x_{2,2i+1}$ and $x_{2,2i}x_{2,2i+2}$ for $i = 1, 2, \ldots, n-1$. (Fig. 3).

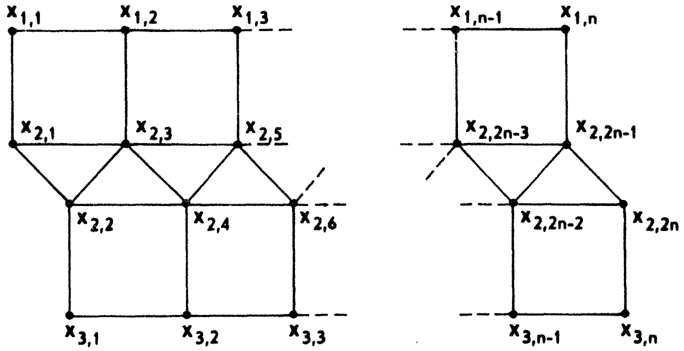

Fig. 3

Define the vertex labelling g_6 and the edge labelling g_7 as follows.

$$g_6(x_{1,i}) = 4n - 2i + 1$$

$$g_6(x_{2,2i-1}) = 2i$$

$$g_6(x_{2,2i}) = 2n - 2i + 1$$
$$g_6(x_{3,i}) = 2n + 2i$$

for $i = 1, 2, \ldots, n$.

$$g_7(x_{1,i}x_{1,i+1}) = 8n + 2i - 4$$
$$g_7(x_{2,2i-1}x_{2,2i+1}) = 6n - 2i$$
$$g_7(x_{2,2i}x_{2,2i+2}) = 6n - 2i - 1$$
$$g_7(x_{3,i}x_{3,i+1}) = 8n + 2i - 3$$
$$g_7(x_{2,2i}x_{2,2i+1}) = 7n + i - 2$$

for $i = 1, 2, \ldots, n - 1$.

$$g_7(x_{1,i}x_{2,2i-1}) = \begin{cases} \left\lfloor \dfrac{21n+1}{2} \right\rfloor + \dfrac{i+1}{2} - 5 & \text{if } i \text{ is odd} \\ \left\lfloor \dfrac{23n+1}{2} \right\rfloor + \dfrac{i}{2} - 5 & \text{if } i \text{ is even} \end{cases}$$

$$g_7(x_{2,2i-1}x_{2,2i}) = 6n + i - 2 \qquad \text{if } 1 \leq i \leq n$$

$$g_7(x_{2,2i}x_{3,i}) = \begin{cases} 10n + \dfrac{i+1}{2} - 5 & \text{if } i \text{ is odd} \\ \left\lfloor \dfrac{21n+1}{2} \right\rfloor + \left\lceil \dfrac{n}{2} \right\rceil + \dfrac{i}{2} - 5 & \text{if } i \text{ is even} \end{cases}$$

Theorem 3. *For $n \geq 2$ the graph D_n has a magic labelling of type* (1,1,1).

Proof. Label the vertices and the edges of D_n by g_6 and g_7, respectively. We obtain the labelling of type (1,1,0), where the weights of 3-sided faces constitute a set of consecutive integers $\{21n, 21n + 1, \ldots, 23n - 3\}$ and the weights of 4-sided faces successively assume the values $43n - 9$, $43n - 8$, ..., $44n - 12$, $44n - 11$, $44n - 9$, $44n - 8$, ..., $45n - 11$ if n is odd and $43n - 10$, $43n - 9$, ..., $44n - 13$, $44n - 12$, $44n - 10$, $44n - 9$, ..., $45n - 12$ if n is even. Hence, if g_8 is the complementary face labelling with values in the set $\{|V(D_n)| + |E(D_n)| + 1, \ldots, |V(D_n)| + |E(D_n)| + |F(D_n)|\}$, then the labellings g_6, g_7 and g_8 combine to a magic labelling of type (1,1,1)

Observe that the external $2n + 4$-sided face is assigned the label $|V(D_n)| + |E(D_n)| + n$.

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О МАГИЧЕСКИХ РАЗМЕТКАХ ТИПА (1,1,1) ДЛЯ ТРЁХ КЛАССОВ ПЛОСКИХ ГРАФОВ

Martin Bača

Резюме

Пусть G — связный плоский граф с $|V(G)|$ вершинами, $|E(G)|$ ребрами и $|F(G)|$ гранями. Разметка типа (1,1,1) приписывает метки из множества $\{1, 2, 3, \dots |V(G)| + |E(G)| + |F(G)|\}$ вершинам, ребрам и граням таким образом, что каждой вершине, ребру и грани приписывается только одна метка, причем каждая метка используется только один раз.

Вес грани относительно данной разметки равен сумме меток, приписанных самой грани и её вершинам и ребрам.

Разметка называется магической, если все грани с одним и тем же числом сторон имеют один и тот же, зависящий от числа сторон, вес. В работе построены магические разметки типа (1,1,1) для трёх классов плоских графов.