Lubomír Kubáček

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ASYMPTOTICAL CONFIDENCE REGION IN A REPLICATED MIXED LINEAR MODEL WITH AN ESTIMATED COVARIANCE MATRIX

LUBOMÍR KUBÁČEK

Introduction

Let $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots$ be independent identically distributed (i.i.d.) random vectors; $\boldsymbol{Y}_j \sim N_n(\mathbf{X}\boldsymbol{\beta}, \Sigma), j = 1, 2, \ldots$; the $n \times k$ matrix $\mathbf{X}$ is known, $\boldsymbol{\beta} \in \mathscr{R}^k$ (k-dimensional Euclidean space) is an unknown vector parameter and the covariance matrix $\boldsymbol{\Sigma}$ is totally or partially unknown.

A confidence region for the parameter $\boldsymbol{\beta}$ (or for its function) based on realizations $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ of the vectors $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$, in the case when the covariance matrix $\boldsymbol{\Sigma}$ is totally unknown is determined in [4] and [2].

The aim of the paper is to find a confidence region when some a priori information on the covariance matrix is available; we shall investigate two following cases:

a) the covariance matrix $\boldsymbol{\Sigma}$ is diagonal with unknown elements

b) the côvariance matrix has the following structure:

$\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, $p \geq 2$, $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)' \in \underline{\boldsymbol{\vartheta}} \subset \mathscr{R}^p$ ($\underline{\boldsymbol{\vartheta}}$ is an open and bounded set), where the $n \times n$ symmetric matrices $\mathbf{V}_i$, $i = 1, \ldots, p$, are known and the components $\vartheta_1, \ldots, \vartheta_p$ are unknown (a mixed linear model).

1. Preliminaries

The notation $\bar{\boldsymbol{Y}} = (1/m) \sum_{j=1}^{m} \boldsymbol{Y}_j$ and $\mathbf{S} = [1/(m-1)] \sum_{j=1}^{m} (\boldsymbol{Y}_j - \bar{\boldsymbol{Y}})(\boldsymbol{Y}_j - \bar{\boldsymbol{Y}})'$ is used in what follows.

Lemma 1.1. *Let* $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$ *be i.i.d. random vectors,* $\boldsymbol{Y}_j \sim N_n(\mathbf{X}\boldsymbol{\beta}, \Sigma)$, $j = 1, \ldots, m$, $R(\mathbf{X}) = k \leq n$ *($R(\mathbf{X})$ is the rank of the matrix $\mathbf{X}$) and let the covariance matrix $\boldsymbol{\Sigma}$ be regular. Let $m > n$ and $\mathbf{G}$ be an $r \times k$ matrix of the rank*

$R(\mathbf{G}) = r$. *Then the confidence ellipsoid for the vector function* $\boldsymbol{g}(\boldsymbol{\beta}) = \mathbf{G}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathscr{R}^k$, *is*

$$\left\{\boldsymbol{u}\colon (\boldsymbol{u} - \mathbf{G}\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{S}})'[\mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{G}']^{-1}(\boldsymbol{u} - \mathbf{G}\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{S}})m\left[1 + \frac{m}{m-1}\hat{\boldsymbol{v}}'\mathbf{S}^{-1}\hat{\boldsymbol{v}}\right]^{-1} \leqslant \right.$$

$$\left. \leqslant \frac{(m-1)r}{m-1-(n-k)} F_{r,m-1-(n-k)}(1-\alpha)\right\},$$

where $\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{S}} = (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\bar{\boldsymbol{Y}}$, $\hat{\boldsymbol{v}} = \bar{\boldsymbol{Y}} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}_{\mathbf{S}}$ *and* $F_{r,m-1-(n-k)}(1-\alpha)$ *is the* $(1-\alpha)$ *th quantile of the Fisher—Snedecor random variable.*

Proof. See Lemma 3b) in [4] or Theorem 2.2 and Corollary 1 in [2].

Lemma 1.2. *Let* $\{\boldsymbol{T}_m\}_{m=1}^{\infty}$ *be a sequence of estimators of a parameter* $\boldsymbol{\Theta} \in \underline{\Theta} \subset \mathscr{R}^s$ *such that* $\sqrt{m}(\boldsymbol{T}_m - \boldsymbol{\Theta}) \xrightarrow[m]{L} N_s(\boldsymbol{0}, \boldsymbol{\Sigma}(\boldsymbol{\Theta}))$. *let* $\underline{\Theta}$ *be an open set in* $\mathscr{R}^s$, *let* $\boldsymbol{\Sigma}(\cdot)\colon \underline{\Theta} \to \mathscr{S}_s$ *(the class of symmetric* $s \times s$ *matrices) be a continuous mapping and let* $\boldsymbol{\Sigma}(\boldsymbol{\Theta})$ *be regular for* $\boldsymbol{\Theta} \in \underline{\Theta}$. *Let a function* $\boldsymbol{g}(\cdot)\colon \underline{\Theta} \to \mathscr{R}^t$ *have continuous partial derivatives* $\partial g_i/\partial \Theta_j$, $i = 1, \ldots, t$, $j = 1, \ldots, s$, *and let the matrix* $(\partial \boldsymbol{g}/\partial \boldsymbol{\Theta}')\boldsymbol{\Sigma}(\boldsymbol{\Theta})\partial \boldsymbol{g}'/\partial \boldsymbol{\Theta}$ *be regular for* $\boldsymbol{\Theta} \in \underline{\Theta}$. *Then*

$$\sqrt{m}[(\partial \boldsymbol{g}/\partial \boldsymbol{T}_m')\boldsymbol{\Sigma}(\boldsymbol{T}_m)\partial \boldsymbol{g}'/\partial \boldsymbol{T}_m]^{-1/2}[\boldsymbol{g}(\boldsymbol{T}_m) - \boldsymbol{g}(\boldsymbol{\Theta})] \xrightarrow[m]{L} N_t(\boldsymbol{0}, \mathbf{I})$$

(here $\mathbf{I}$ *is the identical matrix).*

Proof. See Section 6a.2 in [5].

Lemma 1.3. *Under the assumption of Lemma* 1.2

$$m[\boldsymbol{g}(\boldsymbol{T}_m) - \boldsymbol{g}(\boldsymbol{\Theta})]'[(\partial \boldsymbol{g}/\partial \boldsymbol{T}_m')\boldsymbol{\Sigma}(\boldsymbol{T}_m)\partial \boldsymbol{g}'/\partial \boldsymbol{T}_m]^{-1}[\boldsymbol{g}(\boldsymbol{T}_m) - \boldsymbol{g}(\boldsymbol{\Theta})] \xrightarrow[m]{L} \chi_t^2.$$

Proof. It is a consequence of Lemma 1.2 and the Sverdrup theorem [1, p. 185].

Corollary. *If m is sufficiently large, then the random set (ellipsoid)*

$$\{\boldsymbol{u}\colon [\boldsymbol{u} - \boldsymbol{g}(\boldsymbol{T}_m)]'[(\partial \boldsymbol{g}/\partial \boldsymbol{T}_m')\boldsymbol{\Sigma}(\boldsymbol{T}_m)\partial \boldsymbol{g}'/\partial \boldsymbol{T}_m]^{-1}[\boldsymbol{u} - \boldsymbol{g}(\boldsymbol{T}_m)] \leqslant \chi_t^2(1-\alpha)\, m\}$$

can be considered as a $(1-\alpha)$ *asymptotical confidence ellipsoid for the function* $\boldsymbol{g}(\cdot)$; $\chi_t^2(1-\alpha)$ *is the* $(1-\alpha)$*th quantile of the chi-square distribution with t degrees of freedom.*

Lemma 1.4. *Let* $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$ *be random vectors from Lemma* 1.1. *Let the covariance matrix* $\boldsymbol{\Sigma}$ *be of the structure* $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, *where* $\mathbf{V}_i$, $i = 1, \ldots, p$, *are known symmetric matrices and* $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)'$ *is an unknown vector parameter,* $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$ *(an open and bounded set)* $\subset \mathscr{R}^p$. *Let* $\sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ *be regular for all* $\boldsymbol{\vartheta} \in \underline{\boldsymbol{\vartheta}}$ *and let the matrices* $\mathbf{V}_1, \ldots, \mathbf{V}_p$ *be linearly independent. Then*

(a) *the* ϑ_0*-locally best unbiased estimator of the vector* ϑ *based on the matrix* $\mathbf{S}$ *exists and has the form*

$$\vartheta(\vartheta_0) = \mathbf{S}_{\Sigma_0^{-1}}^{-1}[\mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_1\Sigma_0^{-1}\mathbf{S}), \ldots, \mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_p\Sigma_0^{-1}\mathbf{S})]',$$

where $\Sigma_0 = \sum_{i=1}^{p} \vartheta_{0,i}\mathbf{V}_i$, $\vartheta_0 = (\vartheta_{0,1}, \ldots, \vartheta_{0,p})' \in \underline{\vartheta}$,

$$\{\mathbf{S}_{\Sigma_0^{-1}}\}_{i,j} = \mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{V}_j), \quad i,j = 1, \ldots, p;$$

(b) *if* $\hat{\Sigma} = \sum_{i=1}^{p} \hat{\vartheta}_i(\vartheta_0)\mathbf{V}_i$, $\hat{\vartheta}^* = \hat{\vartheta}[\hat{\vartheta}(\vartheta_0)] = \mathbf{S}_{\hat{\Sigma}^{-1}}^{-1} \cdot [\mathrm{Tr}(\hat{\Sigma}^{-1}\mathbf{V}_1\hat{\Sigma}^{-1}\mathbf{S}), \ldots$ $\ldots, \mathrm{Tr}(\hat{\Sigma}^{-1}\mathbf{V}_p\hat{\Sigma}^{-1}\mathbf{S})]'$, *then*

$$\sqrt{m}[(1/2)\mathbf{S}_{\hat{\Sigma}^{-1}}]^{1/2}(\hat{\vartheta}^* - \vartheta) \xrightarrow[m]{L} N_p(\mathbf{0}, \mathbf{I}).$$

Proof. For (a) see Theorem 3.2 and Remark 3.3 in [3].

(b) As $\mathrm{cov}[\mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{S}), \mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_j\Sigma_0^{-1}\mathbf{S})|\vartheta] =$ $= [2/(m-1)]\,\mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\Sigma\Sigma_0^{-1}\mathbf{V}_j\Sigma_0^{-1}\Sigma) = [2/(m-1)]\{\mathbf{S}_{\Sigma_0^{-1}\Sigma\Sigma_0^{-1}}\}_{i,j}$ $\left(\text{where } \Sigma = \sum_{i=1}^{p} \vartheta_i\mathbf{V}_i\right)$ the covariance matrix of the vector $\hat{\vartheta}(\vartheta_0)$ from (a) is:

$$\mathrm{Var}[\hat{\vartheta}(\vartheta_0)|\vartheta] = [2/(m-1)]\,\mathbf{S}_{\Sigma_0^{-1}}^{-1}\mathbf{S}_{\Sigma_0^{-1}\Sigma\Sigma_0^{-1}}\mathbf{S}_{\Sigma_0^{-1}}^{-1}.$$

Under the given assumption on $\underline{\vartheta}$ the last matrix converges to zero matrix if $m \to \infty$. As the convergence in quadratic mean implies the convergence in probability, $\hat{\vartheta}_i(\vartheta_0) \xrightarrow[m]{P} \vartheta_i$, $i = 1, \ldots, p$, thus $\hat{\Sigma} - \Sigma \xrightarrow[m]{P} \mathbf{0}$. As the matrix $\mathbf{S}_{\Sigma^{-1}}$ is a continuous function of variance components, $\left(\frac{1}{2}\mathbf{S}_{\hat{\Sigma}^{-1}}\right)^{1/2} - \left(\frac{1}{2}\mathbf{S}_{\Sigma^{-1}}\right)^{1/2} \xrightarrow[m]{P} \mathbf{0}$. The last two relationships imply

$$\hat{\vartheta}^* - \hat{\vartheta}(\vartheta) = \mathbf{S}_{\hat{\Sigma}^{-1}}^{-1}[\mathrm{Tr}(\hat{\Sigma}^{-1}\mathbf{V}_1\hat{\Sigma}^{-1}\mathbf{S}), \ldots, \mathrm{Tr}(\hat{\Sigma}^{-1}\mathbf{V}_p\hat{\Sigma}^{-1}\mathbf{S})]' -$$

$$- \mathbf{S}_{\Sigma^{-1}}^{-1}[\mathrm{Tr}(\Sigma^{-1}\mathbf{V}_1\Sigma^{-1}\mathbf{S}), \ldots, \mathrm{Tr}(\Sigma^{-1}\mathbf{V}_p\Sigma^{-1}\mathbf{S})]' \xrightarrow[m]{P} \mathbf{0}.$$

The sequence $\{\sqrt{m}(\mathbf{S} - \Sigma)\}_{m=n+1}^{\infty}$ is obviously asymptotically normal. Thus with respect to Lemma 1.2

$$\sqrt{m}\left(\frac{1}{2}\mathbf{S}_{\Sigma^{-1}}\right)^{1/2}[\hat{\vartheta}(\vartheta) - \vartheta] \xrightarrow[m]{L} N_p(\mathbf{0}, \mathbf{I}).$$

Because of

$$\sqrt{m}\left(\frac{1}{2}\mathbf{S}_{\hat{\Sigma}}^{-1}\right)^{1/2}(\hat{\boldsymbol{\vartheta}}^* - \boldsymbol{\vartheta}) = \sqrt{m}\left(\frac{1}{2}\mathbf{S}_{\Sigma}^{-1}\right)^{1/2}[\hat{\boldsymbol{\vartheta}}(\boldsymbol{\vartheta}) - \boldsymbol{\vartheta}] + \boldsymbol{\varepsilon}_m,$$

where

$$\boldsymbol{\varepsilon}_m = \sqrt{m}\left(\frac{1}{2}\mathbf{S}_{\Sigma}^{-1}\right)^{1/2}[\hat{\boldsymbol{\vartheta}}^* - \hat{\boldsymbol{\vartheta}}(\boldsymbol{\vartheta})] + \sqrt{m}\left[\left(\frac{1}{2}\mathbf{S}_{\hat{\Sigma}}^{-1}\right)^{1/2} - \left(\frac{1}{2}\mathbf{S}_{\Sigma}^{-1}\right)^{1/2}\right]\cdot$$

$$\cdot[\hat{\boldsymbol{\vartheta}}(\boldsymbol{\vartheta}) - \boldsymbol{\vartheta}] + \sqrt{m}\left[\left(\frac{1}{2}\mathbf{S}_{\Sigma}^{-1}\right)^{1/2} - \left(\frac{1}{2}\mathbf{S}_{\Sigma}^{-1}\right)^{1/2}\right][\hat{\boldsymbol{\vartheta}}^* - \hat{\boldsymbol{\vartheta}}(\boldsymbol{\vartheta})] \xrightarrow[m]{P} \boldsymbol{O},$$

the sequences $\left\{\sqrt{m}\left(\frac{1}{2}\mathbf{S}_{\Sigma}^{-1}\right)^{1/2}[\hat{\boldsymbol{\vartheta}}(\boldsymbol{\vartheta}) - \boldsymbol{\vartheta}]\right\}_{m=n+1}^{\infty}$ and $\left\{\sqrt{m}\left(\frac{1}{2}\mathbf{S}_{\Sigma}^{-1}\right)^{1/2}\cdot\right.$ $\left.\cdot(\hat{\boldsymbol{\vartheta}}^* - \boldsymbol{\vartheta})\right\}_{m=n+1}^{\infty}$ have the same asymptotical distribution.

2. The asymptotical confidence ellipsoid for a function of the parameter $\boldsymbol{\beta}$

Theorem 2.1. *Let* $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$ *and* $\boldsymbol{g}(\cdot)$ *be from Lemma* 1.1.
Let $\hat{\hat{\boldsymbol{\beta}}} = (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\bar{\boldsymbol{Y}}$, $\hat{\boldsymbol{v}} = \bar{\boldsymbol{Y}} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}$ *and* $\hat{\mathbf{\Gamma}}_1 = \mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{G}'\cdot$ $\cdot(1 + \hat{\boldsymbol{v}}'\mathbf{S}^{-1}\hat{\boldsymbol{v}})$. *Then*

$$\sqrt{m}\hat{\mathbf{\Gamma}}_1^{-1/2}[\mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\bar{\boldsymbol{Y}} - \mathbf{G}\boldsymbol{\beta}] \xrightarrow[m]{L} N_r(\boldsymbol{O}, \mathbf{I})$$

and

$$m(\mathbf{G}\hat{\hat{\boldsymbol{\beta}}} - \mathbf{G}\boldsymbol{\beta})'\,\hat{\mathbf{\Gamma}}_1^{-1}(\mathbf{G}\hat{\hat{\boldsymbol{\beta}}} - \mathbf{G}\boldsymbol{\beta}) \xrightarrow[m]{L} \chi_r^2.$$

Proof. Let $\boldsymbol{s} = (S_{11}, S_{12}, \ldots, S_{1n}; S_{22}, S_{23}, \ldots, S_{2n}; \ldots; S_{n-1,n-1}, S_{n-1,n}; S_{n,n})'$, $S_{ij} = \{\mathbf{S}\}_{i,j}$, $i,j = 1, \ldots, n$.
Then

$$\sqrt{m}\begin{bmatrix}\bar{\boldsymbol{Y}} - \mathbf{X}\boldsymbol{\beta}\\ \boldsymbol{s} - \boldsymbol{\sigma}\end{bmatrix} \xrightarrow[m]{L} N_{n+r}\left[\boldsymbol{O}; \begin{bmatrix}\Sigma, & \mathbf{0}\\ \mathbf{0}, & \mathbf{\Gamma}_1(\boldsymbol{\sigma})\end{bmatrix}\right],$$

where $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1n}; \ldots; \sigma_{n-1,n-1}, \sigma_{n-1,n}; \sigma_{n,n})'$, $\{\Sigma\}_{i,j} = \sigma_{ij}$, $i,j = 1, \ldots, n$, $r = n(n+1)/2$, $\{\mathbf{\Gamma}_1(\boldsymbol{\sigma})\}_{ij,kl} = \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}$, $i, j, k, l = 1, \ldots, n$.
If $\boldsymbol{g}(\hat{\hat{\boldsymbol{\beta}}}) = \boldsymbol{g}_1(\bar{\boldsymbol{Y}}, \boldsymbol{s}) = \mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\bar{\boldsymbol{Y}}$, then

$$[\partial\boldsymbol{g}_1/\partial(\bar{\boldsymbol{Y}}', \boldsymbol{s}')]\begin{bmatrix}\mathbf{S}, & \mathbf{0}\\ \mathbf{0}, & \hat{\mathbf{\Gamma}}_1(\boldsymbol{s})\end{bmatrix}\partial\boldsymbol{g}_1'/\partial[(\bar{\boldsymbol{Y}}', \boldsymbol{s}')]' = (\partial\boldsymbol{g}_1/\partial\bar{\boldsymbol{Y}}')\mathbf{S}\,\partial\boldsymbol{g}_1'/\partial\bar{\boldsymbol{Y}} +$$

$$+ (\partial\boldsymbol{g}_1/\partial\boldsymbol{s}')\hat{\mathbf{\Gamma}}_1(\boldsymbol{s})\partial\boldsymbol{g}_1'/\partial\boldsymbol{s}$$

and

$$\partial \boldsymbol{g}_1/\partial \bar{\boldsymbol{Y}}' = \mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1} \Rightarrow (\partial \boldsymbol{g}_1/\partial \bar{\boldsymbol{Y}}')\,\mathbf{S}\,\partial \boldsymbol{g}_1'/\partial \bar{\boldsymbol{Y}} = \mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{G}'$$

$$\partial \boldsymbol{g}_1/\partial S_{ij} = \mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}(\partial \mathbf{S}/\partial S_{ij})\,\mathbf{S}^{-1}(\mathbf{X}\hat{\hat{\boldsymbol{\beta}}} - \bar{\boldsymbol{Y}}) \Rightarrow$$

$$\Rightarrow (\partial \boldsymbol{g}_1/\partial \boldsymbol{s}')\,\hat{\Gamma}_1(\boldsymbol{s})\,\partial \boldsymbol{g}_1'/\partial \boldsymbol{s} = \hat{\boldsymbol{v}}'\mathbf{S}^{-1}\hat{\boldsymbol{v}}\mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{G}'.$$

The last implication can be proved in the following way:

$$(\partial \boldsymbol{g}_1/\partial \boldsymbol{s}')\,\hat{\Gamma}_1(\boldsymbol{s})\,\partial \boldsymbol{g}_1'/\partial \boldsymbol{s} = \mathbf{G}(\partial \hat{\hat{\boldsymbol{\beta}}}/\partial \boldsymbol{s}')\,\hat{\Gamma}_1(\boldsymbol{s})(\partial \hat{\hat{\boldsymbol{\beta}}}'/\partial \boldsymbol{s})\,\mathbf{G}'$$

and

$$(\partial \hat{\hat{\boldsymbol{\beta}}}/\partial \boldsymbol{s}')\,\hat{\Gamma}_1(\boldsymbol{s})\,\partial \hat{\hat{\boldsymbol{\beta}}}'/\partial \boldsymbol{s} = (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\sum_{i\leqslant j}^{n}\sum_{k\leqslant l}^{n}(\partial \mathbf{S}/\partial S_{ij})\cdot$$

$$\cdot\mathbf{S}^{-1}\hat{\boldsymbol{v}}(S_{ik}S_{jl} + S_{il}S_{jk})\,\hat{\boldsymbol{v}}'\mathbf{S}^{-1}(\partial \mathbf{S}/\partial S_{kl})\,\mathbf{S}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1},$$

where

$$\partial \mathbf{S}/\partial S_{ij} = \begin{cases} \boldsymbol{e}_i\boldsymbol{e}_i', & i = j, \\ \boldsymbol{e}_i\boldsymbol{e}_j' + \boldsymbol{e}_j\boldsymbol{e}_i', & i \neq j, \end{cases}$$

$$\boldsymbol{e}_i = (0_1, \ldots, 0_{i-1}, 1_i, 0_{i+1}, \ldots, 0_n)', \quad i = 1, \ldots, n.$$

The (r, s)th element $K_{r,s}$ of the matrix

$$\sum_{i\leqslant j}^{n}\sum_{k\leqslant l}^{n}(\partial \mathbf{S}/\partial S_{ij})\,\mathbf{S}^{-1}\hat{\boldsymbol{v}}(S_{ik}S_{jl} + S_{il}S_{jk})\,\hat{\boldsymbol{v}}'\mathbf{S}^{-1}(\partial \mathbf{S}/\partial S_{kl})$$

is

$$K_{r,s} = \boldsymbol{e}_r'\sum_{i\leqslant j}^{n}\sum_{k\leqslant l}^{n}(\partial \mathbf{S}/\partial S_{ij})\,\mathbf{S}^{-1}\hat{\boldsymbol{v}}(S_{ik}S_{jl} + S_{il}S_{jk})\,\hat{\boldsymbol{v}}'\mathbf{S}^{-1}(\partial \mathbf{S}/\partial S_{kl})\,\boldsymbol{e}_s =$$

$$= \{\mathbf{S}\hat{\boldsymbol{v}}'\mathbf{S}^{-1}\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}'\}_{r,s}, \; r, \; s = 1, \ldots, n.$$

As $\mathbf{X}'\mathbf{S}^{-1}\hat{\boldsymbol{v}} = \mathbf{X}'\mathbf{S}^{-1}(\bar{\boldsymbol{Y}} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}) = \boldsymbol{0}$, we have

$$(\partial \hat{\hat{\boldsymbol{\beta}}}/\partial \boldsymbol{s}')\,\hat{\Gamma}_1(\boldsymbol{s})\,\partial \hat{\hat{\boldsymbol{\beta}}}'/\partial \boldsymbol{s} = (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\hat{\boldsymbol{v}}'\mathbf{S}^{-1}\hat{\boldsymbol{v}}\mathbf{S}\mathbf{S}^{-1}\mathbf{X}(\mathbf{X}\mathbf{S}^{-1}\mathbf{X})^{-1} =$$

$$= \hat{\boldsymbol{v}}'\mathbf{S}^{-1}\hat{\boldsymbol{v}}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}$$

and thus

$$(\partial \boldsymbol{g}_1/\partial \boldsymbol{s}')\,\hat{\Gamma}_1(\boldsymbol{s})\,\partial \boldsymbol{g}_1'/\partial \boldsymbol{s} = \hat{\boldsymbol{v}}'\mathbf{S}^{-1}\hat{\boldsymbol{v}}\mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{G}'.$$

The rest of the proof follows from Lemmas 1.2 and 1.3.

Remark 2.1. The confidence ellipsoid for the function $\boldsymbol{g}(\cdot)$ from Lemma 1.1 and the confidence ellipsoid

$$\{u: (u - \mathbf{G}\hat{\hat{\beta}})'[\mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{G}']^{-1}(u - \mathbf{G}\hat{\hat{\beta}})/(1 + \hat{v}'\mathbf{S}^{-1}\hat{v}) \leq \chi_r^2(1-\alpha)\,m\}$$

from Theorem 2.1 have a similar shape after the term $\frac{m}{m-1} - 1$ has been neglected. The ratio of their axes is

$$((m-1)\,rF_{r,m-1-(n-k)}(1-\alpha)/\{[m-1-(n-k)]\,\chi_r^2(1-\alpha)\})^{1/2} =$$
$$= p(1-\alpha, m, r, n-k)$$

(cf. Table 2.1).

Table 2.1

		$1-\alpha = 0.95$	
$n-k$	m	r	$p(1-\alpha, n-k, m, r)$
3	10	2	1.605*)
3	10	1	1.529
3	14	2	1.335
3	14	1	1.296
3	24	2	1.158
3	24	1	1.142
3	50	2	1.068
3	50	1	1.061

*) The "asymptotical" ellipsoid is included into the ellipsoid from Lemma 1.1.

If the number of replications is sufficiently large, the asymptotical procedure can be used for determining an approximate confidence region within the models with a given structure of the covariance matrix; this is shown in Table 2.1.

Theorem 2.2. *Let the assumption of Lemma* 1.1 *be satisfied and the covariance matrix* Σ *be diagonal (in this case the notation* Δ *instead of* Σ *is used). If*

$$\mathbf{D} = \mathrm{Diag}(\mathbf{S}),$$
$$\hat{\hat{\beta}} = (\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}^{-1}\bar{Y},$$
$$\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)' = \bar{Y} - \mathbf{X}\hat{\hat{\beta}},$$
$$\hat{\Gamma}_2 = \mathbf{G}[(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1} + 2(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}^{-1}\mathrm{Diag}(v_1^2, \ldots, v_n^2)\cdot$$
$$\cdot\mathbf{D}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}]\mathbf{G}',$$

then

$$\sqrt{m}\,\hat{\boldsymbol{\Gamma}}_2^{-1/2}(\mathbf{G}\hat{\hat{\boldsymbol{\beta}}} - \mathbf{G}\boldsymbol{\beta}) \xrightarrow[m]{L} N_r(\boldsymbol{0}, \mathbf{I})$$

and

$$m(\mathbf{G}\hat{\hat{\boldsymbol{\beta}}} - \mathbf{G}\boldsymbol{\beta})'\hat{\boldsymbol{\Gamma}}_2^{-1}(\mathbf{G}\hat{\hat{\boldsymbol{\beta}}} - \mathbf{G}\boldsymbol{\beta}) \xrightarrow[m]{L} \chi_r^2.$$

Proof. If $\boldsymbol{d} = (D_{11}, D_{22}, \ldots, D_{nn})'$, $\boldsymbol{\delta} = (\sigma_{11}, \sigma_{22}, \ldots, \sigma_{nn})'$, then

$$\sqrt{m}\begin{bmatrix} \bar{\boldsymbol{Y}} - \mathbf{X}\boldsymbol{\beta} \\ \boldsymbol{d} - \boldsymbol{\delta} \end{bmatrix} \xrightarrow[m]{L} N_{2n}\left[\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}; \begin{bmatrix} \Delta, & \mathbf{0} \\ \mathbf{0}, & 2\Delta^2 \end{bmatrix}\right].$$

If $\boldsymbol{g}(\boldsymbol{\beta}) = \mathbf{G}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathscr{R}^k$ and $\boldsymbol{g}_1(\bar{\boldsymbol{Y}}, \boldsymbol{d}) = \mathbf{G}(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}^{-1}\bar{\boldsymbol{Y}} = \mathbf{G}\hat{\hat{\boldsymbol{\beta}}}$, then obviously

$$[\partial \boldsymbol{g}_1/\partial(\bar{\boldsymbol{Y}}', \boldsymbol{d}')]\begin{bmatrix} \mathbf{D}, & \mathbf{0} \\ \mathbf{0}, & 2\mathbf{D}^2 \end{bmatrix}\partial \boldsymbol{g}_1'/\partial[(\bar{\boldsymbol{Y}}', \boldsymbol{d}')]' =$$

$$= \mathbf{G}[(\partial\hat{\hat{\boldsymbol{\beta}}}/\partial\bar{\boldsymbol{Y}}')\,\mathbf{D}\,\partial\hat{\hat{\boldsymbol{\beta}}}'/\partial\bar{\boldsymbol{Y}} + 2(\partial\hat{\hat{\boldsymbol{\beta}}}/\partial\boldsymbol{d}')\,\mathbf{D}^2\partial\hat{\hat{\boldsymbol{\beta}}}'/\partial\boldsymbol{d}]\,\mathbf{G}',$$

where

$$\partial\hat{\hat{\boldsymbol{\beta}}}/\partial\bar{\boldsymbol{Y}}' = (\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}^{-1},\ \partial\hat{\hat{\boldsymbol{\beta}}}/\partial D_{ii} = -(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}^{-1}\boldsymbol{e}_i\{\mathbf{D}^{-1}\hat{\boldsymbol{v}}\}_i,$$
$$i = 1, \ldots, n.$$

Thus

$$(\partial\hat{\hat{\boldsymbol{\beta}}}/\partial\bar{\boldsymbol{Y}}')\,\mathbf{D}\,\partial\hat{\hat{\boldsymbol{\beta}}}'/\partial\bar{\boldsymbol{Y}} = (\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}$$

and

$$(\partial\hat{\hat{\boldsymbol{\beta}}}/\partial\boldsymbol{d}')\,2\mathbf{D}^2\partial\hat{\hat{\boldsymbol{\beta}}}'/\partial\boldsymbol{d} = 2(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}^{-1}\,\mathrm{Diag}\,(\hat{v}_1^2, \ldots, \hat{v}_n^2)\,\mathbf{D}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}.$$

The rest of the proof follows from Lemmas 1.2 and 1.3.

Theorem 2.3. *Let the assumptions of Lemma* 1.1 *be satisfied, let the covariance matrix* $\boldsymbol{\Sigma}$ *be of the structure* $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ *and fulfil the assumptions of Lemma* 1.4.

If

$$\hat{\vartheta}_m^* = \mathbf{S}_{\hat{\Sigma}_m^{-1}}^{-1}[\mathrm{Tr}\,(\hat{\boldsymbol{\Sigma}}_m^{-1}\mathbf{V}_1\hat{\boldsymbol{\Sigma}}_m^{-1}\mathbf{S}), \ldots, \mathrm{Tr}\,(\hat{\boldsymbol{\Sigma}}_m^{-1}\mathbf{V}_p\boldsymbol{\Sigma}_m^{-1}\mathbf{S})]',$$

$$\hat{\boldsymbol{\Sigma}}_m = \sum_{i=1}^{p} \hat{\vartheta}_i(\vartheta_0)\,\mathbf{V}_i,\quad \hat{\boldsymbol{\Sigma}}^* = \sum_{i=1}^{p} \hat{\vartheta}_i^*\mathbf{V}_i,$$

$$\hat{\hat{\boldsymbol{\beta}}} = (\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}\bar{\boldsymbol{Y}},\ \hat{\boldsymbol{v}} = \bar{\boldsymbol{Y}} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}},$$

$$\hat{\boldsymbol{\Gamma}}_3 = \mathbf{G}[(\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}\mathbf{X})^{-1} + (\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}(\mathbf{V}_1\hat{\boldsymbol{\Sigma}}^{*-1}\hat{\boldsymbol{v}}, \ldots, \mathbf{V}_p\hat{\boldsymbol{\Sigma}}^{*-1}\hat{\boldsymbol{v}})\cdot$$
$$\cdot 2\mathbf{S}_{\hat{\Sigma}^{*-1}}^{-1}(\mathbf{V}_1\hat{\boldsymbol{\Sigma}}^{*-1}\hat{\boldsymbol{v}}, \ldots, \mathbf{V}_p\hat{\boldsymbol{\Sigma}}^{*-1}\hat{\boldsymbol{v}})'\hat{\boldsymbol{\Sigma}}^{*-1}\mathbf{X}(\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}\mathbf{X})^{-1}]\,\mathbf{G}',$$

then

$$\sqrt{m}\hat{\boldsymbol{\Gamma}}_3^{-1/2}(\mathbf{G}\hat{\hat{\boldsymbol{\beta}}} - \mathbf{G}\boldsymbol{\beta}) \xrightarrow[m]{L} N_r(\mathbf{0}, \mathbf{I})$$

and

$$m(\mathbf{G}\hat{\hat{\boldsymbol{\beta}}} - \mathbf{G}\boldsymbol{\beta})'\hat{\boldsymbol{\Gamma}}_3^{-1}(\mathbf{G}\hat{\hat{\boldsymbol{\beta}}} - \mathbf{G}\boldsymbol{\beta}) \xrightarrow[m]{L} \chi_r^2.$$

Proof. It is an analogy of the proofs of the previous theorems; it is sufficient to take into account the following relations:

$$\partial\hat{\hat{\boldsymbol{\beta}}}/\partial\bar{\boldsymbol{Y}}' = (\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1},$$

$$\partial\hat{\hat{\boldsymbol{\beta}}}/\partial\hat{\vartheta}_i^* = -(\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\boldsymbol{\Sigma}}^{*-1}\mathbf{V}_i\hat{\boldsymbol{\Sigma}}^{*-1}\hat{\boldsymbol{v}}, \quad i = 1, \ldots, p,$$

and (b) of Lemma 1.4.

Remark 2.2. If $\mathbf{V}_i = \boldsymbol{e}_i\boldsymbol{e}_i'$, $i = 1, \ldots, p$, then $\hat{\boldsymbol{\Gamma}}_3$ (Theorem 2.3) $= \hat{\boldsymbol{\Gamma}}_2$ (Theorem 2.2).

Example 2.1 (a comparison of the results from Theorems 2.1 and 2.2). Let

$$\mathbf{X} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1, & 0 \\ 0, & 2 \end{bmatrix}, \quad \beta = 8 \text{ and } m = 11.$$

Let

$$\bar{\boldsymbol{Y}} = \begin{bmatrix} 8.3 \\ 7.5 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 1, & 0.5 \\ 0.5, & 2 \end{bmatrix}.$$

a) We know nothing of the matrix $\boldsymbol{\Sigma}$ (Theorem 2.1). Then

$$\hat{\hat{\beta}} = (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\bar{\boldsymbol{Y}} = 8.10,$$

$$\hat{\boldsymbol{v}} = \bar{\boldsymbol{Y}} - \mathbf{X}\hat{\hat{\beta}} = (0.2, \ -0.6)',$$

$$\sqrt{(1 + \boldsymbol{v}'\mathbf{S}^{-1}\boldsymbol{v})(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}/(m-1)} = 1.08/\sqrt{10} = 0.345.$$

b) We know that the matrix $\boldsymbol{\Sigma}$ is diagonal (Theorem 2.2). Then

$$\hat{\hat{\beta}} = (\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}^{-1}\bar{\boldsymbol{Y}} = 8.03,$$

$$\hat{\boldsymbol{v}} = \bar{\boldsymbol{Y}} - \mathbf{X}\hat{\hat{\beta}} = (0.27, \ -0.53)',$$

$$\sqrt{[(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1} + 2(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{D}^{-1}\operatorname{Diag}(\hat{v}_1^2, \hat{v}_2^2)\mathbf{D}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X})^{-1}]/(m-1)} =$$

$$= 0.90/\sqrt{10} = 0.285.$$

The estimates of the accuracy characteristics of $\hat{\hat{\beta}}$ differ, which shows clearly that the a priori information on the structure of the covariance matrix gives a non-negligible effect and therefore it ought to be taken into account.

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Matematický ústav SAV
Obrancov mieru 49
814 73 Bratislava

АСИМПТОТИЧЕСКИЕ ДОВЕРИТЕЛЬНЫЕ ОБЛАСТИ В ПОВТОРЕННОЙ СМЕШАННОЙ ЛИНЕЙНОЙ МОДЕЛИ С ОЦЕНИВАЕМОЙ КОВАРИАЦИОННОЙ МАТРИЦЕЙ

Lubomír Kubáček

Резюме

В регрессионной модели $\boldsymbol{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$ предполагается, что ковариационная матрица неизвестна, или известна только частично (например, она диагональна, или $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, где $\vartheta_1, \ldots, \vartheta_p$ — неизвестные ковариационные компоненты). На основе повторенных реализаций случайного вектора $\boldsymbol{Y}$ найдены границы доверительной области для вектор-функции неизвестного параметра $\boldsymbol{\beta}$.