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HOMOMORPHISMS OF FINITE BIPARTITE GRAPHS ONTO COMPLETE BIPARTITE GRAPHS

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In [1] F. Harary, D. Hsu and Z. Miller have introduced the concepts of a bicomplete homomorphism and bichromaticity of a bipartite graph.

Let B be a connected bipartite graph on the vertex sets C, D. A bicomplete homomorphism of B is a homomorphic mapping φ of B onto a complete bipartite graph $K_{r,s}$ (where r, s are positive integers) with the property that $\varphi(x)=\varphi(y)$ only if either both x, y belong to C, or both x, y belong to D. The bichromaticity $\beta(B)$ of the graph B is the maximum value of $r+s$ for all complete bipartite graphs $K_{r,s}$ onto which B can be mapped by a bicomplete homomorphism. (In [1] only finite graphs are considered.)

In [4] an analogous concept was introduced and studied for infinite graphs. In the present paper we shall study it for finite graphs.

For a connected bipartite graph B the symbol $\beta_0(B)$ denotes the supremum of the values min (r, s) for all complete bipartite graphs $K_{r,s}$ (where r, s are positive integers or infinite cardinal numbers) onto which B can be mapped by a bicomplete homomorphism. This definition was so formulated in order that it might have a sense also for infinite graphs. If we consider only finite graphs, we may say that $\beta_0(B)$ is the maximum value of min (r, s) for all complete bipartite graphs $K_{r,s}$ onto which B can be mapped by a bicomplete homomorphism.

Proposition 1. *Let B be a finite connected bipartite graph. Then $\beta_0(G)$ is equal to the maximal value of r for all complete bipartite graphs $K_{r,r}$ onto which B can be mapped by a bicomplete homomorphism.*

Proof. If r, s are two positive integers, $r\leqq s$, then evidently there exists a bicomplete homomorphism of $K_{r,s}$ onto $K_{r,}K_{r}$. If B can be mapped onto $K_{r,s}$ by a bicomplete homomorphism, we may supperpose this homomorphism with a bicomplete homomorphism of $K_{r,s}$ onto $K_{r,r}$ and thus we obtain a bicomplete homomorphism of B onto $K_{r,r}$, where $r=\min(r, s)$. This implies the assertion.

A matching of a bipartite graph B is a subset M of the edge set of B with the property that no two edges of M have a common end vertex. (This concept was defined in [2] in a slightly different way, but this difference is not essential.)

Proposition 2. *Let B be a finite connected bipartite graph, let k be the number of edges of B. Then*

$$\beta_0(B) \leqq \sqrt{k}$$

Proof. An image of B in a bicomplete homomorphism evidently cannot have more edges that B. The graph $K_{r,r}$, where $r = \beta_0(B)$, has r^2 edges, hence $r^2 \leq k$ and this implies the assertion.

Theorem 1. *Let B be a finite connected bipartite graph, let m be the maximal number of elements of a matching of B Then*

$$[\sqrt{m}] \leqq \beta_0(B) \leq m$$

and this inequality cannot be improved.

Proof. Let the vertex sets of B be C, D. Let M be a matching of B having m elements. Denote $k = [\sqrt{m}]$. Then $k^2 \leqq m$. Choose a subset M_0 of M having k^2 elements. Denote the elements of M_0 by $e(i, j)$, where $1 \leq i \leqq k$, $1 \leq j \leqq k$. For any i, j let $c(i, j)$ (or $d(i, j)$) be the end vertex of the edge $e(i, j)$ belonging to C (or to D respectively). Denote by C_0 (or D_0) the set of all vertices $c(i, j)$ (or $d(i, j)$ respectively) for all pairs i, j. We may define a homomorphic mapping φ of B onto $K_{k,k}$ as follows. For any two vertices $c(i_1, j_1)$, $c(i_2, j\,)$ of C we have $\varphi(c(i_1, j\,)) = \varphi(c(i_2, j_2))$ if and only if $i_1 = i_2$. For any two vertices $d(i_1, j_1)$, $d(i_2, j_2)$ of D we have $\varphi(d(i_1, j_1)) = \varphi(d(i_2, j_2))$ if and only if $j_1 = j_2$. The image in φ of any vertex of $C - C_0$ (or $D - D_0$) is equal to the image of some vertex of C_0 (or D_0 respectively). Evidently φ is a bicomplete homomorphism of B onto $K_{k\,k}$ and therefore $k = [\sqrt{m}] \leqq \beta_0(B)$. On the other hand evidently an image of B in a bicomplete homomorphism cannot have a matching with more elements than m. The maximal number of elements of a matching of $K_{r,s}$ is min (r, s), hence $\beta_0(B) \leq m$.

Now suppose that m is a square of an integer. Consider the bipartite graph B on the vertex sets $C = \{c_1, \ldots, c_m\}$, $D = \{d_1, \ldots, d_m\}$ with the edges $c_i d_i$ and $c_1 d_i$ for $i = 1, \ldots, m$. This graph is in Fig. 1. The maximal number of elements of a matching of B is m. Suppose that $\beta_0(B) \geq \sqrt{m} + 1$. Then B can be mapped by a bicomplete homomorphism φ onto a complete bipartite graph $K_{h\,h}$, where $h = \sqrt{m} + 1$. The degree of any vertex of $K_{h\,h}$ is h. Let C' be the set of images of vertices of C in φ. Each vertex of $C' - \{\varphi(c_1)\}$ must be the image of at least h vertices of $C - \{c_1\}$, because each vertex of $C - \{c_1\}$ has the degree 1 in B. But then $C - \{c_1\}$ must contain at least $h(h-1) - m + \sqrt{m}$ vertices, which is a contradiction. Therefore $\beta_0(B) = \sqrt{m}$ and the lower bound is attained. In the case when $B \cong K_{m\,m}$ the upper bound is attained.

Proposition 3. *For a finite connected bipartite graph B there is $\beta_0(B)=1$ if and only if B is a tres whose diameter is at most* 3.

Proof. If a connected graph is not a tree, then it contains a circuit. If this graph is bipartite, this circuit has an even length at least 4. A circuit of the length 4 is $K_{2,2}$. Consider a circuit C_{2k} of the length $2k$, where k is a positive integer. Let its vertices be $u_1, \ldots, u_{2k}$ and let its edges be $u_i u_{i+1}$ for $i=1, \ldots, 2k-1$ and $u_{2k}u_1$. Let C_4 be

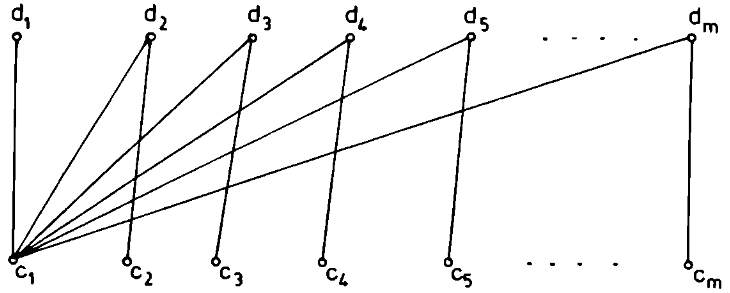

Fig. 1

a circuit of the length 4, let its vertices be v_1, v_2, v_3, v_4 and let its edges be v_1v_2, v_2v_3, v_3v_4, v_4v_1. Define the mapping φ so that $\varphi(u_1)=v_1$, $\varphi(u_2)=v_2$ and for $i \geqq 3$ there is $\varphi(u_i)=v_3$ if i is odd and $\varphi(u_i)=v_4$ if i is even. The mapping φ is a bicomplete homomorphism of C_{2k} onto $C_4 \cong K_{2,2}$. If B contains C_{2k} as a subgraph, then this homomorphism can be easily extended to a bicomplete homomorphism of B onto $K_{2,2}$ and $B_0(B) \geqq 2$. Hence B must be a tree. Suppose that B contains vertices u, v whose distance is 4. As the distance of u, v is even, they may be mapped by a bicomplete homomorphism onto the same vertex and thus a circuit of the length 4 is obtained. Therefore there may be $\beta_0(B)=1$ only if B is a tree of the diameter at most 3. Conversely, if B is a tree of the diameter at most 3, then evidently it can be mapped by no homomorphism onto a graph with a circuit of an even length, therefore $\beta_0(B)=1$.

Proposition 4. *Let B be a finite connected bipartite graph, let B′ be the graph obtained from B by deleting an edge which is not a bridge. Then*

$$\beta_0(B') \geqq \beta_0(B)-1.$$

Proof. Let $\beta_0(B)=r$, let φ be a bicomplete homomorphism of B onto $K_{r,r}$. We may suppose $r \geqq 2$, because etherwise B would be a tree and all its edges would be bridges. Then B' is mapped by φ either also onto $K_{r,r}$, or onto a graph obtained from $K_{r,r}$ by deleting an edge. The graph in the second case can be evidently mapped by a bicomplete homomorphism onto $K_{r\;1,r-1}$. By superposing φ with this homomorphism we obtain a bicomplete homomorphism of B' onto $K_{r-1,r-1}$ and the assertion is proved.

Theorem 2. *Let m, r be positive integers such that $[\sqrt{m}] \leqq r \leqq m$. Then there exists a finite connected bipartite graph B such that $\beta_0(B) = r$ and the maximal number of edges of a matching of B is m.*

Proof. Take a complete bipartite graph $K_{m,m}$ and choose its spanning subraph B_0 isomorphic to the graph in Fig. 1. Each spanning subgraph of $K_{m,m}$ which contains B_0 as a subgraph has a matching with m elements and no matching with more than m elements. We have $\beta_0(K_{m,m}) = m$, $\beta_0(B_0) = [\sqrt{m}]$. If we delete subsequently the edges of $K_{m,m}$ not belonging to B_0, according to Proposition 4 we must obtain graphs of all values of $\beta_0(B)$ which lie between these two numbers.

Theorem 3. *Let C_n be a circuit of the length n, where n is even and $n \geqq 4$. Let r be the greatest integer with the property that either r is even and $r^2 \leqq n$, or n is odd and $r(r+1) \leqq n$. Then*

$$\beta_0(C_n) = r.$$

Proof. Suppose that C_n can be mapped by a bicomplete homomorphism φ onto a complete bipartite graph $K_{h,h}$. Let H be the multigraph obtained from $K_{h,h}$ in such a way that each edge e of $K_{h,h}$ is replaced by k edges, where k is the number of edges of C_n which are mapped by φ onto e. Then there exists a one-to-one correspondence between the edge set of C_n and the edge set of H such that if we go around C_n and take the corresponding edges in H, we obtain a closed Eulerian trail in H. This implies that H is an Eulerian multigraph, i.e. the degrees of all vertices of H are even. Thus the number n of edges of C_n must be greater than or equal to the minimal number of edges of an Eulerian multigraph H whose spanning subgraph is $K_{h,h}$ and which is a bipartite multigraph on the same vertex sets as $K_{h,h}$. If h is even, then such a multigraph is $K_{h,h}$ itself, because it is an Eulerian graph; it has h^2 edges. If h is odd, then such a multigraph is obtained by adding h edges to $K_{h,h}$ (because the degrees of all vertices of $K_{h,h}$ are odd and in H they must be even) and has $h(h+1)$ edges. On the other hand, there exists a homomorphism of C_n onto an arbitrary circuit of an even length less than n in which two vertices have the same image only if their distance is even (it can be constructed analogously to the proof of Proposition 3). Hence if h is even and $h^2 \leqq n$, the graph C_n can be mapped by a bicomplete homomorphism onto $K_{h,h}$; this homomorphism can be constructed by means of an arbitrarily chosen closed Eulerian trail in $K_{h,h}$. Similarly if h is odd and $h(h+1) \leqq n$. This implies the assertion.

Theorem 4. *Let P_n be a snake (path) of the length n. Let r be the greatest integer with the property that either r is even and $r^2 \leqq n$, or r is odd and $r(r+1) \leqq n$. Then*

$$\beta_0(P_n) = r.$$

The proof is analogous to the proof of Theorem 3.

Theorem 5. *Let B be a bipartite graph obtained from a complete bipartite graph $K_{r,r}$, where $r \geqq 3$, by deleting the edges of a linear factor. Then*

$$\beta_0(B) = [\tfrac{3}{4} r].$$

Proof. Let the vertex sets of B be $C = \{c_1, \ldots, c_r\}$, $D = \{d_1, \ldots, d_r\}$ and let the vertices c_i, d_j be adjacent in B if and only if $i \neq j$. Let φ be a bicomplete homomorphism of B onto a complete bipartite graph $K_{h,h}$. Then for each i either c_i, or d_i must have the property that its image in φ is equal to the image of another vertex; otherwise the images of c_i and d_i would not be adjacent in $K_{h,h}$, which is impossible. This implies that $K_{h,h}$ cannot have more than $\frac{3}{2} r$ vertices and thus $h \leqq [\frac{3}{4} r]$ and $\beta_0(B) \leqq [\frac{3}{4} r]$. Let $s = [\frac{1}{4} r]$, $t = r - 4s$. Consider a complete bipartite graph $K_{p,p}$, where $p = [\frac{3}{4} r]$. If $t = 0$ or $t = 1$, then $p = 3s$; if $t = 2$, then $p = 3s + 1$; if $t = 3$, then $p = 3s + 2$. Let the vertex sets of $K_{p,p}$ be $C' = \{c'_1, \ldots, c'_p\}$, $D' = \{d'_1, \ldots, d'_p\}$. Put $\varphi(c_i) = c'_i$, $\varphi(d_i) = d'_i$ for $i = 1, \ldots, 2s$. Further $\varphi(c_i) = c'_{i-2s}$, $\varphi(d_i) = d'_i$ for $i = 2s + 1, \ldots, 3s$ and $\varphi(c_i) = c'_{i-s}$, $\varphi(d_i) = d'_{i-2s}$ for $i = 3s + 1, \ldots, 4s$. If $t = 0$, the mapping φ is ready. If $t = 1$, there is still $\varphi(c_{4s+1}) = c'_p$, $\varphi(d_{4s+1}) = d'_p$. If $t = 2$, then $\varphi(c_{4s+1}) = \varphi(c_{4s+2}) = c'_p$, $\varphi(d_{4s+1}) = \varphi(d_{4s+2}) = d'_p$. If $t = 3$, then $\varphi(c_{4s+1}) = \varphi(c_{4s+2}) = c'_{p-1}$, $\varphi(c_{4s+3}) = c'_p$, $\varphi(c_{4s+1}) = d'_{p-1}$, $\varphi(d_{4s+2}) = \varphi(d_{4s+3}) = d'_p$. The mapping φ thus constructed is a bicomplete homomorphism of B onto $K_{p,p}$ and thus the assertion is proved.

Finally we shall mention direct products of graphs. The direct product $G_1 \times G_2$ of the graphs G_1, G_2 is the graph whose vertex set is the set of all ordered pairs $[u_1, u_2]$, where u_1 is a vertex of G_1 and u_2 is a vertex of G_2 and in which the vertices $[u_1, u_2]$, $[v_1, v_2]$ are adjacent if and only if either $u_1 = v_1$ and the vertices u_2, v_2 are adjacent in G_2, or $u_2 = v_2$ and the vertices u_1, v_1 are adjacent in G_1. In [1] the authors suggested the problem of determining $\beta(B \times K_2)$ in terms of $\beta(B)$. In [3] this problem was solved by determining the lower bound and the upper bound for $\beta(B \times K_2)$ in terms of $\beta(B)$; these bounds cannot be improved. We shall prove an analogous theorem on $\beta_0(B \times K_2)$.

Theorem 6. *Let B be a finite connected bipartite graph. Then*

$$\beta_0(B \times K_2) \geqq \beta_0(B) + 1$$

and this inequality cannot be improved. There exists no upper bound for $\beta_0(B \times K_2)$ in terms of $\beta_0(B)$.

Proof. Let the vertices of K_2 be u_1, u_2. Let B_1 (or B_2) be the subgraph of $B \times K_2$ induced by the set of all vertices of the form $[x, u_1]$ (or $[x, u_2]$ respectively), where x is a vertex of B. Evidently $B_1 \cong B_2 \cong B$. Let $\beta_0(B) = r$. Then also $\beta_0(B_1) = r$ and there exists a bicomplete homomorphism φ_0 of B_1 onto $K_{r,r}$. Consider the complete bipartite graph $K_{r+1,r+1}$ containing $K_{r,r}$ as a subgraph. Then φ_0 can be extended to a bicomplete homomorphism φ of $B\times K_2$ onto $K_{r+1,r+1}$ in such a way that the restriction of φ onto B_2 will be a bicomplete homomorphism of B_2 onto the subgraph of $K_{r+1,r+1}$ induced by the set $\{c, d\}$, where c, d are the vertices of $K_{r+1,r+1}$ which are not contained in $K_{r,r}$. Therefore $\beta_0(B\times K_2) \geqq r+1 = \beta_0(B)+1$. New suppose that $B\cong K_{h,h}$ for some h. In [3] it was proved that in that case $\beta(B\times K_{22}) = \beta(B)+2$. Evidently $\beta(B)\geqq 2\beta_0(B)$ for each finite connected bipartite graph B, hence also $\beta(B\times K_2) \geqq 2\beta_0(B\times K_2)$, which implies $\beta_0(B\times K_2) \leqq \frac{1}{2}\beta(B)+1 = h+1 = \beta_0(B)+1$ and the inequality cannot be improved.

Now let B be a complete bipartite graph $K_{1,m}$. Then $\beta_0(B)=1$. In the graph $B\times K_2$ there is a matching with $m+1$ edges, therefore $\beta_0(B\times K_2) \geqq \sqrt{m+1}$ according to Theorem 1. The number m can be arbitrarily large, hence also $\beta_0(B\times K_2)$ can be arbitrarily large and there is no upper bound for it in terms of $\beta_0(B)$.

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ГОМОМОРФИЗМЫ КОНЕЧНЫХ ДВУДОЛЬНЫХ ГРАФОВ НА ПОЛНЫЕ ДВУДОЛЬНЫЕ ГРАФЫ

Богдан Зелинка

Резюме

Биполный гомоморфизм конечного связного двудольного грава B на множествах вершин C, $Д$ есть гомоморфное отображение φ графа B на полный двудольный граф, такое, что $\varphi(x)=\varphi(y)$ только тогда, когда вершины x, y принадлежат или обе множеству C, или обе множеству D. Число $\beta_0(B)$ есть максимум всех чисел r, таких, что B можно отобразить биполным гомоморфизмом на полный двудольный граф $K_{r,r}$. В статье исследовано число $\beta_0(B)$ для конечных двудольных графов.