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UNIFORM IDEAL COMPLETIONS

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ABSTRACT. The completion of a poset P by ideals (in the sense of Frink) is characterized abstractly as the smallest algebraic complete lattice containing P as a set of compact elements, while the completion by directed ideals is the smallest algebraic poset with the corresponding property. It is known that both types of ideal completions may also be interpreted as certain topological completions. We show that a suitable intermediate completion by so-called Cauchy ideals may be regarded as a uniform completion of P. For lattices, the Cauchy ideal completion coincides with the Frink ideal completion (and with the completion by directed ideals, provided a least element exists).

In the theory of partially ordered sets (*posets*), various types of ideals have been introduced for concrete constructions of certain universal completions. Prominent examples are

$\mathcal{A}P$, the *Alexandroff completion by order ideals* (*lower sets*),
$\mathcal{N}P$, the *Dedekind-MacNeille completion by normal ideals* (*cuts*),
$\mathcal{I}P$, the *completion by Frink ideals,*
$\mathcal{D}P$, the *completion by directed ideals.*

Details on these completions and a general theory of standard completions for partially ordered sets may be found in [1]–[5], but some of the most relevant facts will be sketched below. We adopt the convention that

all (up-)directed sets, all down-directed sets and all lattices are nonempty.

Let P be any poset. Subsets of the form

$$\downarrow Y = \{x \in P : \ x \leq y \text{ for some } y \in Y\} \qquad (Y \subseteq P)$$

are referred to as *lower sets* (*downsets, decreasing sets, order ideals*), while subsets of the form

$$\uparrow Y = \{x \in P : \ x \geq y \text{ for some } y \in Y\} \qquad (Y \subseteq P)$$

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are called *upper sets* (*upsets, increasing sets, order filters*). In particular, for each element $x \in P$, the set

$$(x] = \downarrow x = \downarrow\{x\}$$

is the principal ideal generated by x, and

$$[x) = \uparrow x = \uparrow\{x\}$$

is the principal dual ideal generated by x.

A *normal ideal* or *cut* is an intersection of principal ideals, or equivalently, the set

$$Y_{\downarrow} = \{x \in P:\ x \leq y \text{ for all } y \in Y\} \qquad (Y \subseteq P)$$

of all lower bounds for some subset Y. Dually, the set of all upper bounds for Y is denoted by $Y^{\uparrow}$. The set $\Delta Y = Y^{\uparrow}{}_{\downarrow}$ is the least cut containing Y, in other words, the cut generated by Y. By an *ideal* of P (in the sense of Frink [6]), we mean a subset I containing with any finite subset $F \subseteq I$ the cut generated by F. While the members of $\mathcal{I}P$, i.e. the Frink ideals, are precisely the directed unions of cuts, the members of $\mathcal{D}P$, i.e. the directed lower sets, are the directed unions of principal ideals. Notice that the empty set $\emptyset$ is never directed, but it is an ideal if and only if P has no least element. In case of a (join-semi)lattice L, ideals may be described in the usual way as subsets I of L such that

$$a \in I \text{ and } b \in I \iff a \vee b \in I$$

and $0 \in I$ provided L has a least element 0.

Of course, in the present context, each system of lower sets like $\mathcal{A}P$, $\mathcal{N}P$, $\mathcal{I}P$, $\mathcal{D}P$ etc. is thought to be ordered by set inclusion. $\mathcal{A}P$, $\mathcal{N}P$ and $\mathcal{I}P$ are always closure systems, i.e., closed under arbitrary intersections. Moreover, $\mathcal{A}P$ is closed under arbitrary unions, while $\mathcal{I}P$ and $\mathcal{D}P$ are closed only under directed unions.

An element c of a poset P is *compact* if and only if c belongs to every directed ideal whose join dominates c, and P is *algebraic* (cf. [4], [5], [8]) if it is *up-complete* (i.e., every directed subset has a join) and each element of P is a join of a directed set of compact elements. A map $\varphi\colon P \to Q$ between posets is *join-dense* if each element of Q is a join of elements in the image of φ. A join-dense (order) embedding of a poset P in an up-complete poset Q (or merely the codomain Q of such an embedding) is called an *up-completion* of P; it is said to be *algebraic* provided Q is an algebraic poset and the image of the embedding consists of compact elements of Q. By a *join-completion*, we mean a join-dense embedding in a complete lattice. Furthermore, we call a map between posets $\mathcal{I}$-*continuous* if preimages of Frink ideals are Frink ideals, and $\mathcal{D}$-*continuous* if preimages of directed ideals are directed ideals. The principal ideal embedding

$$i_P\colon P \to \mathcal{I}P\,, \qquad x \mapsto (x]$$

is easily seen to be an $\mathcal{I}$-continuous algebraic join-completion. It will cause no confusion to call both $\mathcal{I}P$ and i_P the *(Frink) ideal completion* of P. This completion may be characterized abstractly by several extremality properties, for example the following (cf. [3; 8.4.6]):

The ideal completion of a poset P is the least algebraic join-completion and the greatest $\mathcal{I}$-continuous join-completion of P.

The collection $\mathcal{D}P$ of all directed ideals, as well as the corresponding embedding

$$d_P : P \to \mathcal{D}P\,, \qquad x \mapsto (x]$$

are referred to as the *ideal up-completion* of P. Clearly, $\mathcal{D}P$ is contained in $\mathcal{I}P$, but it is a complete lattice only if P happens to be a join-semilattice with least element, in which case $\mathcal{D}P$ coincides with $\mathcal{I}P$. Nevertheless, $\mathcal{D}P$ is always an algebraic (up-complete) poset, and consequently, d_P is actually an up-completion. For a comprehensive study on algebraic posets and their generalizations, see [4] and [5] (where the ideal up-completion was denoted by $\mathcal{D}^{\wedge}P$).

Let us mention briefly two properties of $\mathcal{D}P$, resembling those of the ideal completion $\mathcal{I}P$:

The ideal up-completion of a poset P is the least algebraic up-completion and the greatest $\mathcal{D}$-continuous up-completion of P.

Recall that every poset P carries several intrinsic topologies which in turn determine the given order by setting $a \leq b$ if and only if a belongs to the closure of $\{b\}$ (see, for example, [7]). The largest of these topologies is the *Alexandroff topology*, consisting of all upper sets; the smallest one is the *upper topology* generated by the complements of principal ideals; a third one is the *Scott topology*, consisting of those upper sets U which intersect every directed subset of P having a join that lies in U.

It was shown by R.-E. Hoffmann (see [8], [9]) that for any poset P, either of the posets $\mathcal{I}P$ and $\mathcal{D}P$, endowed with the respective Scott topologies, may be regarded as a certain topological completion or reflection of P endowed with the Alexandroff topology.

Our main purpose is to show that an appropriate join-completion between $\mathcal{D}P$ and $\mathcal{I}P$ becomes a uniform completion for P when both posets are equipped with suitable uniformities. This modified ideal completion is defined as follows. By a *Cauchy ideal* or *uniform ideal* of a poset P, we mean a subset I such that $Y^{\uparrow} \not\subseteq {\uparrow} Z$ for each finite subset Y of I and each finite subset Z of $P \setminus I$; in other words, there is a principal ideal containing Y and disjoint from Z. The previous definition is quite flexible: for example, the same condition with arbitrary instead of finite subsets Z characterizes directed ideals, whereas the corresponding condition with singletons Z describes Frink ideals. Hence, every Cauchy ideal is a Frink ideal, and every directed ideal is a Cauchy ideal. In join-semilattices with least elements, all three types of ideals coincide. More generally, it is easy to see that in down-directed join-semilattices, the Frink ideals coincide with the Cauchy ideals, and the same coincidence holds in up-directed meet-semilattices (by other reasons, however). In particular, for any lattice L, the Cauchy ideals are just the ideals in the usual sense (but recall that the empty set is an ideal only if L has no least element).

The collection of all Cauchy ideals of a poset P will be referred to as the *uniform ideal completion* or *Cauchy ideal completion*, and in the present discussion, it will be denoted by $\mathcal{C}P$ (of course, in other contexts, the letter $\mathcal{C}$ may have a different meaning; for example, in [2] and [5], $\mathcal{C}P$ denotes the collection of all chains of P).

In general, the Cauchy ideal completion $\mathcal{C}P$ is not a complete lattice, but it is always up-complete, being closed under directed unions.

Identifying each element x of a poset P with the principal ideal $(x]$ (which makes sense because the map sending x to $(x]$ is an isomorphism between P and the set of all principal ideals, ordered by inclusion), we arrive at the following inclusion chain:

$$P \subseteq \mathcal{D}P \subseteq \mathcal{C}P \subseteq \mathcal{I}P \subseteq \mathcal{A}P .$$

Now, we define for any join-dense subset B of a poset P a uniformity $\mathcal{U} = \mathcal{U}_{P,B}$ by taking as subbasic entourages the sets

$$U_a = \{(x,y) \in P \times P : \ a \le x \iff a \le y\} \qquad (a \in B).$$

Notice that by antisymmetry of the order relation $\le$ and join-density of B, the intersection of all sets U_a is the diagonal of P, so that $\mathcal{U}$ is in fact a Hausdorff (separated) uniformity. Moreover, $\mathcal{U}$ is not only zero-dimensional but also totally bounded, possessing a subbase of equivalence relations each of which has only two equivalence classes, viz. $[a)$ and $P \setminus [a)$. Hence, the uniform space

$$\mathcal{U}(P,B) = (P, \mathcal{U}_{P,B})$$

has an (up to isomorphism unique) uniform completion $(\overline{P}, \overline{\mathcal{U}})$.

Using the 0-1-valued functionals f_a assigning to $x \in P$ the value 1 if and only if $a \le x$, we have the equation

$$U_a = \{(x,y) \in P \times P : \ f_a(x) = f_a(y)\}$$

for each $a \in P$. Consequently, each f_a extends to a unique uniformly continuous 0-1-valued functional $\overline{f}_a$ on the uniform completion $(\overline{P}, \overline{\mathcal{U}})$, and the sets

$$\overline{U}_a = \{(\overline{x}, \overline{y}) \in \overline{P} \times \overline{P} : \ \overline{f}_a(\overline{x}) = \overline{f}_a(\overline{y})\} \qquad (a \in B)$$

generate the uniformity $\overline{\mathcal{U}}$.

Furthermore, a partial order on $\overline{P}$ is defined by setting

$$\overline{x} \leq \overline{y} \iff \overline{f}_a(\overline{x}) \leq \overline{f}_a(\overline{y}) \qquad \text{for all} \quad a \in B.$$

The antisymmetry of the relation $\leq$ is ensured by the Hausdorff separation property of the completion $(\overline{P}, \overline{\mathcal{U}})$. Notice that

$$a \leq b \text{ in } P \quad \text{implies} \quad \overline{f}_b(\overline{x}) \leq \overline{f}_a(\overline{x}) \text{ for all } \overline{x} \in \overline{P}.$$

Indeed, if $\mathcal{F}$ is a (proper!) filter on P converging to $\overline{x}$ then $\overline{f}_b(\overline{x}) = 1$ means that $\overline{x}$ belongs to the open set $\overline{f}_b^{\,-1}[\{1\}]$, and so there is an $F \in \mathcal{F}$ with

$$F \subseteq \overline{f}_b^{\,-1}[\{1\}] \cap P = f_b^{-1}[\{1\}] \subseteq f_a^{-1}[\{1\}] \subseteq \overline{f}_a^{\,-1}[\{1\}].$$

Assuming $\overline{f}_a(\overline{x}) = 0$, we would find a $G \in \mathcal{F}$ with $G \subseteq \overline{f}_a^{\,-1}[\{0\}]$, which leads to the contradiction $\emptyset = F \cap G \in \mathcal{F}$.

From the above monotonicity law it follows that

$$a \leq \overline{x} \iff \overline{f}_a(\overline{x}) = 1$$

for all $a \in P$ and $\overline{x} \in \overline{P}$, and this gives the equations

$$\overline{U}_a = \{(\overline{x}, \overline{y}) \in \overline{P} \times \overline{P} : \ a \leq \overline{x} \iff a \leq \overline{y}\}$$

and

$$\overline{\mathcal{U}_{P,B}} = \mathcal{U}_{\overline{P},B}.$$

In particular, the sets

$$\overline{U}_a(\overline{x}) = [a) \qquad (a \in P \cap (\overline{x}])$$

and

$$\overline{U}_a(\overline{x}) = \overline{P} \setminus [a) \qquad (a \in P \setminus (\overline{x}])$$

form a closed-open neighborhood base at $\overline{x}$ in the topological space associated with $(\overline{P}, \overline{\mathcal{U}})$. Moreover, we have

$$\forall \overline{x}, \overline{y} \in \overline{P} \left(\overline{x} \leq \overline{y} \iff \forall a \in B\, (a \leq \overline{x} \implies a \leq \overline{y})\right).$$

In other words, B (and consequently P) is join-dense in $\overline{P}$. We shall now demonstrate that a concrete completion of the uniform space

$$\mathcal{U}P = \mathcal{U}(P, P)$$

is provided by the Cauchy ideal completion $\mathcal{C}P$.

THEOREM. *For any ordered set P, the ordered uniform space $\mathcal{U}(\mathcal{C}P,P)$ is a completion of the ordered uniform space $\mathcal{U}P$.*

Proof. Every ideal $I \in \mathcal{I}P$ gives rise to a set-theoretical filter $\mathcal{F}_I$, generated by the sets $[a)$ with $a \in I$ and the sets $P \setminus [a)$ with $a \in P \setminus I$. It is easy to see that this filter is proper (i.e. $\emptyset \notin \mathcal{F}_I$) if and only if I is a Cauchy ideal. In that case, $\mathcal{F}_I$ is a Cauchy filter with respect to the uniformity $\mathcal{U}$: for $a \in I$, we have $F = [a) \in \mathcal{F}_I$ and $F \times F \subseteq U_a$, while for $a \in P \setminus I$, we get $G = P \setminus [a) \in \mathcal{F}_I$ and $G \times G \subseteq U_a$. Hence $\mathcal{F}_I$ converges to a unique limit $\overline{x}_I$ in the completion of $(P,\mathcal{U})$. We claim that $\overline{x}_I$ is the join of I in $\overline{P}$.

By join-density of P in $\overline{P}$, it suffices to prove the equivalence $a \in I \iff a \le \overline{x}_I$ for all $a \in P$. This is achieved by using the functionals f_a as follows:

$$\begin{aligned} a \in I &\implies F = [a) \in \mathcal{F}_I \text{ and } f_a(x) = 1 \text{ for all } x \in F \implies \overline{f}_a(\overline{x}_I) = 1 \\ &\implies a \le \overline{x}_I , \\ a \notin I &\implies G = P \setminus [a) \in \mathcal{F}_I \text{ and } f_a(x) = 0 \text{ for all } x \in G \implies \overline{f}_a(\overline{x}_I) = 0 \\ &\implies a \not\le \overline{x}_I . \end{aligned}$$

The above equivalence may be reformulated as an identity, viz.

$$I = P \cap (\overline{x}_I]$$

for each Cauchy ideal I of P. Hence the assignment

$$e \colon \mathcal{C}P \to \overline{P}, \qquad I \mapsto \overline{x}_I$$

is an order embedding. If $\overline{x}$ is an arbitrary element of $\overline{P}$ then $I = P \cap (\overline{x}]$ is a Cauchy ideal of P. For the proof, choose a (proper!) filter $\mathcal{F}$ on P converging to $\overline{x}$ in $(\overline{P},\overline{\mathcal{U}})$. For finite $Y \subseteq I$ and $Z \subseteq P \setminus I$, we have

$$\overline{x} \in U = \bigcap\{[y) : y \in Y\} \cap \bigcap\{\overline{P} \setminus [z) : z \in Z\} = Y^{\uparrow} \setminus \uparrow Z ,$$

where the arrows refer to the poset $\overline{P}$. As we have seen earlier, the set U is a neighborhood of $\overline{x}$ and therefore $U \cap P$ is a member of $\mathcal{F}$, hence not empty, as desired.

Again by join-density, it follows that $\overline{x}$ is equal to $\overline{x}_I$, and consequently, that e is an order isomorphism. But since the uniformity $\overline{\mathcal{U}}$ coincides with $\mathcal{U}_{\overline{P},P}$, the map e is also a uniform isomorphism between $(\overline{P},\overline{\mathcal{U}})$ and $\mathcal{U}(\mathcal{C}P,P)$ whose restriction to P is the identity map or, more precisely, sends each element of P to the principal ideal generated by that element. $\square$

COROLLARY. *For any lattice L, the underlying poset of the uniform completion of $\mathcal{U}L$ is isomorphic to the ideal completion $\mathcal{I}L$, hence an algebraic complete lattice. The same holds for down-directed join-semilattices and for up-directed meet-semilattices.*

A few supplementary comments are in order.

(1) With slightly more effort, one can prove a generalization of the theorem for join-dense subsets B of P such that each of the sets

$$B_x = \{b \in B : \ b \leq x\} \qquad (x \in P)$$

is a Cauchy ideal of B. In that situation, it turns out that $\mathcal{U}(\mathcal{C}B, B)$ is a uniform completion of $\mathcal{U}(P, B)$.

(2) The notation "*Cauchy ideals*" is motivated by the following observation. For any separated uniform space, one may take the minimal Cauchy filters as the points of the completion. In the present context, it turns out that assigning to each Cauchy ideal I the filter $\mathcal{F}_I$ yields a one-to-one correspondence between Cauchy ideals of P (respectively B) and minimal Cauchy filters of $\mathcal{U}P$ (respectively, of $\mathcal{U}(P, B)$).

(3) In contrast to Frink ideals and to directed ideals, the Cauchy ideals do *not* form an algebraic ordered set in general, although $\mathcal{C}P$ is always up-complete and compactly generated (i.e., every element is a join of compact elements). For lattices, the situation is simplified essentially, because then Frink ideals coincide with Cauchy ideals, and one may work with nets instead of filters. For the general case of ordered sets, however, the filter approach is definitely more appropriate: while any directed ideal may be regarded as a net, this is not the case for arbitrary Cauchy ideals.

The following additional fact is easily verified:

LEMMA. *For any join-dense subset B of a meet-semilattice S, the binary meet is uniformly continuous with respect to the uniformity $\mathcal{U}_{S,B}$. In particular, $\mathcal{U}S$ and its completion are uniform semilattices.*

Finally, a few remarks on the topologies induced by uniformities of the form $\mathcal{U}_{P,B}$. By definition, the corresponding partially ordered space (*pospace*) has a subbase consisting of principal dual ideals and their complements. In particular, it is *totally order-disconnected*; i.e., given $x \not\leq y$ in P, there is a closed-open upper set containing x but not y. It follows that the pospace associated with the uniform completion of $\mathcal{U}(P, B)$ is even a totally order-disconnected compact pospace. Such spaces play the role of the duals of bounded distributive lattices in Priestley's duality [11]. In the special case $B = P$, the topology of $\mathcal{U}P$ is the *half-open interval topology* with a subbase consisting of the "half-open" intervals $[a, b[\, = [a) \setminus [b)$ $(a, b \in P)$.

The study of such (ordered) topological spaces and their compactifications, as well as a more comprehensive investigation of completions for posets with certain distinguished join-bases, is deferred to a forthcoming note. Here we mention only one interesting fact concerning the *Lawson topology* λ_P, which is the join of the Scott topology and the lower topology on P (the upper topology on the dual of P). Using the *Fundamental theorem for compact totally disconnected semilattices* (see [7; VI-3.13], and [10]), one proves easily:

PROPOSITION. *If P is a poset whose Cauchy ideal completion is a complete lattice (for example, if P is a lattice) then the topology induced by the completion of the uniform space $\mathcal{U}P$ is the Lawson topology on the ideal completion $\mathcal{C}P = \mathcal{I}P$. The corresponding compactification is a compact totally disconnected topological unital meet-semilattice. Conversely, up to isomorphism, every compact totally disconnected topological meet-semilattice with greatest element arises in this way from a unique join-semilattice with least element.*

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