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SYMMETRIC HOMOTOPIES FOR SOLVING SYSTEMS OF POLYNOMIAL EQUATIONS

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1. Introduction

During the past decade several numerical methods for the solution of the following problem were suggested.

Problem 1. *Find all isolated solution of a system of equations*

$$P(x) = 0, \tag{1}$$

where P: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ *is a polynomial map of degree* $d = (d_1, \ldots, d_n)$ *(i.e. the k-th component* $P_k(x)$ *of* $P(x)$ *is a polynomial of degree* $d_k \geq 1$ *for all* $k = 1, \ldots, n$*).*

The numerical methods for solving Prob. 1 are based on the homotopy approach and differ from one another mainly by the particular form of the homotopy map used and by the techniques used to prove the convergence of the particular method. (Under (theoretical) convergence of a homotopy method for solving Prob. 1 we understand that each isolated solution of (1) can be approximated with arbitrary precision by a point of at least one homotopy path.)

It is known (Bezout's theorem and its generalizations, see e.g. [8]) that the (reachable) upper bound on the number of different isolated solutions of (1) is given by the (Bezout) number $B = d_1 . d_2 \ldots d_n$. Hence any generally applicable homotopy for solving (1) must follow at least B homotopy paths.

In different fields of application there are problems where the system (1) possesses a special property: it has a symmetric solution set (see Sec. 2 and 3). We shall call such systems symmetric (this corresponds to the concept of equivariancy from [7]). As B may be very large even for relatively few equations and small degrees, it is desirable to have an appropriate homotopy method which could effectively utilize the known symmetry of the system (1).

Li [3] used a homotopy map (with its zero-set symmetric with respect to all permutations of components of solutions) in order to obtain all roots of a single polynomial of degree r by following only one homotopy path in $\boldsymbol{C}^r$. Li, Sauer and Yorke [4] constructed a homotopy map using a special property of a different nature: keeping the zero-set in the hyperplane at infinity unchanged during the continuation they follow less than B homotopy paths in the proper space.

By the results of Zulehner [9] it is sufficient to find a particular polynomial map R: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ of the given degree d such that the system $R(x) = 0$ has exactly B known different solutions which are regulr (i.e. the Jacobi matrix $\mathrm{D}R$ is regular at the solution). By [9] the momotopy map

$$H(x, t) = (1 - t)\,\alpha R(x) + tP(x) \tag{2}$$

$(0 \leq t \leq 1)$ yields for almost all choices of the complex parameter $\alpha \in \boldsymbol{C}$ a convergent homotopy method for Prob. 1. So the problem of the construction of an appropriate homotopy map for solving a particular symmetric system (1) is reduced to

Problem 2. *Find a polynomial map R: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ of given degree d such that $R^{-1}((0)$ consists of exactly B distinct points x at which $\mathrm{D}R(x)$ is regular and the system $R(x) = 0$ is symmetric with respect to the same symmetry as the system* (1) *in Prob.* 1.

In Sec. 2 we discuss a particular case of symmetry — sign-symmetry — and we show that in general Prob. 2 need not have a solution. Sec. 3 is devoted to the general case: we prove the main result (if Prob. 2 has a solution, then almost every polynomial map of degree d can be used to construct a solution of Prob. 2) and describe a probability-one procedure for the construction of homotopy maps (2) for solving Prob. 1 based on our result.

2. Sign-symmetry

An arbitrary change of signs of components x_k of $x \in \boldsymbol{C}^n$ can be expressed by the matrix-vector product $\mathbf{V}x$, where $\mathbf{V}$ is a diagonal $n \times n$ matrix with ± 1 on the diagonal, i.e. $\mathbf{V} = \mathrm{diag}_n(\pm 1)$. Let $\boldsymbol{S}$ denote the group of all $n \times n$ matrices $\mathbf{V} = \mathrm{diag}_n(\pm 1)$ and let $\boldsymbol{G} \subset \boldsymbol{S}$ be a subgroup of $\boldsymbol{S}$.

Definition 1. *Let P: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ be a polynomial map of degree d and let $\boldsymbol{G} \subset \boldsymbol{S}$ be a subgroup. We say that the system* (1) *is $\boldsymbol{G}$-sign-symmetric* (*or that the map P is $\boldsymbol{G}$-sign-symmetric*) *if for all $\mathbf{V} \in \boldsymbol{G}$ there holds*

$$\mathbf{V}P(x) = P(\mathbf{V}x) \quad \textit{for all } x \in \boldsymbol{C}^n.$$

Example 1. An odd polynomial map P is $\boldsymbol{G}$-sign-symmetric, where $\boldsymbol{G} = \{-\boldsymbol{E}, \boldsymbol{E}\}$ and $\boldsymbol{E}$ denotes the unit matrix.

From now on we assume in this section that the map P in Prob. 1 is $\boldsymbol{G}$-sign-symmetric for some $\boldsymbol{G} \subset \boldsymbol{S}$. We have the trivial

Lemma 1. *The solution set $P^{-1}(0)$ of the $\boldsymbol{G}$-sign-sysmmetric system* (1) *is a union of orbits $\boldsymbol{G}[x] = \{\mathbf{V}x | \mathbf{V} \in \boldsymbol{G}\}$, or equivalently,*

$$x \in P^{-1}(0) \Leftrightarrow \forall \mathbf{V} \in \boldsymbol{G}; \qquad \mathbf{V}x \in P^{-1}(0).$$

The main idea of utilizing the symmetry of the system (1) to construct a homotopy (2) for solving Prob. 1 is simple: Let us choose R to be $\boldsymbol{G}$-sign-symmetric. Then for each $t \in [0, 1]$ fixed the map $H_t = H(., t)$ is $\boldsymbol{G}$-sign-symmetric (in x). So it is sufficient to follow numerically only one homotopy path from those starting at points in the hyperplane $t = 0$ which belong into the same orbit. In this way the number of homotopy paths to be numerically followed is equal to the number of orbits of solution of $R(x) = 0$.

Let us now return to Prob. 2. First we summarize all the properties the desired polynomial map R: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ should have to be the solution of Prob. 2 (and hence to yield a method for solving Prob. 1):

$$\text{degree of } R \text{ is } d = (d_1, \ldots, d_n), \tag{3}$$

$$|R^{-1}(0)| = B = d_1 . d_2 \ldots d_n, \tag{4}$$

$$\begin{array}{l}\text{rank } \mathrm{D}R(x) = n \qquad \text{for all } x \in R^{-1}(0) \\ \text{(i.e. } 0 \in \boldsymbol{C}^n \text{ is a regular value of } R),\end{array} \tag{5}$$

$$R \text{ is } \boldsymbol{G}\text{-sign-symmetric} \tag{6}$$

where by $|A|$ we denote the number of elements of a set A).

The following example shows that for some choices of n, d, $\boldsymbol{G}$ there is no solution of Prob. 2.

Example 2. Let $n = 3$, all d_k be even ($k = 1, 2, 3$) and $\boldsymbol{G} = \{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{E}\}$, where

$$\mathbf{V}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{V}_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{V}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It can be easily checked for this example (by comparing the coefficients of the corresponding monoms in $\mathbf{V}R(x)$ and $R(\mathbf{V}x)$) that the kth component R_k of R can contain only monoms (i.e. they appear with nonzero coefficient $c \in \boldsymbol{C}$) of the following form

$$c . x_k^{2u} . x_i^{2v+1} . x_j^{2w+1},$$

where $u, v, w \in \{0, 1, 2, \ldots\}$ and $2u + 2v + 1 + 2w + 1 \le d_k$. Then, however, the system $R(x) = 0$ has always nonisolated solutions: the subspaces $\{x \in \boldsymbol{C}^3 | x_1 = x_2 = 0\}$, $\{x \in \boldsymbol{C}^3 | x_1 = x_3 = 0\}$, $\{x \in \boldsymbol{C}^3 | x_2 = x_3 = 0\}$ are contained in $R^{-1}(0)$.

Denote $N^- = \{k \in \{1, \ldots, n\} | \exists\, \mathbf{V} \in \boldsymbol{G}$ such that the kth diagonal entry of $\mathbf{V}$ is $-1\}$. The above example shows: if there is a $k \in N^-$ such that d_k is even, then there may be no solution of Prob. 2. In the opposite case it is a rather trivial task to prove (by verifying (3—6))

Theorem 1. (*Solution of Prob.* 2 — *the case of sign-symmetry.*) *Let the* $\boldsymbol{G}$*-sign-symmetric system* (1) *in Prob.* 1 *satisfy*: *for all* $k \in N^-$ *the degrees* d_k *are odd. Let* $a = (a_1, \ldots, a_n)^T \in \boldsymbol{C}^n$ *be any constant vector satisfying*: $a_k \neq 0$ (*for all* $k = 1, \ldots, n$) *and* $a_k \neq 1$ (*if* $d_k = 1$). *Then the polynomial map* $R\colon \boldsymbol{C}^n \to \boldsymbol{C}^n$, *where*

$$R_k(x) = x_k^{d_k} - a_k \,.\, x_k \qquad (k = 1, \ldots, n)$$

is a solution of Prob. 2.

It is clear that $y = (y_1 \ldots, y_n)^T \in R^{-1}(0)$ if and only if for each $k = 1, \ldots, n$ there holds: $y_k = 0$ or (if $d_k \geq 2$) $y_k = |a_k|^{1/(d_k - 1)} \,.\, \exp(i(\varphi_k + 2\pi j)/d_k - 1))$ for some $0 \leq j < d_k - 1$, where $a_k = |a_k| \,.\, \exp(i\varphi_k)$ and $|a_k|$ denotes the modulus of $a_k \in \boldsymbol{C}$. By [9] the above choice of R into (2) leads to a convergent homotopy method for Prob. 1.

This approach was used in [5] to solve the following Prob. 1: the number of equations is $n = 6$, the degrees are $d_k = 3$ ($k = 1, \ldots, 6$) and $\boldsymbol{G}$ is generated by the following three matrices

$$\mathsf{V}_1 = \begin{pmatrix} -1 & & & & & \\ & 1 & & & 0 & \\ & & -1 & & & \\ & & & -1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad \mathsf{V}_2 = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & 0 \\ & & 1 & & & \\ & & & -1 & & \\ & 0 & & & -1 & \\ & & & & & 1 \end{pmatrix},$$

$$\mathsf{V}_3 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & -1 & & \\ & 0 & & & 1 & \\ & & & & & -1 \end{pmatrix}.$$

All components of the map P in Prob. 1 are of the form

$$P_k(x) = x_k\left(1 - \frac{\lambda}{\lambda_k}\right) + \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq 6} c(k, i_1, i_2, i_3)\, x_{i_1} x_{i_2} x_{i_3}\,.$$

Here λ, λ_k, $c(k, i_1, i_2\, i_3)$ are fixed coefficients and $x_1, \ldots, x_6$ the unknowns. The $\boldsymbol{G}$-sign-symmetry of P is implied by the zero-nonzero structure of the coefficients $c(k, i_1, i_2, i_3)$. This system was obtained by substituting a truncated expansion of the solution (in terms of approprite eigenfunctions) into a von Kármán equation, which describes the buckling of a flat eastic plate that is square in shape, simply supported along its edges and subjected to a constant compressive thrust applied normal to two of its opposite edges (modulus of the compressive thrust is proportional to λ; λ_k ($k = 1, \ldots, 6$) are the six smallest characteristic values; x_k ($k = 1, \ldots, 6$) are the coefficients of the truncated expansion of the solution).

The computer program was based on the bounded homotopy numerical algorithm [1]. We have obtained approximations of all $B = 3^6 = 729$ isolated solutions (all proper). The whole solution set consists of 116 orbits of solutions (including the singleton-orbit of the trivial zero-solution). Numerically we had to follow only 115 homotopy paths, 11 of which led to orbits of real solutions (5 orbits per 4 solutions, 6 orbits per 2 solutions) and 104 to orbits of complex solutions (72 orbits per 8 solutions, 28 orbits per 4 solutions, 4 orbits per 2 solutions). Using the sign-symmetry of the original system we have decreased the amount of computations almost 8-times.

3. General symmetry

In this section we shall generalize the ideas introduced in the previous section. Concerning the symmetry we shall follow the approach from [7] (equivariancy from [7] corresponds to our concept of symmetry).

First let us introduce some notations. By $\boldsymbol{GL}(\boldsymbol{C}, n)$ we denote the group of all regulr square matrices with complex entries. For an arbitrary group $\boldsymbol{G}$ a homeomorphism $V: \boldsymbol{G} \to \boldsymbol{GL}(\boldsymbol{C}, n)$ will be referred to as a matrix representation of $\boldsymbol{G}$ (or simply a $\boldsymbol{C}^n$-representation of $\boldsymbol{G}$). Clearly $V(e) = \mathbf{E}$ (where $e \in \boldsymbol{G}$ is the unit-element of $\boldsymbol{G}$) and $V(g^{-1}) = (V(g))^{-1}$ (where g^{-1} is the inverse to $g \in \boldsymbol{G}$). We shall write $\mathbf{V}_g$ instead of $V(g)$.

Definition 2 (*cf.* [7, *p.* 94]). *Let $P: \boldsymbol{C}^n \to \boldsymbol{C}^n$ be a polynomial map of degree $d = (d_1, \ldots, d_n)$ and $\boldsymbol{G}$ a group. Let V, W be $\boldsymbol{C}^n$-representations of $\boldsymbol{G}$. We say that P is $(\boldsymbol{G}, V, W)$-symmetric if for all $g \in \boldsymbol{G}$ there holds*

$$\mathbf{W}_g P(x) = P(\mathbf{V}_g x) \qquad \textit{for all } x \in \boldsymbol{C}^n .$$

Examples.

3. Let $\boldsymbol{G} \subset \boldsymbol{S}$ (see Sec. 2). Then P is $\boldsymbol{G}$-sign-symmetric if and only if P is $(\boldsymbol{G}, V, W)$-symmetric, where for all $g \in \boldsymbol{G}$ there holds $\mathbf{V}_g = g$, $\mathbf{W}_g = g$.

4. An even polynomial map is $(\boldsymbol{G}, V, W)$-symmetric, where $\boldsymbol{G} = \{-\mathbf{E}, \mathbf{E}\}$, $\mathbf{V}_g = g$, $\mathbf{W}_g = \mathbf{E}$ (for all $g \in \boldsymbol{G}$). An even polynomial map is not $\boldsymbol{G}$-sign-symmetric (unless identically zero).

5. The symmetry of the polynomial system considered in [3] is defined by $\boldsymbol{G} = \{$set of all permutation matrices $n \times n\}$, the corresponding representations are $\mathbf{V}_g = g$, $\mathbf{W}_g = \mathbf{E}$ (for all $g \in \boldsymbol{G}$).

Analogously to Lemma 1 we have the trivial

Lemma 2. *Let* P: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ *be a* $(\boldsymbol{G}, V, W)$*-symmetric polynomial map. Then* $P^{-1}(0)$ *is a union of orbits* $\boldsymbol{G}_V[x] = \{\mathbf{V}_g x | g \in \boldsymbol{G}\}$ *or, equivalently,*

$$x \in P^{-1}(0) \Leftrightarrow \forall g \in \boldsymbol{G}; \qquad \mathbf{V}_g x \in P^{-1}(0).$$

As we are studying methods for solving Prob. 1 (i.e. only isolated solutions are sought), it is natural to assume the group $\boldsymbol{G}$ to be finite. Moreover, from now on we assume that the map P in Prob. 1 is $(\boldsymbol{G}, V, W)$-symmetric, where V, W are known $\boldsymbol{C}^n$-representations of $\boldsymbol{G}$.

The solution R of the corresponding Prob. 2 is characterized by (3—5) and

$$R \text{ is } (\boldsymbol{G}, V, W)\text{-symmetric}. \tag{7}$$

We know already from Sec. 2 that in general Prob. 2 need not have a solution. A natural question arises: How to find a solution of Prob. 2 if a solution exists? The answer is based on the following theorem, which is the main result of this paper.

Before stating the main result we briefly clarify the structure of the set of polynomial maps. We identify any particular polynomial map P: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ of degree $d = (d_1, \ldots, d_n)$ with its coefficient-vector $c = (c_1, \ldots, c_r)^T$ (we suppose a fixed correspondence between particular components c_i of c and coefficients of particular monoms in components of P). Thus the set $\mathscr{P}(n, d)$ of all polynomial maps $\boldsymbol{C}^n \to \boldsymbol{C}^n$ of degree at most d (i.e. the kth component is of degree at most d_k for each $k = 1, \ldots, n$) is isomorphic to $\boldsymbol{C}^r$.

Theorem 2. (*Main result.*) *Let there be a solution of Prob.* 2 (*i.e. there is a polynomial map R satisfying* (3, 4, 5, 7)). *Then there is an open, dense subsest* $\mathscr{P}^*$ *of* $\mathscr{P}(n, d)$ *such that for all* $R \in \mathscr{P}^*$ *the map*

$$R_{\boldsymbol{G}} = \frac{1}{|\boldsymbol{G}|} \sum_{g \in \boldsymbol{G}} \mathbf{W}_g^{-1} R \mathbf{V}_g \tag{8}$$

is a solution of prob. 2.

The relation (8) is a standard symmetrization as used in [7].

First we discuss some consequence of Thm. 2 and prove it in the end of this section.

Theorem 2 has a constructive character: If there is a solution of Prob. 2 (and we do not know any particular solution of it), then for almost every $R \in \mathscr{P}(n, d)$ we obtain by (8) a solution of Prob. 2. (Note that the right-hand site of (8) can be evaluated for any $x \in \boldsymbol{C}^n$ as $\boldsymbol{G}$ is finite and $\mathbf{V}_g$ and $\mathbf{W}_g$ are known.)

An immediate consequence of Theorem 2 is

Procedure (*for solving Prob.* 2):

Step 1. *Choose at random* $R \in \mathscr{P}(n, d)$.

Step 2. *Construct* $R_{\boldsymbol{G}}$ *according to* (8).

Step 3. *Solve Prob.* 1, *where* $P = R_G$, *using any numerical method* (*e.g.* [1], [9]).
Step 4. *a*) *If* (3, 4, 5) *are not satisfied for* R_G, *then with probability one there is no solution of Prob.* 2.
b) *If* (3, 4, 5) *are satisfied for* R_G, *then* R_G *can be used in* (2) *instead of* R *to define a* $(\boldsymbol{G}, V, W)$*-symmetric homotopy for solving Prob.* 1 (*starting points for homotopy paths are chosen from the solutions obtained in Step* 3).

Note that there is no need to check the assumption of Thm. 2 before starting our procedure. The termination in Step 4a expresses the fact that either Prob. 2 has no solution (and Procedure surely terminates at Step 4a) or Prob. 2 has a solution (and in this case Procedure terminates with probability zero at Step 4a).

Remarks.

1. Let us suppose that we have to solve several Problems 1 with the same n, d and the same symmetry (e.g. a parametrized sytem, where the symmetry is retained for any value of the parameter). In this case our procedure is very efficient, as we have to solve one Prob. 1 using a general homotopy (following all B paths) and subsequently several symmetric Problems 1 (following due to the symmetry each time less than B paths). Of course, in case we have to solve one particular symmetric Prob. 1 and we can not find an approprite symmetric map R into (2) without our procedure, it is more effective to apply a general homotopy method directly to the original Prob. 1.
2. The computational expense of Step 3 and of the subsequent applications of the symmetric homotopy to symmetric Probs. 1 consists not only in the large number of homotopy paths but also in the complicated computation of $R_G(x)$. (The random choice of R in Step 1 leads to a map with a large number of monoms.) Hence, although we have a generally applicable and generically succesful procedure for solving Prob. 2, it may be effective in case of a simple symmetry to try to construct a simple solution of Prob. 2 "by hand". Such simple solutions of Prob. 2 are given particularly in the case of sign-symmetry in Thm. 1.

Suppose we used the bounded homotopy numerical method [1] for solving Prob. 1. Then we solve instead of the system (1) in $\boldsymbol{C}^n$ a corresponding homogeneous system

$$\tilde{P}(x_0, x) = 0 \tag{1'}$$

in $\boldsymbol{C}^{n+1}$, where the homogenization $\tilde{P}$ of P is defined by

$$\tilde{P}_k(x_0, x_1, \ldots, x_n) = x_0^{d_k} . P_k\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \qquad (k = 1, \ldots, n).$$

The zero set $\tilde{P}^{-1}(0) \subset \boldsymbol{C}^{n+1}$ is homogeneous (i.e. $(x_0, x) \in \tilde{P}^{-1}(0) \Leftrightarrow \forall\, 0 \neq \lambda \in \boldsymbol{C}$;

$\lambda(x_0, x) \in \tilde{P}^{-1}(0))$, hence we can project it to the complex projective space $\boldsymbol{CP}^n$ by the natural projection

$$\varrho_{n+1}: \boldsymbol{C}^{n+1}\backslash\{0\} \to \boldsymbol{CP}^n,$$

where $\varrho_{n+1}(x_0, x)$ is the equivalence class of vectors differing from (x_0, x) only by a complex scalar multiple. A proper solution x of (1) corresponds to a solution $(1, x)$ of (1′). If $\tilde{P}(0, x) = 0$, then we call x an improper solution of (1) (corresponding to the solution $(0, x)$ of (1′)). We say that a (proper or improper) solution of (1) is isolated if it corresponds to a solution (x_0, x) of (1′) projected on an isolated point $\varrho_{n+1}(x_0, x)$ of $\varrho_{n+1}(\tilde{P}^{-1}(0))$ (cf. [1]).

In the bounded homotopy method we do not distinguish proper and improper solutions, so it is sufficient to require instead of (4) and (5)

$$|\varrho_{n+1}(\tilde{R}^{-1}(0))| = B \tag{4′}$$

and

$$\operatorname{rank} D_{x_0,x}\tilde{R}(x_0, x) = n, \tag{5′}$$

respectively.

Let us denote by Problem 1′the Prob. 1 with the above introduced concept of an isolated solution of (1) and by Problem 2′ the problem of finding a polynomial map R: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ satisfying (3,4′,5′,7). For these modified problems we have

Theorem 3. (*Main result for the bounded homotopy.*) *Let there be a polyomial map R satisfying* (3,4′,5′,7). *Then there is an open, dense subset* $\mathscr{P}^*$ *of* $\mathscr{P}(n, d)$ *such that for all* $R \in \mathscr{P}^*$ *the map* $R_{\boldsymbol{G}}$ *given by* (8) *satisfies* (3,4′,5′7).

Remark 3. Note that each solution of Prob. 2 is also a solution of Prob. 2′ (the converse is in general not true). However, already a solution of Prob. 2′ can be used to construct a bounded homotopy method (with H based on (2)) for solving Prob. 1′ and hence Prob. 1 too.

Proof of Thm. 3. Let us denote the set of all $(\boldsymbol{G}, V, W)$-symmetric polynomial maps from $\mathscr{P}(n, d)$ by $\mathscr{P}_{\boldsymbol{G}}(n, d)$. This set is clearly a linear subspace of $\mathscr{P}(n, d)$. The map

$$\sigma: \mathscr{P}(n, d) \to \mathscr{P}_{\boldsymbol{G}}(n, d)$$

defined by

$$\sigma(R) = R_{\boldsymbol{G}} \quad \text{(where } R_{\boldsymbol{G}} \text{ is given by (8))}$$

is a linear projection onto $\mathscr{P}_{\boldsymbol{G}}(n, d)$. So $\mathscr{P}(n, d)$ is the direct sum of $\mathscr{P}_{\boldsymbol{G}}(n, d)$ and $\operatorname{Ker}\sigma$ and there holds $\dim \mathscr{P}_{\boldsymbol{G}}(n, d) + \dim \operatorname{Ker}\sigma = \dim \mathscr{P}(n, d) = r$. Suppose now that we have an open, dense set $\mathscr{P}^*_{\boldsymbol{G}}$ in $\mathscr{P}_{\boldsymbol{G}}(n, d)$, then the set

$\mathscr{P}^* = \mathscr{P}_G^* + \operatorname{Ker}\sigma$ is also open and dense in $\mathscr{P}(n, d)$. Hence it is sufficient to prove

Lemma 3. *Let there be a map* $R^* \in \mathscr{P}_G(n, d)$ *such that* (3,4′,5′) *are satisfied. Then the set* $\mathscr{P}_G^*$ *of all maps from* $\mathscr{P}_G(n, d)$ *which satisfy* (3,4′, 5′) *is open and dense in* $\mathscr{P}_G(n, d)$.

Let us first introduce some useful notations and recall some basic facts from algebraic geometry. We denote $\dim \mathscr{P}_G(n, d) = s$, i.e. $\mathscr{P}_G(n, d)$ is isomorphic to $\boldsymbol{C}^s$. Like in the algebraic geometry we shall also use the Zariski topology in a complex projective space besides the classical topology. Open sets in the classical topology in $\boldsymbol{CP}^{m-1}$ are the sets $\varrho_m(N)$, where N is a homogeneous (i.e. $x \in N \Leftrightarrow \forall \lambda \in \boldsymbol{C}; \lambda x \in N$), open subset of $\boldsymbol{C}^m$. The Zariski topology is defined by specifying all closed sets (Zariski-closed); open sets in this topology (Zariski-open) are hence complements of Zariski-closed sets. Zariski-closed sets are the sets $\varrho_m(N)$, where $N = \{x \in \boldsymbol{C}^m | p_i(x) = 0 \; (i = 1, \ldots, m_N)\}$ for some homogeneous polynomials p_i in $x_1, \ldots, x_m$. A basic result in algebraic geometry (see, e.g., [6; pp. 21—24]) is that each Zariski-closed sets is a finite union of varieties $\mathscr{V}_i$ such that $\mathscr{V}_i \not\subset \mathscr{V}_j$ for all $i \neq j$. Moreover, varieties $\mathscr{V}_i$ in this union are determined uniquely up to the order of their appearence in the union (A variety is a Zariski-closed set which is not a union of two proper Zariski-closed subsets).

Proof of Lemma 3. Denote $M = \{(x_0, x, c) \in \boldsymbol{C}^{n+1} \times \boldsymbol{C}^s | \tilde{R}(x_0, x) = 0, R \in \mathscr{P}_G(n, d)\}$, where $c \in \boldsymbol{C}^s$ denotes the coordinate-vector of R as an element of the space $\mathscr{P}_G(n, d)$ (i.e. the coefficients of R are linear functions of $c \in \boldsymbol{C}^s$. We shall analyse the set M using arguments analogous to [9; Lemma 2]. Clearly $\tilde{R}(x_0, x)$ is homogeneous separately in $(x_0, x_1, \ldots, x_n)$ and in $(c_1, \ldots, c_s)$. So by [6; Def. 2.9] we can consider the set $\tilde{M} = \{(\varrho_{n+1}(x_0, x), \varrho_s(c)) | (x_0, x, c) \in M\}$ as a subset of $\boldsymbol{CP}^n \times \boldsymbol{CP}^{s-1}$. Let us denote $\varrho(x_0, x, c) = (\varrho_{n+1}(x_0, x) \; \varrho_s(c))$.

Assumption (3) is equivalent to: in each component R_k not all coefficients of the monoms of highest degree are zero; i.e. R does not satisfy (3) if it is from a Zariski-closed set and hence the set $\mathscr{P}_G^1$ of all $R \in \mathscr{P}_G(n, d)$ satisfying (3) is Zariski-open.

Let π_1 be the natural projection of $\tilde{M}$ to $\boldsymbol{CP}^n$ and π_2 be the natural projection of $\tilde{M}$ to $\boldsymbol{CP}^{s-1}$. Let us assume the following decomposition of $\tilde{M}$ to varieties

$$\tilde{M} = \left(\bigcup_{j=1}^{m_x} X_j\right) \cup \left(\bigcup_{i=1}^{m_y} Y_i\right), \tag{9}$$

where by Y_i $(i = 1, \ldots, m_y)$ we denote varieties for which $\pi_2(Y_i) = \boldsymbol{CP}^{s-1}$, by X_j $(j = 1, \ldots, m_x)$ varieties for which $\pi_2(X_j) \subsetneq \boldsymbol{CP}^{s-1}$.

For each $R \in \mathscr{P}(n, d)$ the set $\tilde{R}^{-1}(0)$ is an intersection of zero-sets of n polynomials $\tilde{R}_k(x_0, x)$ so by [2; Cor. IV.3.2] $\dim \varrho_{n+1}(\tilde{R}^{-1}(0)) \geq 0$ and hence $\tilde{R}^{-1}(0)$ is nonempty. This implies that $\pi_2(\tilde{M}) = \boldsymbol{CP}^{s-1}$ and $m_y \geq 1$.

By [6; Thm.2.23] the projection $\pi_2(X_j)$ is a Zariski-closed proper subset of

$\boldsymbol{CP}^{s-1}$ for each $j = 1, \ldots, m_x$, hence so is $X = \cup \pi_2(X_j)$, where the union goes through all j.

Let $\varrho(x_0^*, x^*, c^*)$ be a point where the coordinate-vector c^* corresponds to R^* and $(x_0^*, x^*) \in (\tilde{R}^*)^{-1}(0)$.

From $\pi_2(Y_i) = \boldsymbol{CP}^{s-1}$ it follows that each Y_i contains at least one point $\varrho(x_0^*, x^*, c^*)$.

For all varieties Y_i there obviously holds that $\dim Y_i \geq s - 1$. From the definition of the dimension of a projective variety [6; Def. 2.7] and from (5′) it follows

$$\dim Y_i = \min_{\varrho(x_0, x, c) \in Y_i} \dim T_{\varrho(x_0, x, c)} Y_i \leq \dim T_{\varrho(x_0^*, x^*, c^*)} Y_i = s - 1\,,$$

where $T_{\varrho(x_0, x, c)} Y_i$ is the Zariski tangent space to Y_i at $\varrho(x_0, x, c) \in Y_i$. Hence, we have $\dim Y_i = s - 1$ for all $i = 1, \ldots, m_y$ and, moreover, $\varrho(x_0^*, x^*, c^*) \in Y_i$ is a smooth point of Y_i (i.e. $\dim Y_i = \dim T_{\varrho(x_0^*, x^*, c^*)} Y_i$).

The set $\operatorname{NoReg} Y_i = \{\varrho(x_0, x, c) \in Y_i | \operatorname{rank} D_{x_0, x} \tilde{R}(x_0, x) < n$; i.e. the relation (5′) is not satisfied$\}$ is a Zariski-closed subset of Y_i and $\operatorname{NoReg} Y_i \subsetneqq Y_i$ (as by (5′) $\varrho(x_0^*, x^*, c^*) \notin \operatorname{NoReg} Y_i$). By [6; Prop. 1.14) $\dim \operatorname{NoReg} Y_i < s - 1$ and then also $\dim \pi_2(\operatorname{NoReg} Y_i) < s - 1$. Finally, the set

$$\mathscr{P}_{\boldsymbol{G}}^2 = \boldsymbol{CP}^{s-1} \Big\backslash \left[X \cup \left(\bigcup_{i=1}^{m_y} \pi_2(\operatorname{NoReg} Y_i) \right) \right]$$

is Zariski-open and nonempty (as it is a complement of a proper Zariski-closed set).

We show now that $\varrho(c^*) \in \mathscr{P}_{\boldsymbol{G}}^2$. To do so, it suffices to prove that none of the points $\varrho(x_0^*, x^*, c^*)$ lies in any X_j ($j = 1, \ldots, m_x$). Let us suppose the contrary, i.e. $\varrho(x_0^*, x^*, c^*) \in X_j$. Then (analogous as for Y_i) we have $\dim X_j = s - 1$. By the implicit function theorem it follows from (5′) that near $\varrho(x_0^*, x^*, c^*)$ the variety X_j is parametizable by $\varrho_s(c)$ from a neighbourhood of $\varrho_s(c^*)$ so there holds $\dim \pi_2(X_j) = s - 1$. The last is, however, a contradiction to $\pi_2(X_j) \subsetneqq \boldsymbol{CP}^{s-1}$.

The set $\mathscr{P}_{\boldsymbol{G}}^* = \mathscr{P}_{\boldsymbol{G}}^1 \cap \mathscr{P}_{\boldsymbol{G}}^2$ is Zariski-open in $\boldsymbol{CP}^{s-1}$ and by [6; Thm. 2.33] also open and dense in the classical topology in $\boldsymbol{CP}^{s-1}$. Moreover, the set $\mathscr{P}_{\boldsymbol{G}}^*$ is the set of all maps R (coordinate-vectors) with are of degree d, with all solutions of $\tilde{R}(x_0, x) = 0$ regular in the sense of (5′).

It is clear that from (5′) it follows that $\varrho_{n+1}(\tilde{R}^{-1}(0)$ is a set of isolated points in $\boldsymbol{CP}^n$ and hence it is finite (by compactness of $\boldsymbol{CP}^n$). From [4; Thm. 2.1] it follows that the multiplicity of each point in $\varrho_{n+1}(\tilde{R}^{-1}(0))$ is exactly one and the total number of them is $B = d_1 . d_2 \ldots d_n$.

We conclude this section with the

Proof of Theorem 2. This proof is fully analogous to the proof of Thm. 3.

An analogous auxiliary lemma to Lemma 3 is used (it differs only in the change of (4′,5′) to (4,5)), which can be proved almost step-by-step like Lemma 3. The only differences are the use of (4), (5) instead of (4′), (5′), respectively, and a modified definition of

$$\operatorname{NoReg} Y_i = \{\varrho(x_0, x, c) \in Y_i | \det \mathrm{D}_x \tilde{R}(x_0, x) = 0,\ x_0 = 0\}$$

(which is a Zariski-closed proper subset of Y_i too).

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СИММЕТРИЧЕСКИЕ ГОМОТОПИИ ДЛЯ РЕШЕНИЯ СИСТЕМ МНОГОЧЛЕННЫХ УРАВНЕНИЙ

Pavol Meravý

Резюме

В статье излагается один подход к конструкции гомотопических отображений, являющихся основой вычислительных методов для отыскания всех изолированных решений систем симметрических многочленных уравнений. Обсуждается возможность этой конструкции в общем случае. Для знако-симметрических систем описано одно простое подходящее гомотопическое отображение.