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SEMIGROUPS CONTAINING COVERED ONE-SIDED IDEALS

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In [3] a notion of a covered ideal was introduced. The aim of the present paper is to show some other properties and the mutual relation betwen covered ideals and bases of semigroups.

Definition 1. *A proper left ideal L of S is called a covered left ideal (briefly a CL-ideal) if $L \subset S(S-L)$.* Analogously a covered right ideal (CR-ideal) is defined. The case of two-sided ideals will be treated later.

Clearly if S contains a zero element 0 and card $|S| \geq 2$, then 0 is a CL-ideal. Note that by definition S itself is not a CL-ideal.

Lemma 1. *If S contains two different left ideals L_1 and L_2 such that $L_1 \cup L_2 = S$, then none of the ideals L_1, L_2 is a CL-ideal.*

Proof. If $L_1 \cup L_2 = S$, then $S - L_2 \subset L_1$, and $S - L_1 \subset L_2$. Now $L_1 \subset S(S-L_1)$ mplies $L_1 \subset SL_2 \subset L_2$ and $L_2 \subset S(S-L_2)$ implies $L_2 \subset SL_1 \subset L_1$, hence $L_1 = L_2$ a contradiction.

Corollary. *If S contains more than one maximal left ideal, then none of them is a CL-ideal of S.*

If L is a left ideal of S and $L \subsetneqq Sa$, then L is certainly a CL-ideal. (For, in this case we have $a \in S - L$.) In particular if $L = Sa \cap Sb$ is a proper subset of Sa or Sb then L is a CL-ideal of S.

A semigroup in which a is not contained in Sa (i.e. $a \in S - Sa$) contains CL-ideals, since for the left ideal $L = Sa$ we have $L = Sa \subset S(S-L)$.

In a semigroup which does not contain a CL-ideal, the ideal Sa cannot contain a proper left ideal of S, hence *Sa is a minimal left ideal for every $a \in S$.* In such a semigroup for any $a \neq b$ we have either $Sa = Sb$ or $Sa \cap Sb = \emptyset$. Moreover, $a \in Sa$ for every $a \in S$.

Lemma 2. *A semigroup S with* card $|S| > 1$ *contains no Cl-ideals iff S is a union of (disjoint) minimal left ideals.*

Proof. 1. It has been just remarked that such a semigroup is necessarily of the form: $S = \bigcup_{i \in I} Sa_i$, where each summand is a minimal left ideal.

2. Conversely, let be $S=\bigcup_{\iota\in I} L_\iota$, where each L_ι is a minimal left ideal of S. Any left ideal of S is a union of some minimal left ideals. Write $A=\bigcup_{\iota\in K} L_\iota$, $B=\bigcup_{\iota\in I-K} L_\iota$, then $S=A\cup B$. By Lemma 1 neither A nor B is a CL-ideal of S.

If S is a union of its minimal left ideals, it is known that S is simple.

In the following when speaking about CL-ideals we shall suppose that such ideals exist i.e. S is not a simple semigroup (without zero) containing a minimal left ideal.

Lemma 3. *If L_1 and L_2 are two CL-ideals of S, then $L_1\cup L_2$ is a CL-ideal of S.*

Proof. We have to show that $L_1\cup L_2\subset S[S-(L_1\cup L_2)]$. Note that by Lemma 1, $S-(L_1\cup L_2)\neq\emptyset$. Let x be any element from L_1. $L_1\subset S(S-L_1)$ implies that there is $a\in S-L_1$ such that $x\in Sa$.

1. If $a\in S-L_1-L_2$, then $x\in S(S-L_1-L_2)$
2. If $a\in(S-L_1)\cap L_2$, we have $a\in L_2\subset S(S-L_2)$. Hence there is $k_2\in S-L_2$ such that $a\in Sk_2$. The element k_2 cannot be contained in L_1 since otherwise we would have $a\in Sk_2\subset SL_1\subset L_1$, a contradiction with $a\in S-L_1$. Hence $k_2\in(S-L_1\cap(S-L_2)=S-(L_1\cup L_2)$. Therefore, $x\in Sa\in SSk_2\subset Sk_2\subset S[S-(L_1\cup L_2)]$.

We have proved $L_1\subset S[S-(L_1\cup L_2)]$ and by the same argument $L_2\subset S[S-(L_1\cup L_2)]$, so that

$$L_1\cup L_2\subset S[S-(L_1\cup L_2)].$$

Lemma 4. *If L_1, L_2 are two CL-ideals of S and $L_1\cap L_2\neq\emptyset$ then $L_1\cap L_2$ is a CL-ideal of S.*

Proof. $L_1\subset S(S-L_1)$ implies $L_1\cap L_2\subset S(S-L_1)\subset S[S-(L_1\cap L_2)]$.

If we consider the empty set $\emptyset$ as a CL-ideal, we may state:

Theorem 1. *The set of all CL-ideals of S (including $\emptyset$) is a sublattice of the lattice of all left ideals of S (including $\emptyset$).*

Example 1. Let $S=\{a, b, c, d\}$ with the multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	d
d	a	b	c	d

S has the lattice of all left ideals given in fig. 1, while fig. 2 gives the lattice of all CL-ideals.

Definition 2. *A left ideal L of S and $L \neq S$ is called the greatest left ideal of S if L contains any proper left ideal of S.*

Example 2. Let $S_0 = \langle 0, 1 \rangle$ with the usual multiplication of real numbers and $S_1 = \{a_1, 0\}$, $a_1^2 = a_1$ and 0 having the properties of a zero. Let S be the 0-direct union of S_0 and S_1. Then S contains a unique maximal ideal, namely S_0. But S_0 is not the greatest ideal of S, since S_0 does not contain the ideal $\{0, a_1\}$.

If S contains the greatest left ideal of S, this ideal will be denotet by L^*. Clearly if S contains L^*, then L^* is a maximal left ideal of S.

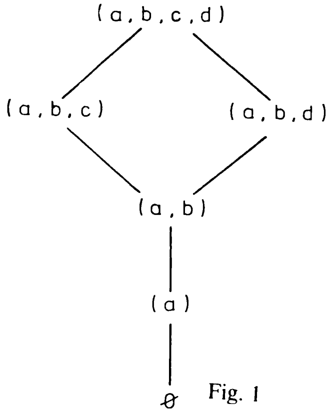

Fig. 1

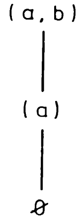

Fig. 2

Theorem 2. *A maximal left ideal L of S is a CL-ideal of S iff S contains L^* and in this case $L = L^*$.*

Proof. 1. By Lemma 1 a maximal left ideal L of S can be CL-ideal only if for any left ideal l of S we have $l \subset L$ (For otherwise $L \cup l$ would be equal to S). Since L is maximal, necessarily $L = L^*$.

2. Conversely, suppose that L^* exists. We prove that $L^* \subset S(S - L^*)$. Since $S(S - L^*)$ is a left ideal of S we have either $S(S - L^*) = S$, or $S(S - L^*) \subset L^*$. In the first case $L^* \subset S = S(S - L^*)$.

In the second case $S(S - L^*) \subset L^*$ and $L^* \subset SL^*$ imply $S^2 = S[(S - L^*) \cup L^*] \subset L^*$. If $S - S^2 = \{a, b, c, \ldots\}$, then any set $S - a$, $S - b$, ... is a left ideal of S. Hence, since L^* exists we have card $(S - S^2) = 1$. Denote $S - S^2 = \{a\}$. Then $L^* = S - a$ and $S = L^* \cup \{a\}$. Now $a \cup Sa$ is a left ideal of S and since it is not contained in L^* we have $a \cup Sa = S$. The equalities $a \cup L^* = a \cup Sa = S$ (since $a \bar{\in} L^*$ and $a \bar{\in} Sa$) imply $L^* = Sa$, so that $L^* \subset S(S - L^*)$. This proves our statement.

2.

We now treat the case that S contains more than one maximal left ideal.

Definition 3. *A CL-ideal L is called a greatest covered left ideal of S if L contains every covered left ideal of S.*

If S contains the greatest covered left ideal of S, this ideal will be denoted by L^g.

Suppose that S contains maximal left ideals and $\{L_\alpha / \alpha \in I\}$ is the totality of all such ideals. Denote $\hat{L} = \bigcap_{\alpha \in I} L_\alpha$ and suppose $\hat{L} \neq \emptyset$ (i.e. S is not a simple semigroup containing a minimal left ideal).

If L^g exists, we have necessarily $L^g \subset \hat{L}$. For if there is at least one L_α such that L^g is not contained in L_α, then $L_\alpha \cup L^g = S$ and by Lemma 1 L^g cannot be a CL-ideal.

Unfortunately $\hat{L}$ need not be a covered left ideal.

Example 3. Let S_0 be the multiplicative semigroup of real numbers from the half-open interval $\langle 0, 1)$ and $S_1 = \{0, a_1\}$ $S_2 = \{0, a_2\}$, $a_1^2 = a_1$, $a_2^2 = a_2$, the element 0 having the usual properties of multiplicative zero. The 0-direct union $S = S_0 \cup S_1 \cup S_2$ contains two maximal ideals, namely $L_1 = S - \{a_1\}$, $L_2 = S - \{a_2\}$. The ideal $S_1 \cup S_2$ is not contained in a maximal ideal of S. $\hat{L} = S_0$, $S(S - \hat{L}) = \{0, a_1, a_2\}$ so $\hat{L} \not\subset S(S - \hat{L})$.

Example 4. Modify the foregoing example by taking for S_0 the closed interval $\langle 0, 1 \rangle$. Then S contains a further maximal ideal, namely $L_3 = S - \{1\}$, and $\hat{L} = \langle 0, 1)$. In this case $S - \hat{L} = \{a_1, a_2, 1\}$ and $S\{a_1, a_2, 1\} = S$, so that $\hat{L} \subset S(S - \hat{L})$. Hence $\hat{L}$ is a covered left ideal.

An $\mathscr{L}$-class (the set of all elements of S generating the same principal left ideal) containing a given element a will be denoted by L^a.

An $\mathscr{L}$-class L^a is a maximal one, if $(a)_L$ is not a proper subset of any principal left ideal of S.

In [1] it is proved that a complement of a maximal left ideal is a maximal $\mathscr{L}$-class.

We shall denote maximal left ideals by L_α and corresponding maximal $\mathscr{L}$-classes by L^α.

Now we introduce a partial ordering $<$ between *$\mathscr{L}$-classes namely*: $L^a < L^b$ if $(a)_L \subset (b)_L$.

A non-empty subset A of S is a right base of S if

(1) $A \cup SA = S$

(2) there is no proper subset $B \subsetneqq A$ such that $B \cup Sb = S$

Consider a quasi-ordering in S, namely: $a \leqslant b$ means $(a)_L \subset (b)_L$.

Lemma 4 [6]. *A non-empty subset A of S is a right base of S iff*

(1) *for any $x \in S$ there is $a \in A$ such that $x \leqslant a$,*

(2) *for any two distinct elements $a_1, a_2 \in A$ neither $a_1 \leqslant a_2$, nor $a_2 \leqslant a_1$.*

Remark. Lemma 4 implies that a right base A consists of elements from all maximal $\mathscr{L}$-classes.

Lemma 5 [5]. *Let S contain maximal left ideals. Then the intersection of all maximal left ideals $\bigcap_{\alpha \in \lambda} L_\alpha = \emptyset$ iff S is a simple semigroup (without zero) containing a minimal left ideal.*

Theorem 3. *A semigroup S contains L^g iff*
(1) *S is not a simple semigroup, containing a minimal left ideal,*
(2) *S contains a right base A.*

Prof. (a) Suppose that S satisfies (1), (2). Then (see [3], Theorem 1) S contains maximal left ideals. Denote by $\hat{L}=\bigcap_{\alpha\in\lambda} L_\alpha$ the intersection of all maximal left ideals. $\hat{L}\neq\emptyset$ by (1). As we know from [4] $L_\alpha = S-L^\alpha(\alpha\in\lambda)$ and L^α is a maximal $\mathcal{L}$-class of S. Then $\hat{L} = \bigcap_{\alpha\in\lambda} L_\alpha = \bigcap_{\alpha\in\lambda}(S-L^\alpha) = S-\bigcup_{\alpha\in\lambda} L^\alpha$. So, $S-\bigcup_{\alpha\in\lambda} L^\alpha=\hat{L}$. This implies that no element from $L^\alpha(\alpha\in\lambda)$ and therefore from the right base A belongs to $\hat{L}$.

Let $x\in\hat{L}$ by any element. By (1) of Lemma 4 there is $a\in A$ such that $x\leqslant a$, i.e $(x)_L\subset(a)_L$, or in another form:

$$\bigcup_{x\in\hat{L}}[x\cup Sx]\subset\bigcup_{a\in A}[a\cup Sa]=S.$$

Hence, we have $\hat{L}\subset SA\subset S(S-\hat{L})$, so $\hat{L}$ is a CL-ideal of S. It remains to show that any CL-ideal is contained in $\hat{L}$. Let L be any left ideal of S, which is not contained in $\hat{L}$, so $L\cap(\bigcup_{\alpha\in\lambda} L^\alpha)\neq\emptyset$, i. e. $L^\alpha\subset L$ at least for one $\alpha\in\lambda$. Let $L^\beta\subset L$ (L^β is a maximal $\mathcal{L}$class of S). We shall show that L is not a CL-ideal of S. Let $b\in L^\beta\subset L$, so $(b)_L\subset L$. In $S-L$ are $\mathcal{L}$-classes either from $\hat{L}$, or from $S-\hat{L}$, except L^β. Therefore, there is no $\mathcal{L}$-class L^u in $S-L$ such that $L^\beta<L^u$. So we have proved that any left ideal which is not contained in $\hat{L}$ cannot be a CL-ideal of S. Since $\hat{L}$ is a CL-ideal, we conclude that L^g exists and $\hat{L}=L^g$.

(b) Now suppose that S contains L^g. We show that (1) and (2) are satisfied. It is known that any left ideal of S is a union of certain $\mathcal{L}$-classes of S, so its complement must be a union of the remaining $\mathcal{L}$-classes. Let us construct a subset A in the following way: exactly one element is chosen into A from each $\mathcal{L}$-class in $S-L^g$. We show that A satisfies (1) and (2) of Lemma 4.

Let $x\in S$ be any element. Then either $x\in L^g$, or $x\in S-L^g$. If $x\in L^g$, then $L^g\subset S(S-L^g)$ implies that there is $a\in S-L^g$ such that $x\in Sb$ and $b\in L^a$. From $x\in Sb$ we have $(x)_L\subset(b)_L=(a)_L$, so $x\leqslant a$. If $x\in S-L^g$, then $x\in L^b$ and $x\leqslant b$. Therefore, (1) is satisfied in both cases.

Let a, $b\in A$, $a\neq b$. We shall show that neither $a\leqslant b$ nor $b\leqslant a$ holds. If $a\leqslant b$, then $a\cup Sa\subset b\cup Sb$. Since $a\neq b$, we have $a\in Sb$. This implies $(a)_L\subset Sb$ ($b\,\bar{\in}\,(a)_L$), therefore $(a)_L$ is a CL-ideal of S. Then $L^g\cup(a)_L$ is a CL-ideal of S, properly containing L^g, which is a contradiction. Similarly we can prove that $b\leqslant a$ does not hold. Hence A satisfies the condition (2) of Lemma 4. We have proved that S contains a right base.

It remains to show that S is not simple, containing a minimal left ideal. According to Lemma 5 it suffices to show that the intersection of all maximal left ideals is non-empty. This follows from our assumption that S contains L^g and from the fact that we always have $L^g \subset \hat{L}$.

Corollary. *If S contains L^g, then L^g is of the form: $L^g = \bigcap_{\alpha \in \lambda} L_\alpha$, i.e. L^g is the intersection of all maximal left ideals of S.*

Theorem 4. *Every left ideal of a semigroup S is covered iff either there is a chain of principal left ideals such that the union of its elements is S,, or S contains L^*.*

Proof. (a) Let every left ideal of S be covered. Let L be any left ideal of S, and $a \in L$. Since every left ideal is covered, we have $(a)_L \subset S[S-(a)_L]$. It implies $a \in Sb$, for $b \in S-(a)_L$, hence $(a)_L \subset (b)_L$. So, we can construct a chain of principal left ideals. By Hausdorff Theorem any chain is contained in a maximal one. Denote by $U\{(a_i)_L\}$ $(i \in I)$ a maximal chain of proper principal left ideals of S and $\bigcup_{i \in I}(a_i)_L = L_1$. If $L_1 = S$ there is nothing to prove more. $L_1 \subsetneqq S$ we shall show that S contains L^*. If $L_1 \subsetneqq S$ holds, then $S - L_1 \neq \emptyset$. L_1 is a left ideal of S and therefore (by supposition) a covered one, so $L_1 \subset S(S-L_1)$. For every $i \in I$ $(a_i)_L \subset S(S-L_1)$, there is an element $c \in S - L_1$ such that $a_i \in Sc$, therefore $(a_i)_L \subset (c)_L$. We shall show that $(c)_L = S$. If this were not true, then $(c)_L \subsetneqq S$ and since $(a_i)_L \subset (c)_L$, then $(c)_L$ would belong to the chain U. But it is a contradiction with our assumption that U is a maximal chain. Hence $c \cup Sc = S$. The $\mathscr{L}$-class containing c is a maximal one. Denote it by L^β. Then $S - L^\beta = L_\beta$ is a maximal left ideal. Every left ideal T which is not contained in L_β meets L^β, hence $T \cap L^\beta \neq \emptyset$, so that $T = S$. It means that L_β is such a maximal left ideal that every proper left ideal of S is contained in L , hence $L_\beta = L^*$.

(b) If S contains L^*, then L^* is a CL-ideal and for any proper left ideal L we have:

$$L \subset L^* \subset S(S-L^*) \subset S(S-L).$$

Hence L is a CL-ideal.

Let S contain a chain U of principal left ideals $(a_j)_L$ $j \in I$, and $\bigcup_{j \in I}(a_j)_L = S$. Let L be any left ideal of S. Recall that every left ideal is a union of principal left ideals generated by its elements. Let $b \in S - L$. Since $\bigcup_{j \in I}(a_j)_L = S$, then there exists an index $i \in I$ such that $b \in (a_i)_L$ and $(b)_L \subset (a_i)_L$. The element $a_i \bar{\in} L$, since $a \in L$ would imply $(a_i)_L \subset L$ and $(b)_L \subset (a_i)_L \subset L$ implies $b \in L$, what is a contradiction with a choice of b. Denote by K the set of indices of all elements of U that are contained in $(a_i)_L$. Clearly $\bigcup_{j\in I-K}(a_j)_L = S$. All elements a_j, $j\in I-K$, belong to $S-L$ and $\bigcup_{j\in I-K}(a_j\cup Sa_j) = S$.

Now $L\subset \bigcup_{j\in I-K}(a_j\cup Sa_j)$. But $a_j\in S-L$ for $j\in I-K$, hence $L\subset \bigcup_{j\in I-K} Sa_j\subset S(S-L)$, so that L is a CL-ideal of S. This proves Theorem 4.

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ПОЛУГРУППЫ СОДЕРЖАНИЕ ЗАКРЫТЫЕ ОДНОСТОРОННИЕ ИДЕАЛЫ

Имрих Фабрици

Резюме

Левый (правый) идеал L (R) называется закрытым, если

$$L\subset S(S-L),\quad (R\subset (S-R)R).$$

В настоящей работе доказаны утверждения, касающиеся строения полугрупп, имеющих односторонные закрытые идеалы. Следующие утверждения являются главными:

1. Множество всех закрытых левых (правых) идеалов (включая $\emptyset$) является подструктурой структуры всех левых (правых) идеалов (включая $\emptyset$).

2. Приведено необходимое и достаточное условие для того, что бы:

а) полугруппа содержала самый большой закрытый левый (правый) идеал

б) всякий левый (правый) идеал полугруппы был закрытым.