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SUFFICIENT FAMILIES AND ENTROPY OF INVERSE LIMIT

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ABSTRACT. The purpose of the present paper is to study the metric entropy $h(\phi,\mathcal{N})$ of an F-measure preserving transformation ϕ relative to a σ-algebra $\mathcal{N}$ of an F-dynamical system. Concepts of sufficient families and generators are introduced and a few results are proved. Finally, the entropy of the inverse limit of an inverse spectrum of F-dynamical systems is obtained.

1. Introduction

Subsequent to the introduction of a new mathematical model of the statistical quantum theory, called F-quantum space, by Riečan and Dvurečenskij [14], [15], several studies in this direction have been made by different workers [2], [4], [8] – [13]. An F-quantum space is a couple $(X,\mathcal{M})$ where $\mathcal{M}$ ($\subseteq [0,1]^X$) is a σ-algebra of fuzzy events on a nonempty set X. A probability (normalized) measure, called an F-state $m\colon \mathcal{M} \to [0,1]$ is then defined on $\mathcal{M}$. Different sets of axioms for these basic notions of fuzzy σ-algebra and of F-state were proposed in [2], [4], [8] – [13].

Following Piasecki [13], Markechová studied entropy of complete partitions, and entropy of an F-dynamical system in [10], [11]. At the same time, Dumitrescu [4] developed fuzzy partition theory in a different way using triangular norms (see also Butnariu [2]). The major difficulty in defining fuzzy partition relates to "the disjointness of fuzzy sets"; to overcome this difficulty different methods have been chosen. In what follows some crucial tools and results are missing in each of the respective theories. We ([7], [17] – [21]) have adopted the basic definition of fuzzy σ-algebra and F-measure due to Klement [8], and have developed a theory of F-dynamical systems and entropy using the

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concept of an atom in a fuzzy σ-algebra which we introduce to circumvent lacunae in other approaches to such a theory. Appropriate generalizations of the definitions lead to a very satisfactory theory which contains the corresponding classical theory as a particular case.

The present paper is devoted to the study of metric entropy and the entropy of the inverse spectrum. In Section 2, prerequisites are collected. In Section 3, the concepts of the conditional entropy and metric entropy are studied and some new results are obtained which include the corresponding classical results as a particular case (cf. [1], [5], [6]). In Section 5, the entropy of the inverse limit of an inverse spectrum $\{\Phi_\alpha : \alpha \in J\}$ of F-dynamical systems is investigated using the concepts of sufficient families and generators in an F-dynamical system, introduced in Section 4. It is proved that if $\mathcal{L} \in \mathcal{F}(\mathcal{M}_\alpha)$ has a generator then the entropy $h(\hat{\Phi}, [\mathcal{L}])$ of the inverse limit $\hat{\Phi}$ on $[\mathcal{L}]$ is the limit of the entropies of Φ_α on $[\mathcal{L}]$.

2. Prerequisites

2.1. A *fuzzy set* in a nonempty set X is an element of I^X where $I = [0,1]$ is the closed unit interval of the real line $\mathbb{R}$. A fuzzy set which assigns to each x in X the constant value t, $t \in I$, is denoted by $\boldsymbol{t}$. If a sequence $\{\lambda_i(x)\}_{i=1}^{\infty}$ is monotonic increasing and converges to $\lambda(x)$ for each $x \in X$, then we say that $\{\lambda_i\}_{i=1}^{\infty}$ *increases to* λ, $\lambda \in I^X$; we then write $\lambda_i \uparrow \lambda$.

The set of all positive integers is denoted by $\mathbb{N}$ and $\mathbb{Z}$ denotes the set of integers.

2.2. ([8]) A *fuzzy σ-algebra* on a set X is a subfamily of I^X which satisfies the following conditions:

(i) $\mathbf{1} \in \mathcal{M}$,
(ii) $\lambda \in \mathcal{M} \implies \mathbf{1} - \lambda \in \mathcal{M}$,
(iii) if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{M}$, then $\bigvee_{i=1}^{\infty} \lambda_i \equiv \sup_i \lambda_i \in \mathcal{M}$.

If $\mathcal{N}_i$, $i = 1, 2$, are fuzzy sub-σ-algebras on X, then $\mathcal{N}_1 \vee \mathcal{N}_2$ is the smallest fuzzy σ-algebra on X containing $\mathcal{N}_1 \cup \mathcal{N}_2$.

2.3. ([8]) An *F-probability measure* m on $\mathcal{M}$ is a function $m\colon \mathcal{M} \to I$ which satisfies:

(i) $m(\mathbf{1}) = 1$,
(ii) $m(\mathbf{1} - \lambda) = 1 - m(\lambda)$, $\lambda \in \mathcal{M}$,
(iii) for $\lambda, \mu \in \mathcal{M}$, $m(\lambda \vee \mu) + m(\lambda \wedge \mu) = m(\lambda) + m(\mu)$,

(iv) if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{M}$ such that $\lambda_i \uparrow \lambda$, then $m(\lambda) = \sup_i m(\lambda_i)$.

The triple $(X, \mathcal{M}, m)$ is called an *F-probability measure space.*

2.4. *If m and n are F-probability measures on a fuzzy σ-algebra $\mathcal{M}$, then, for $p \in [0,1]$, $pm + (1-p)n$ is also an F-probability measure on $\mathcal{M}$.*

2.5. ([19]) For $\lambda, \mu \in \mathcal{M}$, define an equivalence relation on $\mathcal{M}$ as follows:

$$\lambda = \mu \pmod{m} \iff m(\lambda) = m(\mu) = m(\lambda \vee \mu).$$

Denote by $\tilde{\mathcal{M}}$ the set of all equivalence classes induced by this relation; $\tilde{\mu}$ stands for the equivalence class containing μ. Elements λ, μ of $\mathcal{M}$ are called *m-disjoint* if $m(\lambda \wedge \mu) = 0$, i.e. $\lambda \wedge \mu = \mathbf{0} \pmod{m}$.

2.6. ([19]) Let $(X, \mathcal{M}, m)$ be an F-probability measure space, and let $\mathcal{N}$ be a fuzzy sub-σ-algebra of $\mathcal{M}$. An element $\tilde{\mu} \in \tilde{\mathcal{N}}$ is called an *atom* of $\mathcal{N}$ if $m(\mu) > 0$, and for $\tilde{\lambda} \in \tilde{\mathcal{N}}$,

$$m(\lambda \wedge \mu) = m(\lambda) \neq m(\mu) \implies m(\lambda) = 0.$$

The set of all atoms of $\mathcal{N}$ is denoted by $\overline{\mathcal{N}}$ and $\mathcal{F}(\mathcal{M})$ stands for the family of all fuzzy sub-σ-algebras of $\mathcal{M}$ having finitely many atoms.

2.7. For an F-measure preserving transformation $\phi\colon (X, \mathcal{M}, m) \to (X, \mathcal{M}, m)$ (i.e. $\phi^{-1}(\mathcal{M}) \subset \mathcal{M}$ and $m(\phi^{-1}(\mu)) = m(\mu)$ for all $\mu \in \mathcal{M}$), the quadruple $\Phi = (X, \mathcal{M}, m, \phi)$ is known as an *F-dynamical system.* The transformation ϕ is called an *invertible measure preserving transformation* if it is bijective and ϕ^{-1} is also measure preserving. If ϕ is invertible then Φ is called an *invertible F-dynamical system.*

2.8. ([19]) *If ϕ is F-measure preserving and $\mathcal{N}$ is a fuzzy sub-σ-algebra of $\mathcal{M}$, then $\overline{\phi^{-1}(\mathcal{N})} = \phi^{-1}(\overline{\mathcal{N}})$.*

2.9. ([19]) Let $(X, \mathcal{M}, m)$ be an F-probability measure space and $\mathcal{N}_1$, $\mathcal{N}_2$ be fuzzy sub-σ-algebras of $\mathcal{M}$. Then $\mathcal{N}_2$ is called an *m-refinement* of $\mathcal{N}_1$ (or $\mathcal{N}_1$ is *subordinate* to $\mathcal{N}_2$), written as $\mathcal{N}_1 \leq_m \mathcal{N}_2$, if for $\mu \in \overline{\mathcal{N}}_2$, there exists $\lambda \in \overline{\mathcal{N}}_1$ such that $m(\lambda \wedge \mu) = m(\mu)$.

The fuzzy sub-σ-algebras $\mathcal{N}_1$ and $\mathcal{N}_2$ are called *m-equivalent,* written as $\mathcal{N}_1 \approx_m \mathcal{N}_2$, if

$$m\Big(\lambda \wedge \Big(\bigvee\{\mu :\ \mu \in \overline{\mathcal{N}}_2\}\Big)\Big) = m(\lambda) \qquad \text{for each} \quad \lambda \in \overline{\mathcal{N}}_1$$

and

$$m\Big(\mu \wedge \Big(\bigvee\{\lambda :\ \lambda \in \overline{\mathcal{N}}_1\}\Big)\Big) = m(\mu) \qquad \text{for each} \quad \mu \in \overline{\mathcal{N}}_2.$$

The relation of "m-equivalence" is an equivalence relation on $\mathcal{F}(\mathcal{M})$. We denote by $[\mathcal{N}]$ the set of all m-equivalent fuzzy sub-σ-algebra in $\mathcal{F}(\mathcal{M})$.

3. The metric entropy $h_m(\phi, \mathcal{N})$

3.1. Let $(X, \mathcal{M}, m)$ be an F-probability measure space. For $\mathcal{N} \in \mathcal{F}(\mathcal{M})$, the *entropy* $H(\mathcal{N})$ of $\mathcal{N}$ is given by [19]

$$H_m(\mathcal{N}) \equiv H(\mathcal{N}) = -\sum_{\mu \in \overline{\mathcal{N}}} g(m(\mu)),$$

where the convex function $g\colon [0, \infty) \to R$ is Shannon's function, defined by

$$g(x) = \begin{cases} x \log x, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

For $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\mathcal{M})$ with $\overline{\mathcal{N}}_1 = \{\lambda_i : 1 \le i \le r\}$ and $\overline{\mathcal{N}}_2 = \{\mu_j : 1 \le j \le s\}$, we define the *conditional entropy* $H(\mathcal{N}_1 \mid \mathcal{N}_2)$ by

$$H_m(\mathcal{N}_1 \mid \mathcal{N}_2) \equiv H(\mathcal{N}_1 \mid \mathcal{N}_2) = -\sum_j \sum_i m(\mu_j) g(m(\lambda_i \mid \mu_j)),$$

where $m(\lambda_i \mid \mu_j) = \frac{m(\lambda_i \wedge \mu_j)}{m(\mu_j)}$ ([7]).

3.2. *Let $(X, \mathcal{M}, m)$ be an F-probability measure space. Then, for $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \in [\mathcal{N}]$, the following hold:*

(i) $\mathcal{N}_1 \le_m \mathcal{N}_2 \implies H(\mathcal{N}_1) \le H(\mathcal{N}_2)$ ([19]).
(ii) $H(\mathcal{N}_1 \vee \mathcal{N}_2) = H(\mathcal{N}_1) + H(\mathcal{N}_2 \mid \mathcal{N}_1)$ ([17]).
(iii) $H(\mathcal{N}_1 \vee \mathcal{N}_2 \mid \mathcal{N}_3) = H(\mathcal{N}_1 \mid \mathcal{N}_3) + H(\mathcal{N}_2 \mid \mathcal{N}_1 \vee \mathcal{N}_3)$ ([7]).
(iv) *If ϕ is an F-measure preserving transformation on X, then*

$$H(\phi^{-k+1}(\mathcal{N})) = H(\mathcal{N}), \qquad k \in \mathbb{N}.$$

(v) $d_R(\mathcal{N}_1 \mid \mathcal{N}_2) = H(\mathcal{N}_1 \mid \mathcal{N}_2) + H(\mathcal{N}_2 \mid \mathcal{N}_1)$ *defines a pseudo metric on* $[\mathcal{N}]$.

For the relation $\sim$ of equivalence modulo 0 ($\mathcal{N}_1 \sim \mathcal{N}_2 \iff \mathcal{N}_1 \le_m \mathcal{N}_2$ *and* $\mathcal{N}_2 \le_m \mathcal{N}_1$), d_R *defines a metric on* $[\mathcal{N}]/\sim$ ([17]).

3.3. ([19]) Let $\Phi = (X, \mathcal{M}, m, \phi)$ be an F-dynamical system. For $\mathcal{N} \in \mathcal{F}(\mathcal{M})$ define the metric entropy $h(\phi, \mathcal{N})$ of ϕ relative to $\mathcal{N}$ by

$$h_m(\phi, \mathcal{N}) \equiv h(\phi, \mathcal{N}) = \lim_{k \to \infty} \frac{1}{k} H(\mathcal{N}^{k,\phi}),$$

where $\mathcal{N}^{k,\phi} = \mathcal{N} \vee \phi^{-1}(\mathcal{N}) \vee \cdots \vee \phi^{-k+1}(\mathcal{N})$, $k \in \mathbb{N}$.

For $\mathcal{L} \in \mathcal{F}(\mathcal{M})$, the entropy $h(\Phi, [\mathcal{L}])$ of Φ on $\mathcal{L}$ is given by

$$h(\phi, [\mathcal{L}]) \equiv h(\Phi, [\mathcal{L}]) = \sup\{h(\phi, \mathcal{N}) : \mathcal{N} \in [\mathcal{L}]\}.$$

3.4. ([19]) *For* $\mathcal{N}_1, \mathcal{N}_2 \in [\mathcal{N}]$,

$$\mathcal{N}_1 \leq_m \mathcal{N}_2 \implies h(\phi, \mathcal{N}_1) \leq h(\phi, \mathcal{N}_2).$$

3.5 PROPOSITION. *Let* $(X, \mathcal{M}, m, \phi)$ *and* $(X, \mathcal{M}, n, \phi)$ *be* F*-dynamical systems. Then, for every fuzzy sub-*σ*-algebra* $\mathcal{N} \in \mathcal{F}(\mathcal{M})$ *and for* $p \in I$, *we have*

(i) $pH_m(\mathcal{N}) + (1-p)H_n(\mathcal{N}) \leq H_{pm+(1-p)n}(\mathcal{N})$;
(ii) $ph_m(\phi, \mathcal{N}) + (1-p)h_n(\phi, \mathcal{N}) \leq h_{pm+(1-p)n}(\phi, \mathcal{N})$.

Proof.
(i) By the convexity of g, we have

$$\begin{aligned} pH_m(\mathcal{N}) + (1-p)H_n(\mathcal{N}) &= -p \sum_{\mu \in \overline{\mathcal{N}}} g(m(\mu)) - (1-p) \sum_{\mu \in \overline{\mathcal{N}}} g(n(\mu)) \\ &= -\sum_{\mu \in \overline{\mathcal{N}}} \big[pg(m(\mu)) + (1-p)g(n(\mu))\big] \\ &\leq -\sum_{\mu \in \overline{\mathcal{N}}} g\big(pm(\mu) + (1-p)n(\mu)\big) \\ &= -\sum_{\mu \in \overline{\mathcal{N}}} g\big((pm + (1-p)n)(\mu)\big) \\ &= H_{pm+(1-p)n}(\mathcal{N}). \end{aligned}$$

(ii) Follows from (i). □

3.6. PROPOSITION. *Let* $\mathcal{N}$ *be a fuzzy sub-*σ*-algebra of an* F*-dynamical system* $\Phi = (X, \mathcal{M}, m, \phi)$ *having finitely many atoms. Then*

$$h(\phi, \mathcal{N}) = h\Big(\phi, \bigvee_{i=0}^{p} \phi^{-i}(\mathcal{N})\Big), \qquad p \in \mathbb{N}.$$

Furthermore, if Φ *is invertible, then*

$$h(\phi, \mathcal{N}) = h\Big(\phi, \bigvee_{i=-p}^{p} \phi^{i}(\mathcal{N})\Big), \qquad p \in \mathbb{N}.$$

Proof. Since, for $p, k \in \mathbb{N}$,

$$\Big(\bigvee_{i=0}^{p} \phi^{-i}(\mathcal{N})\Big)^{k,\phi} = \mathcal{N}^{p+k,\phi},$$

we get

$$h\Big(\phi, \bigvee_{i=0}^{p} \phi^{-i}(\mathcal{N})\Big) = \lim_{k\to\infty} \frac{1}{k} H\big(\mathcal{N}^{p+k,\phi}\big) = \lim_{k\to\infty} \frac{1}{p+k} H\big(\mathcal{N}^{p+k,\phi}\big) = h(\phi, \mathcal{N}).$$

If Φ is invertible, then since

$$\Big(\bigvee_{i=-p}^{p}\phi^i(\mathcal{N})\Big)^{k,\phi}=\Big(\bigvee_{i=-p}^{p}\phi^i\Big)(\mathcal{N}^{k,\phi}),$$

we have

$$H\bigg(\Big(\bigvee_{i=-p}^{p}\phi^i(\mathcal{N})\Big)^{k,\phi}\bigg)=H\bigg(\Big(\bigvee_{i=-p}^{p}\phi^i\Big)(\mathcal{N}^{k,\phi})\bigg)=H(\mathcal{N}^{k,\phi}).$$

Therefore

$$h\Big(\phi,\bigvee_{i=-p}^{p}\phi^i(\mathcal{N})\Big)=h(\phi,\mathcal{N}).$$

□

3.7. Proposition. *Let $(X,\mathcal{M},m,\phi)$ be an F-dynamical system. Then, for $\mathcal{N}\in\mathcal{F}(\mathcal{M})$ and $p\in\mathbb{N}$,*

$$h(\phi^p,\mathcal{N})=ph(\phi,\mathcal{N}),$$

and consequently $h(\phi^p,[\mathcal{N}])=ph(\phi,[\mathcal{N}])$.

Proof. By Proposition 3.6, we have

$$\begin{aligned}h(\phi^p,\mathcal{N})&=h\Big(\phi^p,\bigvee_{i=0}^{p}\phi^{-i}(\mathcal{N})\Big)\\&=\lim_{k\to\infty}\frac{1}{k}H\bigg(\Big(\bigvee_{i=0}^{p}\phi^{-i}(\mathcal{N})\Big)^{k,\phi^p}\bigg)=\lim_{k\to\infty}\frac{1}{k}H\bigg(\bigvee_{j=0}^{k-1}\phi^{-jp}\Big(\bigvee_{i=0}^{p}\phi^{-i}(\mathcal{N})\Big)\bigg)\\&=\lim_{k\to\infty}\frac{1}{k}H\Big(\bigvee_{i=0}^{kp-1}\phi^{-i}(\mathcal{N})\Big)=p\lim_{k\to\infty}\frac{1}{kp}H(\mathcal{N}^{kp,\phi})=ph(\phi,\mathcal{N}).\end{aligned}$$

□

3.8. Proposition. *Let $(X,\mathcal{M},m,\phi)$ be an invertible F-dynamical system. For $\mathcal{N}\in\mathcal{F}(\mathcal{M})$, we have*

$$h(\phi,\mathcal{N})=h(\phi^{-1},\mathcal{N}).$$

Consequently

$$h(\phi,[\mathcal{N}])=h(\phi^{-1},[\mathcal{N}]).$$

Proof. Since $\mathcal{N}^{k,\phi}=\phi^{-k+1}(\mathcal{N}^{k,\phi^{-1}})$, we get

$$H(\mathcal{N}^{k,\phi})=H(\phi^{-k+1}(\mathcal{N}^{k,\phi^{-1}})).$$

Therefore, by 3.2(iv), we get

$$h(\phi,\mathcal{N})=\lim_{k\to\infty}\frac{1}{k}H(\phi^{-k+1}(\mathcal{N}^{k,\phi^{-1}}))=\lim_{k\to\infty}\frac{1}{k}H(\mathcal{N}^{k,\phi^{-1}})=h(\phi^{-1},\mathcal{N}).$$

□

3.9. Proposition. *Let $(X,\mathcal{M},m,\phi)$ be an invertible F-dynamical system. For $\mathcal{N}\in\mathcal{F}(\mathcal{M})$ and $p\in\mathbb{Z}$, we have*

$$h(\phi^p,[\mathcal{N}]) = |p|h(\phi,[\mathcal{N}])\,.$$

Proof. Follows from Proposition 3.7 and Proposition 3.8. □

3.10. Proposition. *Let $\Phi = (X,\mathcal{M},m,\phi)$ be an F-dynamical system. Then, for $\mathcal{N}_1,\mathcal{N}_2\in[\mathcal{N}]$,*

$$h(\phi,\mathcal{N}_1)\le h(\phi,\mathcal{N}_2)+H(\mathcal{N}_1\mid\mathcal{N}_2)\,.$$

Proof. By 3.2(i), (ii), and (iii), we get

$$\begin{aligned}
H(\mathcal{N}_1^{k,\phi}) &\le H(\mathcal{N}_1^{k,\phi}\vee\mathcal{N}_2^{k,\phi})\\
&= H(\mathcal{N}_2^{k,\phi})+H(\mathcal{N}_1^{k,\phi}\mid\mathcal{N}_2^{k,\phi})\\
&= H(\mathcal{N}_2^{k,\phi})+H(\mathcal{N}_1\mid\mathcal{N}_2^{k,\phi})+H(\phi^{-1}(\mathcal{N}_1^{k-1,\phi})\mid(\mathcal{N}_1\vee\mathcal{N}_2^{k,\phi}))\\
&\le H(\mathcal{N}_2^{k,\phi})+H(\mathcal{N}_1\mid\mathcal{N}_2)+H(\phi^{-1}(\mathcal{N}_1)\mid\phi^{-1}(\mathcal{N}_2))\\
&\qquad\qquad+H(\phi^{-2}(\mathcal{N}_1^{k-2,\phi})\mid\mathcal{N}_2^{k,\phi})\\
&\le H(\mathcal{N}_2^{k,\phi})+kH(\mathcal{N}_1\mid\mathcal{N}_2)\,.
\end{aligned}$$

Hence

$$h(\phi,\mathcal{N}_1)\le\lim_{k\to\infty}\frac{1}{k}H(\mathcal{N}_2^{k,\phi})+H(\mathcal{N}_1\mid\mathcal{N}_2)=h(\phi,\mathcal{N}_2)+H(\mathcal{N}_1\mid\mathcal{N}_2)\,.$$

□

4. Sufficient families and generators

4.1. Definition. Let $\Phi=(X,\mathcal{M},m,\phi)$ be an F-dynamical system, and let $\mathcal{L}\in\mathcal{F}(\mathcal{M})$. A subfamily Θ of $[\mathcal{L}]$ is called *sufficient* for $[\mathcal{L}]$ with respect to ϕ if

(i) ϕ is noninvertible and fuzzy sub-σ-algebras in $[\mathcal{L}]$ subordinate to $\bigvee_{i=0}^{p}\phi^{-i}(\mathcal{N})$, $\mathcal{N}\in\Theta$, $p\in\mathbb{N}$, forms a dense set in the space $[\mathcal{L}]$.

(ii) ϕ is invertible and fuzzy sub-σ-algebras in $[\mathcal{L}]$ subordinate to $\bigvee_{i=-p}^{p}\phi^{i}(\mathcal{N})$, $\mathcal{N}\in\Theta$, $p\in\mathbb{N}$, forms a dense subset in the space $[\mathcal{L}]$.

4.2. PROPOSITION. *Let $\Phi = (X, \mathcal{M}, m, \phi)$ be an F-dynamical system, and let $\mathcal{L} \in \mathcal{F}(\mathcal{M})$. For an arbitrary sufficient family Θ for $[\mathcal{L}]$ with respect to ϕ, we have*

$$h(\phi, [\mathcal{L}]) = \sup\{h(\phi, \mathcal{N}) : \mathcal{N} \in \Theta\}.$$

Proof. Let $\mathcal{S} \in [\mathcal{L}]$, and $\varepsilon > 0$. Choose $\mathcal{N} \in \Theta$ and $k \in \mathbb{N}$ such that

$$d_R(\mathcal{S}, \mathcal{K}) \equiv H(\mathcal{S} \mid \mathcal{K}) + H(\mathcal{K} \mid \mathcal{S}) < \varepsilon,$$

for some $\mathcal{K} \in [\mathcal{L}]$, $\mathcal{K} \leq_m \bigvee_{i=0}^{p} \phi^{-i}(\mathcal{N})$ if ϕ is noninvertible, and $\mathcal{K} \leq_m \bigvee_{i=-p}^{p} \phi^{i}(\mathcal{N})$ if ϕ is invertible. Hence by Proposition 3.10, 3.4, and Proposition 3.6, we obtain

$$\begin{aligned} H(\phi, \mathcal{S}) &= h(\phi, \mathcal{K}) + H(\mathcal{S} \mid \mathcal{K}) \\ &< h(\phi, \mathcal{K}) + \varepsilon \\ &\leq \begin{cases} h\Big(\phi, \bigvee_{i=0}^{p} \phi^{-i}(\mathcal{N})\Big) + \varepsilon & \text{if } \phi \text{ is noninvertible,} \\ h\Big(\phi, \bigvee_{i=-p}^{p} \phi^{i}(\mathcal{N})\Big) + \varepsilon & \text{if } \phi \text{ is invertible} \end{cases} \\ &= h(\phi, \mathcal{N}) + \varepsilon. \end{aligned}$$

Thus

$$h(\phi, \mathcal{S}) \leq h(\phi, \mathcal{N}) \leq \sup\{h(\phi, \mathcal{N}) : \mathcal{N} \in \Theta\},$$

and so

$$h(\phi, [\mathcal{L}]) \leq \sup\{h(\phi, \mathcal{N}) : \mathcal{N} \in \Theta\}.$$

Also, since $\Theta \subset [\mathcal{L}]$, $h(\phi, [\mathcal{L}]) \geq \sup\{h(\phi, \mathcal{N}) : \mathcal{N} \in \Theta\}$, and hence the result follows. □

4.3. DEFINITION. A fuzzy sub-σ-algebra $\mathcal{N} \in [\mathcal{L}]$ of an F-dynamical system $(X, \mathcal{M}, m, \phi)$ is called a *generator* for $[\mathcal{L}]$ with respect to ϕ if $\Theta = \{\mathcal{N}\}$ is a sufficient family for $[\mathcal{L}]$ with respect to ϕ.

5. Entropy of the inverse limit

5.1. DEFINITION. ([20]) Let J be a directed set (cf. [3]), and $\{\Phi_\alpha : \alpha \in J\}$ be a family of F-dynamical systems. Then the triple $(J, \Phi_\alpha, \Psi_{\alpha\beta})$ is called an *inverse spectrum* if, for each $\alpha, \beta \in J$ with $\alpha < \beta$, there is an F-measure preserving transformation $\psi_{\beta\alpha} : \Phi_\beta \to \Phi_\alpha$ satisfying:

$$\alpha < \beta < \gamma \implies \psi_{\gamma\alpha} + \psi_{\beta\alpha} \circ \psi_{\gamma\beta}.$$

An *upper bound* for an inverse spectrum $(J, \Phi_\alpha, \psi_{\beta\alpha})$ is an F-dynamical system with a set of homomorphisms $\psi_\alpha : \Phi \to \Phi_\alpha$, $\alpha \in J$, such that, whenever $\alpha < \beta$, $\beta \in J$, $\psi_{\beta\alpha} \circ \psi_\beta = \psi_\alpha$.

An *inverse limit* $\hat{\Phi}$ of the inverse spectrum $(J, \Phi_\alpha, \psi_{\beta\alpha})$ is an upper bound for the spectrum with homomorphism $\hat{\psi}_\alpha : \hat{\Phi} \to \Phi_\alpha$, $\alpha \in J$, such that for any upper bound Φ with homomorphism $\psi_\alpha : \Phi \to \Phi_\alpha$, there exists a homomorphism $\rho : \Phi \to \hat{\Phi}$ such that $\hat{\psi}_\alpha \circ \rho = \psi_\alpha$, for each $\alpha \in J$. We then write $\hat{\Phi} = \operatorname{inv} \lim_{\alpha \in J} \Phi_\alpha$.

The inverse limit $\hat{\Phi}$ of the inverse spectrum can be identified with $(X, \hat{\mathcal{M}}, m, \phi)$ where $\hat{\mathcal{M}}$ is the fuzzy σ-algebra generated by $\bigcup_\alpha \mathcal{M}_\alpha$.

5.2. THEOREM. *Let $\{\Phi_\alpha : \alpha \in J\}$ be an inverse spectrum of F-dynamical system indexed by a directed set J; $\Phi_\alpha = (X, \mathcal{M}_\alpha, m, \phi)$. Let $\tilde{\mathcal{M}} = \bigcup \mathcal{M}_\alpha$, and $\mathcal{L} \in \mathcal{F}(\mathcal{M}_\alpha)$ for some $\alpha \in J$. If $\mathcal{L}$ has a generator with respect to ϕ, then*

$$h(\hat{\Phi}, [\mathcal{L}]) = \lim_{\alpha \in J} h(\Phi_\alpha, [\mathcal{L}]) .$$

Proof. Since $h(\Phi_\alpha, [\mathcal{L}])$ is a monotone net $\lim_{\alpha \in J} h(\Phi_\alpha, [\mathcal{L}])$ exists and is equal to $\sup_{\alpha \in J} h(\Phi_\alpha, [\mathcal{L}])$. Moreover,

$$h(\Phi_\alpha, [\mathcal{L}]) \leq h(\hat{\Phi}, [\mathcal{L}]) ,$$

for all $\alpha \in J$ and so

$$h(\hat{\Phi}, [\mathcal{L}]) \geq \lim_{\alpha \in J} h(\Phi_\alpha, [\mathcal{L}]) . \tag{4.5.1}$$

Also, if $\mathcal{N}$ is a generator for $[\mathcal{L}]$ with respect to ϕ, then

$$h(\phi, \mathcal{N}) \leq h(\Phi_\alpha, [\mathcal{L}]) \equiv \sup\{h(\phi, \mathcal{N}) : \mathcal{N} \in [\mathcal{L}]\} .$$

Hence,

$$h(\hat{\Phi}, [\mathcal{L}]) = h(\phi, \mathcal{N}) \leq \sup_{\alpha \in J} h(\Phi_\alpha, [\mathcal{L}]) = \lim_{\alpha \in J} h(\Phi_\alpha, [\mathcal{L}]) . \tag{4.5.2}$$

Combining (4.5.1) and (4.5.2) we get the result. □

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REFERENCES

[1] BROWN, J. R.: *Ergodic Theory and Topological Dynamics*, Academic Press, Inc., London, 1976.

[2] BUTNARIU, D.: *Additive fuzzy measures and integrals*, J. Math. Anal. Appl. **93** (1983), 436–452.

[3] DUGUNDJI, J.: *Topology*, Prentice Hall of India Pvt. Ltd., New Delhi, 1975.

[4] DUMITRESCU, D.: *Fuzzy measures and the entropy of fuzzy partitions*, J. Math. Anal. Appl. **176** (1993), 359–373.

[5] HALMOS, P. R.: *Measure Theory*, Van Nostrand Reinhold, Princeton, NJ, 1950.

[6] KATOK, A.—HASSELBLATT, B.: *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1995.

[7] KHARE, M.: *Fuzzy σ-algebras and conditional entropy*, Fuzzy Sets and Systems **102** (1999), 287-292.

[8] KLEMENT, E. P.: *Fuzzy σ-algebras and fuzzy measurable functions*, Fuzzy Sets and Systems **4** (1980), 83–93.

[9] MALIČKÝ, P.—RIEČAN, B.: *On the entropy of dynamical systems.* In: Proc. Conf. Ergodic Theory and Related Topics II (Georgenthal 1986), Teubner, Leipzig, 1987, pp. 135–138.

[10] MARKECHOVÁ, D.: *The entropy of fuzzy dynamical systems and generators*, Fuzzy Sets and Systems **48** (1992), 351–363.

[11] MARKECHOVÁ, D.: *Entropy of complete fuzzy partitions*, Math. Slovaca **43** (1993), 1–10.

[12] MESIAR, R.: *The Bayes principle and the entropy on fuzzy probability spaces*, Internat. J. Gen. Systems **20** (1991), 67–72.

[13] PIASECKI, K.: *Probability of fuzzy events defined as denumerable additive measure*, Fuzzy Sets and Systems **17** (1985), 271–284.

[14] RIEČAN, B.: *A new approach to some notions of statistical quantum mechanics*, Busefal **35** (1988), 4-6.

[15] RIEČAN, B.—DVUREČENSKIJ, A.: *On randomness and fuzziness.* In: Progress in Fuzzy Sets in Europe, 1986, Polska Akademia Nauk, Warszawa, 1988, pp. 321–326.

[16] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure and Ordering*, Kluwer Acad. Publ.; Ister Press, Dordrecht; Bratislava, 1997.

[17] SRIVASTAVA, P.—KHARE, M.: *Conditional entropy and Rokhlin metric*, Math. Slovaca (1999), 433–441.

[18] SRIVASTAVA, P.—KHARE, M.—SRIVASTAVA, Y. K.: *A fuzzy measure algebra as a metric space*, Fuzzy Sets and Systems **79** (1996), 395–400.

[19] SRIVASTAVA, P.—KHARE, M.—SRIVASTAVA, Y. K.: *m-equivalence, entropy and F-dynamical systems*, Fuzzy Sets and Systems (To appear).

[20] SRIVASTAVA, P.—KHARE, M.—SRIVASTAVA, Y. K.: *Fuzzy dynamical systems — inverse and direct spectra*, Fuzzy Sets and Systems (To appear).

[21] SRIVASTAVA, Y. K.: *Fuzzy Probability Measures and Dynamical Systems.* D. Phil. Thesis, Allahabad University, 1993.

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