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NOTE ON RAMSEY NUMBERS AND SELF-COMPLEMENTARY GRAPHS

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ABSTRACT. We prove that there exists a self-complementary graph G with $\frac{1}{4}\frac{1+o(1)}{e\sqrt{2}}k2^{k/2}$ vertices which does not contain a clique of size k.

1. Introduction

The following concept was introduced in [8] (cf. also [7]).

DEFINITION 1.1. A graph $G = (V, E)$ is called *self-complementary* if there exists a permutation $\pi\colon V \to V$ such that

$$\{x, y\} \in E \iff \{\pi(x), \pi(y)\} \notin E$$

for every pair of vertices x and y. Such a permutation π is called the *generator* of the graph G.

In other words, the graph G is self-complementary if it is isomorphic to its complement $\overline{G}$. It was proved in [8] that generators can be only permutations on sets V of size $4n$ or $4n+1$.

Let k be a positive integer. We denote by $r(k,k)$ the diagonal Ramsey number, i.e., the smallest integer n with the property that for any graph G on n vertices either G or its complement contains a clique of size k.

Self-complementary graphs have been used in the past to give lower bounds for the Ramsey numbers $r(k,k)$, by Greenwood and Gleason [6], Burling and Reyner [1] and Clapham [3] for small values of k.

DEFINITION 1.2. Let $s(k)$ be the largest integer n such that there exists a self-complementary graph G with n vertices which does not contain a clique of size k.

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Clearly $s(k) \le r(k,k) - 1$. On the other hand, it is well known (cf. [6]) that $s(k) = r(k,k) - 1$ for $k = 3$ and $k = 4$, and this fact suggests that investigating of $s(k)$ is of independent interest. Chvátal, Erdös and Herdlín [2] proved that

$$4 \cdot 2^{k/4} \le s(k).$$

The aim of this note is to improve this lower bound and establish the following theorem:

THEOREM 1.3.

$$s(k) \ge \frac{1}{4}\frac{1+o(1)}{\mathrm{e}\sqrt{2}} k 2^{k/2}.$$

Note that this bound is, up to the constant factor, the same as the best known current lower bound for Ramsey numbers $r(k,k)$ (cf. [5], [9]).

2. Proof of Theorem 1.3

Let A, B, C, D be pairwise disjoint sets of cardinality n. Set $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_n\}$, $C = \{c_1, c_2, \dots, c_n\}$, $D = \{d_1, d_2, \dots, d_n\}$ and $V = A \cup B \cup C \cup D$. Let $\pi\colon V \to V$ be a permutation consisting of n 4-cycles (a_i, b_i, c_i, d_i).

Set

$$c_k = \left(\frac{2^9}{\mathrm{e}^4 k^4}\right)^{1/4}. \tag{0}$$

We will show that, if

$$4n < \frac{1}{4}\frac{c_k}{\mathrm{e}\sqrt{2}} k 2^{k/2}, \tag{1}$$

then there exists a self-complementary graph $G = (V, E)$ with generator π which does not contain a k-clique.

Let $\sim$ be an equivalence on a set $[V]^2$ of all unordered pairs of distinct elements of V defined by

$$\{x,y\} \sim \{z,u\} \iff \{z,u\} = \{\pi^m(x), \pi^m(y)\}$$

for some integer m.

Obviously, the pairs $\{a_i,a_j\}$, $\{a_i,b_j\}$, $\{a_i,c_j\}$, $\{a_i,d_j\}$, $(i,j = 1,2,\dots,n$, $i \ne j)$ and $\{a_i,b_i\}$, $\{a_i,c_i\}$ $(i = 1,2,\dots,n)$ are in different classes of $\sim$. In order to define a self-complementary graph, it is sufficient to decide for representants of the classes whether or not they are edges of G. This, with respect to the fact that π is a generator of G, determines the edge set of G in the following way: If $\{x,y\}$ is a representant of the equivalence class and $\{x,y\} = \{\pi^m(z), \pi^m(u)\}$, then

$$\begin{aligned} \{x,y\} \in E &\iff \{z,u\} \in E \text{ provided } m \text{ is even},\\ \{x,y\} \in E &\iff \{z,u\} \notin E \text{ provided } m \text{ is odd}. \end{aligned} \tag{2}$$

Note that due to the form of the permutation π, we have $\{\pi^m(z), \pi^m(u)\} \neq \{\pi^p(z), \pi^p(u)\}$ whenever m is even and p is odd, and thus the above definition is correct.

We will find convenient to consider the self-complementary graphs $G = (V, E)$ generated by π which satisfy the following additional condition

$$\{a_i, a_j\} \in E \iff \{a_i, c_j\} \notin E \tag{3}$$

for $i, j = 1, 2, \ldots, n$, $i \neq j$.

In view of the condition (3), consider a new equivalence $\sim'$ on $[V]^2$ defined as the finest equivalence which is coarser than $\sim$ and in addition satisfies $\{a_i, a_j\} \sim' \{a_i, c_j\}$. The equivalence $\sim'$ has $2n + 3\binom{n}{2}$ equivalence classes. Define the random graph G by deciding whether a fixed representant of the equivalence class of $\sim'$ is an edge. We will make these $2n + 3\binom{n}{2}$ decisions independently, each with probability $\frac{1}{2}$.

In order to conclude the proof of Theorem 1.3, it will be sufficient to prove the following claim.

CLAIM 2.1. Prob(G *contains clique of size* k) < 1.

P r o o f o f C l a i m 2.1. Suppose that G contains a clique with vertex set $\widetilde{A} \cup \widetilde{B} \cup \widetilde{C} \cup \widetilde{D}$, where $\widetilde{A} \subset A$, $\widetilde{B} \subset B$, $\widetilde{C} \subset C$, and $\widetilde{D} \subset D$. Set

$$A' = \{i \in [1, n],\ a_i \in \widetilde{A}\},$$
$$B' = \{i \in [1, n],\ b_i \in \widetilde{B}\},$$
$$C' = \{i \in [1, n],\ c_i \in \widetilde{C}\},$$
$$D' = \{i \in [1, n],\ d_i \in \widetilde{D}\}.$$

Suppose that $\{i, j\} \subseteq A' \cap B'$, then $\{a_i, a_j\} \in E$ and $\{a_i, b_j\} = \{\pi(a_i), \pi(a_j)\} \in E$, which contradicts (2).

Thus

$$\begin{aligned} &|A' \cap B'| \leq 1, \qquad \text{and similarly,} \\ &|B' \cap C'| \leq 1, \\ &|C' \cap D'| \leq 1, \\ &|D' \cap A'| \leq 1. \end{aligned} \tag{4}$$

Suppose on the other hand that $\{i, j\} \subseteq A'$ and $j \in A' \cap C'$, $i \neq j$; then both $\{a_i, a_j\}$ and $\{a_i, c_j\}$ are edges of G which contradicts to (3).

This however means that

$$\begin{aligned}&\text{either}\quad A'\cap C'=\emptyset\quad\text{or}\quad A'=C'=\{i\}\ \text{ for some } i\in[1,n],\\&\text{and similarly,}\\&\text{either}\quad B'\cap D'=\emptyset\quad\text{or}\quad B'=D'=\{j\}\ \text{ for some } j\in[1,n].\end{aligned}\tag{5}$$

Set

$$\begin{aligned}A_A&=\widetilde{A},\\A_B&=\{a_i\in A;\ i\in B'\},\\A_C&=\{a_i\in A;\ i\in C'\},\\A_D&=\{a_i\in A;\ i\in D'\}.\end{aligned}$$

In view of (4), one of the following cases happens

a) $A'\cap C'=\emptyset$ and $B'\cap D'=\emptyset$,
b) $A'=C'=\{i\}$ and $B'\cap D'=\emptyset$,
c) $A'\cap C'=\emptyset$ and $B'=D'=\{i\}$,
d) $A'=C'=\{i\}$ and $B'=D'=\{j\}$.

The cases b) and c) are analogous, and the case d) implies that $k\le 4$, and hence is not interesting. Thus we will analyse the first two cases (depicted on Figure 1 and 2) only.

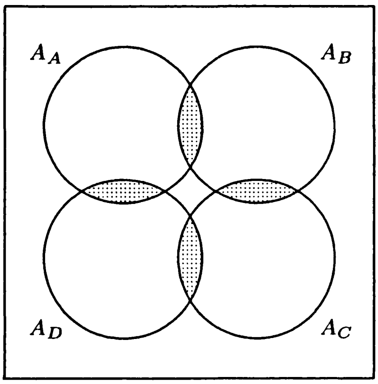

Figure 1.

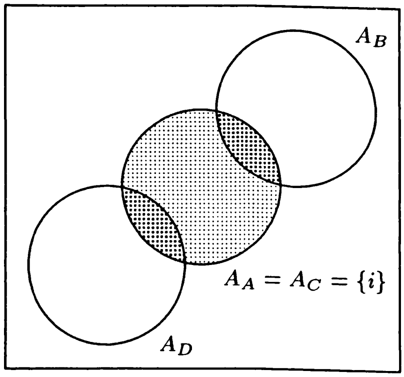

Figure 2.

The following numbers express the number of choices of a set $X=A_A\cup A_B\cup A_C\cup A_D$ such that no, one, two, three or four dashed areas in Fig. 1 contain precisely one point:

$$\begin{gathered}\binom{n}{k}4^k,\quad \binom{n}{k-2}4^{k-2}(n-k+2)4,\quad \binom{n}{k-4}4^{k-4}\binom{n-k+4}{2}\binom{4}{2},\\ \binom{n}{k-6}4^{k-6}\binom{n-k+6}{3}\binom{4}{3},\quad \binom{n}{k-8}4^{k-8}\binom{n-k+8}{4}\binom{4}{4},\end{gathered}\tag{6}$$

Similarly

$$\binom{n}{k-1}2^{k-2}(k-1), \qquad \binom{n}{k-2}2^{k-2}(k-2) \tag{7}$$

are numbers of choices of a set $X \subseteq A$ so that $(A_A \cap A_D = \emptyset$ and $A_A \cap A_B = \emptyset)$ or $\big(A_A - (A_B \cup A_D) = \emptyset\big)$.

Suppose that the subgraph $G(\widetilde{X})$ of G induced on a set $\widetilde{X} = \widetilde{A} \cup \widetilde{B} \cup \widetilde{C} \cup \widetilde{D}$ is a clique; then

$$\begin{array}{lll}
\{a_i, a_j\} \in E \text{ for } i,j \in A',\ i \neq j & & (\text{as } G(\widetilde{A}) \text{ is a clique}),\\
\{a_i, a_j\} \notin E \text{ for } i,j \in B',\ i \neq j & & (\text{as } G(\widetilde{B}) \text{ is a clique}),\\
\{a_i, a_j\} \in E \text{ for } i,j \in C',\ i \neq j & & (\text{as } G(\widetilde{C}) \text{ is a clique}),\\
\{a_i, a_j\} \notin E \text{ for } i,j \in D',\ i \neq j & & (\text{as } G(\widetilde{D}) \text{ is a clique}),\\
\{a_i, a_j\} \notin E \text{ for } i \in A',\ j \in C' & & (\text{consequence of } (3)),\\
\{a_i, a_j\} \in E \text{ for } i \in B',\ j \in D' & & (\text{consequence of } (3)),\\
\{a_i, b_j\} \in E \text{ for } i \in A',\ j \in B' & & (\text{as } G(\widetilde{A} \cup \widetilde{B}) \text{ is a clique}),\\
\{a_i, b_j\} \notin E \text{ for } i \in B',\ j \in C' & & (\text{as } G(\widetilde{B} \cup \widetilde{C}) \text{ is a clique}),\\
\{a_i, b_j\} \in E \text{ for } i \in C',\ j \in D' & & (\text{as } G(\widetilde{C} \cup \widetilde{D}) \text{ is a clique}),\\
\{a_i, b_j\} \notin E \text{ for } i \in D',\ j \in A' & & (\text{as } G(\widetilde{A} \cup \widetilde{D}) \text{ is a clique}).
\end{array}$$

For all pairs $\{i,j\} \in [X]^2$ we have a condition of “type” $\{a_i, a_j\}$ or $\{a_i, b_j\}$. As every pair of such type is a representant of a different equivalence class of $\sim'$, the events that corresponding pairs $\{a_i, a_j\}$ and $\{a_i, b_j\}$ are (or not are) edges are independent.

Thus

$$\operatorname{Prob}\big(G(\widetilde{X}) \text{ is a clique}\big) \leq 2^{-\binom{|X|}{2}}. \tag{8}$$

Let P be the probability that G contains a k-clique. Then in view of (6), (7) and (8)

$$P \leq P_1 + P_2,$$

where

$$\begin{aligned}
P_1 &= \sum_{j=0}^{4} \binom{n}{k-2j} 4^{k-2j} \binom{n-k+2j}{j} \binom{4}{j} 2^{-\binom{k-j}{2}}\\
&< \sum_{j=0}^{4} \frac{1}{j!} \left(\frac{k-2j}{4\mathrm{e}}\right)^{j} \left(\frac{4n\mathrm{e}}{k-2j}\right)^{k-j} \binom{4}{j} 2^{-\binom{k-j}{2}},
\end{aligned}$$

and

$$P_2 = 2\left(\binom{n}{k-2}2^{k-2}(k-2)2^{-\binom{k-2}{2}} + \binom{n}{k-1}2^{k-2}(k-1)2^{-\binom{k-1}{2}}\right)$$
$$< \sum_{j=1}^{2}\binom{n}{k-2j}4^{k-2j}\binom{n-k+2j}{j}\binom{4}{j}2^{-\binom{k-j}{2}}.$$

Thus

$$P < 2\sum_{j=0}^{4}\frac{1}{j!}\left(\frac{k-2j}{4\,\mathrm{e}}\right)^{j}\left(\frac{4n\,\mathrm{e}}{k-2j}\right)^{k-j}\binom{4}{j}2^{-\binom{k-j}{2}}. \tag{9}$$

As for $k > k_0$

$$\frac{1}{j!}\left(\frac{k-2j}{4\,\mathrm{e}}\right)^{j}\binom{4}{j} < \frac{1}{24}\left(\frac{k}{4\,\mathrm{e}}\right)^{4}$$

holds, we infer that $k > k_0$ implies

$$P < \frac{1}{12}\left(\frac{k}{4\,\mathrm{e}}\right)^{4}\sum_{j=0}^{4}\left(\frac{4n\,\mathrm{e}}{k-2j}2^{-\frac{k-j-1}{2}}\right)^{k-j}. \tag{10}$$

We will prove that for k sufficiently large each summand on the right-hand side of (10) is bounded from above by $\frac{1}{5}$. This means that for k large enough the probability that G contains no k-clique is positive which concludes the proof:

In view of (1), we have

$$\begin{aligned}
&\frac{1}{12}\left(\frac{k}{4\,\mathrm{e}}\right)^{4}\left(\frac{4n\,\mathrm{e}}{k-2j}2^{-\frac{k-j-1}{2}}\right)^{k-j}\\
&\leq \frac{1}{12}\left(\frac{k}{4\,\mathrm{e}}\right)^{4}\left(\frac{k}{4\sqrt{2}}c_k\frac{2^{k/2}}{k-2j}2^{-\frac{k-j-1}{2}}\right)^{k-j}\\
&\leq \frac{1}{12}\left(\frac{k}{4\,\mathrm{e}}\right)^{4}\left(\frac{1}{4}c_k\frac{k}{k-2j}2^{j/2}\right)^{k-j}.
\end{aligned} \tag{11}$$

Due to the condition on j, we have that $\frac{1}{4}2^{j/2} \leq 1$, and thus we bound the right-hand side of (11) by

$$\frac{1}{12}\left(\frac{k}{4\,\mathrm{e}}\right)^{4}c_k^{k}c_k^{-j}\left(\frac{k}{k-2j}\right)^{k-j}. \tag{12}$$

As $\lim_{k\to\infty} c_k^{-j} = 1$ and $\lim_{k\to\infty}\left(\frac{k}{k-j}\right)^{k-j} = \mathrm{e}^{2j}$, we bound (12) from above by

$$\frac{1}{12}\left(\frac{k}{4\,\mathrm{e}}\right)^{4}\frac{2^9}{\mathrm{e}^4 k^4}\,\mathrm{e}^{2j}\big(1+o(1)\big) \leq \frac{1}{6}\big(1+o(1)\big) < \frac{1}{5}.$$

□

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