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Mathematica Slovaca, Vol. 45 (1995), No. 1, 29--38

Persistent URL: http://dml.cz/dmlcz/130473

Math. Slovaca, **45** (1995), No. 1, 29–38

Mathematica Slovaca
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Mathematical Institute
Slovak Academy of Sciences

Dedicated to Professor Ján Jakubík on the occasion of his 70th birthday

LEXICOGRAPHIC PRODUCTS OF CYCLICALLY ORDERED GROUPS

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(*Communicated by Tibor Katriňák*)

ABSTRACT. Refinements of lexicographic product decompositions of cyclically ordered groups and the cancellation in these decompositions are studied.

Basic results on cyclically ordered groups are due to L. Rieger [7] and S. Swierczkowski [8]. Further results in this field are obtained, e.g., in [4], [5], [9]–[11].

In this paper, the notion of a *cyclically ordered group* will be used in a more general sense. This notion will be applied in the same sense as in the papers [12] [14] of S. D. Želeva.

J. Jakubík [6] introduced the notion of an *extended cyclic order*, defined the concepts of an *ec-group* (a group with an extended cyclic order), a *dc-group* and studied *direct product decompositions* of ec-groups.

In the present paper, it is defined and investigated a *lexicographic product* of cyclically ordered groups. It is shown that two finite lexicographic product decompositions of isomorphic cyclically ordered groups $\boldsymbol{G}$ and $\boldsymbol{H}$,

$$\boldsymbol{G} = \boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n\,, \qquad \boldsymbol{H} = \boldsymbol{H}_1 \circ \boldsymbol{H}_2 \circ \cdots \circ \boldsymbol{H}_m$$

have always isomorphic refinements, provided that all $\boldsymbol{G}_i$, $\boldsymbol{H}_j$ $(i = 1, 2, \dots, n;$ $j = 1, 2, \dots, m)$ are dc-groups.

In particular, it is proved that if $\boldsymbol{G}_i$, $\boldsymbol{H}_j$ $(i = 1, 2, \dots, n;$ $j = 1, 2, \dots, m)$ are *lc-groups*, then $m = n$, and $\boldsymbol{G}_i$ is isomorphic with $\boldsymbol{H}_i$ $(i = 1, 2, \dots, n)$. Further, the cancellation in lexicographic product decompositions of cyclically ordered groups is studied.

AMS Subject Classification (1991): Primary 06F15.
Key words: Cyclically ordered group, Directed group, Lexicographic product, Refinement.

1. Preliminaries

For the sake of completeness, we recall some definitions concerning cyclically ordered groups (see Želeva [12]; our definition is more general than that applied by Rieger [7] and Fuchs [2]).

Let $(G,+)$ be a group, and let C be a ternary relation on G satisfying the following conditions:

I. if $(x,y,z) \in C$, then $(z,y,x) \notin C$;
II. if $(x,y,z) \in C$, then $(z,x,y) \in C$;
III. if $(x,y,z) \in C$ and $(x,z,u) \in C$, then $(x,y,u) \in C$;
IV. if $(x,y,z) \in C$, then $(a+x+b,\ a+y+b,\ a+z+b) \in C$ for each $a,b \in G$.

Then C is called a *cyclic order* on G and the triple $\boldsymbol{G} = (G,+,C)$ is said to be a *cyclically ordered group.*

If $x,y,z \in G$, $(x,y,z) \in C$, then using I and II we get that x, y, z are different elements of G.

Every subgroup of a cyclically ordered group is considered as cyclically ordered under the induced cyclic order.

An element x of a cyclically ordered group $\boldsymbol{G}$ is called *isolated* (cf. [6]) if there are no elements $y,z \in G$ with the property $(x,y,z) \in C$. If every element of $\boldsymbol{G}$ is isolated, then $\boldsymbol{G}$ will be called isolated.

An *isomorphism* of cyclically ordered groups is defined in a natural way. The fact that cyclically ordered groups $\boldsymbol{G}$ and $\boldsymbol{H}$ are isomorphic will be written as $\boldsymbol{G} \simeq \boldsymbol{H}$.

We say that a cyclically ordered group $\boldsymbol{G}$ is an *lc-group* (see [6]) if $\operatorname{card} G > 2$, and if the following condition is fulfilled: whenever x, y, z are distinct elements of G, then either $(x,y,z) \in C$ or $(z,y,x) \in C$.

A cyclically ordered group $\boldsymbol{G}$ will be called a *dc-group* if for each $x,y \in G$, $x \neq y$, there exists an element $z \in G$ such that either $(x,y,z) \in C$ or $(x,z,y) \in C$ (see [6]).

Let $(G,+,\leq)$ be a partially ordered group. For the basic definitions concerning *partially ordered* (*directed*, *linearly ordered*) *groups* we refer to L. Fuch [2] and G. Birkhoff [1]. Define a ternary relation $C_{\leq}$ on G in the following manner. For elements $x,y,z \in G$ we put $(x,y,z) \in C_{\leq}$ if and only if $x < y <$ or $y < z < x$ or $z < x < y$. Then $(G,+,C_{\leq})$ is a cyclically ordered group.

Let A, B be subgroups of a partially ordered group $(G,+,\leq)$ such that the following conditions hold:

(i) for each $g \in G$ there exist uniquely determined elements $a \in A$, $b \in B$ such that $g = a + b$;
(ii) if $g_i = a_i + b_i$, $a_i \in A$, $b_i \in B$ $(i = 1,2)$, then $g_1 + g_2 = (a_1 + a_2) + (b_1 + b_2)$;

(iii) for each $g \in G$, $g = a + b$, $a \in A$, $b \in B$, we have $g \geq 0$ if and only if $a > 0$, or $a = 0$ and $b \geq 0$.

Under these assumptions, we write $(G, +, \leq) = A \circ B$ and this equation is called a *lexicographic product decomposition* of a partially ordered group $(G, +, \leq)$.

2. Lexicographic product decompositions of cyclically ordered groups

Let $\boldsymbol{A} = (A, +_1, C_1)$ and $\boldsymbol{B} = (B, +_2, C_2)$ be cyclically ordered groups. We form the (*external*) *direct product* G of groups A and B. The operation $+$ on G is performed componentwise. Let $g_i \in G$, $g_i = (a_i, b_i)$, $a_i \in A$, $b_i \in B$ $(i = 1, 2, 3)$. Define a ternary relation C on G as follows. We put

$$(g_1, g_2, g_3) \in C \iff (a_1, a_2, a_3) \in C_1 \quad \text{or} \quad a_1 = a_2 = a_3 \text{ and } (b_1, b_2, b_3) \in C_2 .$$

It can be easily verified that C is a cyclic order on G and $\boldsymbol{G} = (G, +, C)$ is a cyclically ordered group. Then $\boldsymbol{G}$ is said to be a (*external*) *lexicographic product* of cyclically ordered groups $\boldsymbol{A}$ and $\boldsymbol{B}$. We shall use the notation $\boldsymbol{G} = [\boldsymbol{A} \circ \boldsymbol{B}]$.

Put

$$A_1 = \{(a, 0) : a \in A\}, \qquad B_1 = \{(0, b) : b \in B\}. \tag{$*$}$$

A_1 and B_1 are subgroups of G and $\boldsymbol{A} \simeq \boldsymbol{A}_1$, $\boldsymbol{B} \simeq \boldsymbol{B}_1$.

Assume that A and B are subgroups of a cyclically ordered group $\boldsymbol{G}$ such that the conditions (i) and (ii) are satisfied, and that, moreover, the following condition holds:

(iii) if g_1, g_2, g_3 are distinct elements of G, $g_i = a_i + b_i$ $(i = 1, 2, 3)$, then $(g_1, g_2, g_3) \in C$ if and only if either $(a_1, a_2, a_3) \in C$ or $a_1 = a_2 = a_3$ and $(b_1, b_2, b_3) \in C$.

Under these assumptions, we write $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$. This equation is called a lexicographic product decomposition of $\boldsymbol{G}$ with factors $\boldsymbol{A}$ and $\boldsymbol{B}$.

If $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$, then clearly the mapping φ: $[\boldsymbol{A} \circ \boldsymbol{B}] \to \boldsymbol{G}$ defined by $\varphi\big((a, b)\big) = a + b$ is an isomorphism of $[\boldsymbol{A} \circ \boldsymbol{B}]$ onto $\boldsymbol{G}$.

Assume that $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$. Denote by $\overline{g}$ an element of the factor group G/B containing $g \in G$. Define a ternary relation $\overline{C}$ on G/B by the following rule: for distinct elements $\overline{g}_1$, $\overline{g}_2$, $\overline{g}_3$ of G/B we put $(\overline{g}_1, \overline{g}_2, \overline{g}_3) \in \overline{C}$ if and only if there exist elements $g'_i \in G$, $g'_i \in \overline{g}_i$ $(i = 1, 2, 3)$ such that $(g'_1, g'_2, g'_3) \in C$.

Now, we intend to show that if $(\overline{g}_1, \overline{g}_2, \overline{g}_3) \in \overline{C}$, then $(x, y, z) \in C$ for each $x \in \overline{g}_1$, $y \in \overline{g}_2$, $z \in \overline{g}_3$.

In fact, let $(\overline{g}_1, \overline{g}_2, \overline{g}_3) \in \overline{C}$, $g_i = a_i + b_i$, $a_i \in A$, $b_i \in B$ $(i = 1, 2, 3)$. Assume that $x \in \overline{g}_1$, $y \in \overline{g}_2$, $z \in \overline{g}_3$. Then $x = a_1 + b_1''$, $y = a_2 + b_2''$, $z = a_3 + b_3''$, $b_i'' \in B$ $(i = 1, 2, 3)$. There are elements $g_i' \in \overline{g}_i$, $g_i' = a_i + b_i'$, $b_i' \in B$ such that $(g_1', g_2', g_3') \in C$. This implies that either $(a_1, a_2, a_3) \in C$ or $a_1 = a_2 = a_3$ and $(b_1', b_2', b_3') \in C$. If $(a_1, a_2, a_3) \in C$, then $(x, y, z) \in C$. If $a_1 = a_2 = a_3$, then $\overline{g}_1 = \overline{g}_2 = \overline{g}_3$, which is impossible.

It is easy to verify that $\boldsymbol{G}/\boldsymbol{H} = \left(G/H, +, \overline{C}\right)$ is a cyclically ordered group.

2.1. Lemma. *Let $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$. Then $\boldsymbol{G}/\boldsymbol{B} \simeq \boldsymbol{A}$.*

Proof. Let $\overline{g} \in G/B$, $g = a + b$, $a \in A$, $b \in B$. The mapping $\varphi\colon G/B \to A$, defined by $\varphi(\overline{g}) = a$, is an isomorphism of the group G/B onto A.

Let $\overline{C}$ be a cyclic order on $\boldsymbol{G}/\boldsymbol{B}$ as above and let $\overline{g}_i \in G/B$, where $g_i = a_i + b_i$, $a_i \in A$, $b_i \in B$ $(i = 1, 2, 3)$. Now we show that $(\overline{g}_1, \overline{g}_2, \overline{g}_3) \in \overline{C}$ if and only if $(a_1, a_2, a_3) \in C$.

Let $(\overline{g}_1, \overline{g}_2, \overline{g}_3) \in \overline{C}$. Then $(g_1, g_2, g_3) \in C$ is valid. Since a_1, a_2, a_3 are distinct elements, we get $(a_1, a_2, a_3) \in C$. The converse is analogous.

We conclude that φ is an isomorphism of $\boldsymbol{G}/\boldsymbol{B}$ onto $\boldsymbol{A}$.

2.2. Lemma. *Let $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$. Then $\boldsymbol{G}$ is a dc-group if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are dc-groups.*

Proof. Let $\boldsymbol{G}$ be a dc-group. Assume that $a_1, a_2 \in A$, $a_1 \neq a_2$. Choose an arbitrary element $b \in B$. Therefore elements $g_1, g_2 \in G$, $g_1 = a_1 + b$, $g_2 = a_2 + b$ are distinct and there exists $g_3 \in G$, $g_3 = a_3 + b_3$, $a_3 \in A$, $b_3 \in B$ such that either $(g_1, g_2, g_3) \in C$ or $(g_1, g_3, g_2) \in C$. Hence either $(a_1, a_2, a_3) \in C$ or $(a_1, a_3, a_2) \in C$. Now assume that $b_1, b_2 \in B$, $b_1 \neq b_2$. Choose $a \in A$ arbitrarily. Then $g_1 = a + b_1$, $g_2 = a + b_2$ are distinct elements of G. There is an element $g_3 \in G$, $g_3 = a_3 + b_3$ such that either $(g_1, g_2, g_3) \in C$ or $(g_1, g_3, g_2) \in C$. Therefore $a_3 = a$, and either $(b_1, b_2, b_3) \in C$ or $(b_1, b_3, b_2) \in C$. We conclude that $\boldsymbol{A}$ and $\boldsymbol{B}$ are dc-groups.

Conversely, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be dc-groups. Assume that $g_1, g_2 \in G$, $g_1 \neq g_2$, $g_1 = a_1 + b_1$, $g_2 = a_2 + b_2$, $a_1, a_2 \in A$, $b_1, b_2 \in B$. Suppose $a_1 \neq a_2$. Then there exists $a_3 \in A$ such that either $(a_1, a_2, a_3) \in C$ or $(a_1, a_3, a_2) \in C$, which implies that $(g_1, g_2, a_3) \in C$ or $(g_1, a_3, g_2) \in C$.

Next suppose that $a_1 = a_2$. Since $b_1 \neq b_2$, there exists $b_3 \in B$ such that either $(b_1, b_2, b_3) \in C$ or $(b_1, b_3, b_2) \in C$. Hence either $(g_1, g_2, a_1 + b_3) \in C$ or $(g_1, a_1 + b_3, g_2) \in C$. Therefore $\boldsymbol{G}$ is a dc-group.

3. Refinements of lexicographic product decompositions

This section deals with *refinements* of finite lexicographic product decompositions of a cyclically ordered group $\boldsymbol{G}$. Further the cancellation in lexicographic product decompositions of cyclically ordered groups is investigated. Related questions have been studied for partially ordered groupoids in [3]. [3; Example 3.7] is applied. Also the procedure concerning the operation $+$, which is applied in the proofs of 3.4 and 3.9, is analogous to that dealt with in [3].

3.1. Lemma. *Let* $\boldsymbol{G} = (\boldsymbol{A} \circ \boldsymbol{B}) \circ \boldsymbol{D}$. *Then* $\boldsymbol{G} = \boldsymbol{A} \circ (\boldsymbol{B} \circ \boldsymbol{D})$.

Proof. Denote

(1) $\boldsymbol{E} = \boldsymbol{A} \circ \boldsymbol{B}$,
(2) $\boldsymbol{F} = \boldsymbol{B} \circ \boldsymbol{D}$.

Then

(3) $\boldsymbol{G} = \boldsymbol{E} \circ \boldsymbol{D}$.

We have to show that $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{F}$. From associativity of the direct product decomposition of groups it follows that every element $g \in G$ can be uniquely expressed in the form $g = a + f$, $a \in A$, $f \in F$. Let $g_i \in G$, $g_i = a_i + f_i$, $a_i \in A$, $f_i \in F$ $(i = 1, 2, 3)$. We have only to show that $(g_1, g_2, g_3) \in C$ if and only if either $(a_1, a_2, a_3) \in C$ or $a_1 = a_2 = a_3$ and $(f_1, f_2, f_3) \in C$.

In view of (2), we get $f_i = b_i + d_i$, $b_i \in B$, $d_i \in D$ $(i = 1, 2, 3)$. By (3), (2) and (1), we obtain $g_i = a_i + f_i = a_i + (b_i + d_i) = (a_i + b_i) + d_i = e_i + d_i$, where $a_i + b_i = e_i \in E$.

Assume that $(g_1, g_2, g_3) \in C$. With respect to (3), we obtain $(e_1, e_2, e_3) \in C$ or $e_1 = e_2 = e_3$, and $(d_1, d_2, d_3) \in C$.

Suppose that $(e_1, e_2, e_3) \in C$. According to (1), we have $(a_1, a_2, a_3) \in C$ or $a_1 = a_2 = a_3$, and $(b_1, b_2, b_3) \in C$. Then (2) implies $(f_1, f_2, f_3) \in C$.

Let $e_1 = e_2 = e_3$ and $(d_1, d_2, d_3) \in C$. By (1), we get $a_1 = a_2 = a_3$ and $b_1 = b_2 = b_3$. Then (2) yields $(f_1, f_2, f_3) \in C$.

The converse is analogous. Therefore $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{F}$ is valid. □

In an analogous way, we can show that from $\boldsymbol{G} = \boldsymbol{A} \circ (\boldsymbol{B} \circ \boldsymbol{D})$ it follows $\boldsymbol{G} = (\boldsymbol{A} \circ \boldsymbol{B}) \circ \boldsymbol{D}$.

3.2. Remark. According to 3.1, brackets can be omitted, and we can write $\boldsymbol{A} \circ \boldsymbol{B} \circ \boldsymbol{D}$ instead of $(\boldsymbol{A} \circ \boldsymbol{B}) \circ \boldsymbol{D}$. This result can be generalized by induction and we need not use brackets in expressions of the form $\boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n$.

Now we can define a lexicographic product decomposition with a finite number of factors.

Let $G_1, G_2, \dots, G_n$ be subgroups of G such that $\boldsymbol{G} = \boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n$. Such an expression of $\boldsymbol{G}$ is called a lexicographic product decomposition of $\boldsymbol{G}$ with factors $\boldsymbol{G}_1, \boldsymbol{G}_2, \dots, \boldsymbol{G}_n$.

3.3. Lemma. *Let*

(i) $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$,

(ii) $\boldsymbol{G} = \boldsymbol{D} \circ \boldsymbol{E}$,

and let $\boldsymbol{B}$ *and* $\boldsymbol{E}$ *be dc-groups. Then either* $B \subseteq E$ *or* $E \subseteq B$.

Proof. Assume that $E \nsubseteq B$. Then there exists $e \in E$, $e \notin B$. Because E is a dc-group and $e \neq 0$, there exists $e' \in E$ such that either $(0, e', e) \in C$ or $(0, e, e') \in C$. Suppose that

(1) $(0, e', e) \in C$.

The case $(0, e, e') \in C$ is analogous. Therefore $e' \notin B$. Indeed, if $e' \in B$, then (i) implies that elements 0, e', e are uniquely expressed in the form $0 = 0 + 0$, $e' = 0 + e'$, $e = a_1 + b_1$, $a_1 \in A$, $b_1 \in B$. Whence (1) yields that $a_1 = 0$, and so $e \in B$, which gives a contradiction. Let $e' = a' + b'$, $a' \in A$, $b' \in B$. Since $a_1 \neq 0$, using (1) we get

(2) $(0, a', a_1) \in C$.

Assume that $b \in B$, $b = d_1 + e_1$, $d_1 \in D$, $e_1 \in E$. From (i) and (2), we infer that $(b, e', e) \in C$. Then using (ii) we get $d_1 = 0$, and so $b \in E$. We have proved that $B \subseteq E$. $\square$

Let $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$, $X \subseteq G$. Denote

$$X(A)(\boldsymbol{A} \circ \boldsymbol{B}) = \{a \in A : \text{ there exist } x \in X, \ b \in B, \ x = a + b\}.$$

$X(B)(\boldsymbol{A} \circ \boldsymbol{B})$ is defined analogously.

3.4. Lemma. *Let*

(i) $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$,

(ii) $\boldsymbol{G} = \boldsymbol{D} \circ \boldsymbol{E}$,

and let $E \subseteq B$. *Then* $\boldsymbol{B} = (\boldsymbol{B} \cap \boldsymbol{D}) \circ \boldsymbol{E}$ *and* $B \cap D = B(D)(\boldsymbol{D} \circ \boldsymbol{E})$.

Proof. We put $F = B(D)(\boldsymbol{D} \circ \boldsymbol{E})$. First we will show that $F \subseteq B \cap D$. Choose any element $f \in F$. Then $f \in D$ and there exist $b \in B$, $e \in E$ such that $b = f + e$. With respect to (i), elements e and f can be uniquely expressed in the forms $e = 0 + e$, $f = a_1 + b_1$, $a_1 \in A$, $b_1 \in B$. The group operation is performed componentwise, thus $b = f + e = (a_1 + 0) + (e + b_1) = a_1 + (e + b_1)$. Since $b \in B$, we obtain $a_1 = 0$. Hence $f \in B$.

Now we intend to show that $B \cap D \subseteq F$. Assume that $g \in B \cap D$. Since $g \in B$, $g \in D$, $g = g + 0$, in view of (ii), we get $g \in F$.

We have proved that $F = B \cap D$.

It remains to show that $\boldsymbol{B} = \boldsymbol{F} \circ \boldsymbol{E}$. It is clear that E and F are subgroups of B. According to (ii), for each element $b \in B$ there exist uniquely determined elements $d \in D$, $e \in E$ satisfying the equation $b = d + e$. Hence $d \in F$. Let

$b_i \in B$, $b_i = f_i + e_i$, $f_i \in F$, $e_i \in E$ $(i = 1,2,3)$. Again according to (ii), we get that $(b_1, b_2, b_3) \in C$ if and only if either $(f_1, f_2, f_3) \in C$ or $f_1 = f_2 = f_3$ and $(b_1, b_2, b_3) \in C$. Therefore $\boldsymbol{B} = \boldsymbol{F} \circ \boldsymbol{E}$. □

3.5. THEOREM. *Let $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$, $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{D}$, and let $\boldsymbol{B}$ and $\boldsymbol{D}$ be dc-groups. Then $B = D$.*

Proof. In view of 3.3, we obtain either $B \subseteq D$ or $D \subseteq B$. Suppose that $B \subseteq D$. With respect to 3.4, we have $\boldsymbol{D} = (\boldsymbol{A} \cap \boldsymbol{D}) \circ \boldsymbol{B}$. Since $A \cap D = \{0\}$, $D = B$ holds true. If $D \subseteq B$, the proof is analogous. □

3.6. Remark. Let $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$, $\boldsymbol{G} = \boldsymbol{D} \circ \boldsymbol{B}$, and let $\boldsymbol{B}$ be a dc-group. Then $A = D$ need not be true in general.

3.7. Example. Let $(Z, +, \leq)$ be the additive group of all integers with the natural linear order, $\boldsymbol{Z} = (Z, +, C_{\leq})$, and let $\boldsymbol{G} = [\boldsymbol{Z} \circ \boldsymbol{Z}]$. It suffices to put $A = \{(a, 0) : a \in Z\}$, $B = \{(0, b) : b \in Z\}$ and $D = \{(a, b) : a, b \in Z, \ a = b\}$.

However, the following assertion is valid.

3.8. THEOREM. *Let*

(i) $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$,

(ii) $\boldsymbol{G} = \boldsymbol{D} \circ \boldsymbol{B}$.

Then $\boldsymbol{A} \simeq \boldsymbol{D}$.

Proof. By the assumptions and 2.1, we get $\boldsymbol{G}/\boldsymbol{B} \simeq \boldsymbol{A}$, and $\boldsymbol{G}/\boldsymbol{B} \simeq \boldsymbol{D}$. Hence $A \simeq \boldsymbol{D}$. □

Let

(1) $\boldsymbol{G} = \boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \cdots \circ \boldsymbol{G}_n$,

and let $\boldsymbol{G}_i = \boldsymbol{G}_{i1} \circ \boldsymbol{G}_{i2} \circ \cdots \circ \boldsymbol{G}_{im(i)}$ $(i = 1, 2, \dots, n)$. Applying (1) we get

(2) $\boldsymbol{G} = \boldsymbol{G}_{11} \circ \boldsymbol{G}_{12} \circ \cdots \circ \boldsymbol{G}_{nm(n)}$.

The lexicographic product decomposition (2) of $\boldsymbol{G}$ is said to be a *refinement of the lexicographic product decomposition* (1).

Let

(3) $\boldsymbol{H} = \boldsymbol{H}_1 \circ \boldsymbol{H}_2 \circ \cdots \circ \boldsymbol{H}_m$.

We say that the lexicographic decompositions (1) and (3) are *isomorphic* whenever $m = n$ and $\boldsymbol{G}_i \simeq \boldsymbol{H}_i$ $(i = 1, 2, \dots, n)$.

3.9. THEOREM. *Let $\boldsymbol{G}$, $\boldsymbol{H}$, $\boldsymbol{G}_i$, $\boldsymbol{H}_j$ $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, m)$ be dc-groups such that $\boldsymbol{G} \simeq \boldsymbol{H}$. Suppose* (1) *and* (3)*. Then the lexicographic product decompositions* (1) *and* (3) *possess isomorphic refinements.*

Proof. We proceed by induction with respect to $m+n$. We have $m+n \geq 2$. For $m + n = 2$ the assertion is evident. Assume that $m + n > 2$. Let φ be an isomorphism of $\boldsymbol{G}$ onto $\boldsymbol{H}$ and let $\varphi(G_i) = G'_i$ $(i = 1,2,\dots,n)$. Then $\boldsymbol{H} = \boldsymbol{G}'_1 \circ \boldsymbol{G}'_2 \circ \dots \circ \boldsymbol{G}'_n$. By 3.3, we obtain that $G'_n \subseteq H_m$ or $H_m \subseteq G'_n$. Suppose that $G'_n \subseteq H_m$ (the case $H_m \subseteq G'_n$ is analogous). By 3.4, we get $\boldsymbol{H}_m = (\boldsymbol{H}_m \cap \boldsymbol{G}'_1 \circ \boldsymbol{G}'_2 \circ \dots \circ \boldsymbol{G}'_{n-1}) \circ \boldsymbol{G}'_n$. If we put $\boldsymbol{A} = \boldsymbol{H}_m \cap \boldsymbol{G}'_1 \circ \boldsymbol{G}'_2 \circ \dots \circ \boldsymbol{G}'_{n-1}$, then $\boldsymbol{H} = \boldsymbol{H}_1 \circ \boldsymbol{H}_2 \circ \dots \circ \boldsymbol{H}_{m-1} \circ \boldsymbol{A} \circ \boldsymbol{G}'_n$. With respect to 3.8, we get $\boldsymbol{G}'_1 \circ \boldsymbol{G}'_2 \circ \dots \circ \boldsymbol{G}'_{n-1} \simeq \boldsymbol{H}_1 \circ \boldsymbol{H}_2 \circ \dots \circ \boldsymbol{H}_{m-1} \circ \boldsymbol{A}$. An application of the induction hypothesis completes the proof.

4. lc-groups, directed groups and dc-groups

Now the previous results will be applied to lc-groups. Evidently, every lc-group is a dc-group.

4.1. Lemma. *Let $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$. If $\boldsymbol{D}$ is an lc-subgroup of $\boldsymbol{G}$, then either $B \cap D = \{0\}$ or $D \subseteq B$.*

Proof. Assume that $B \cap D \neq \{0\}$. Then there exists $d_1 \in B$, $d_1 \in D$, and $d_1 \neq 0$. Let $d \in D$, $d \neq 0$ and $d \neq d_1$. Therefore either $(0,d,d_1) \in C$ or $(0,d_1,d) \in C$. Suppose that $(0,d,d_1) \in C$ (the case $(0,d_1,d) \in C$ is analogous). From the assumption $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$, it follows that elements d and d_1 can be uniquely written in the form $d = a + b$, $a \in A$, $b \in B$, $d_1 = 0 + d_1$. Whence we obtain $a = 0$, and so $d \in B$. Thus $D \subseteq B$.

4.2. Theorem. *Let*

(i) $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$,
(ii) $\boldsymbol{G} = \boldsymbol{D} \circ \boldsymbol{E}$.

If $\boldsymbol{B}$ and $\boldsymbol{E}$ are lc-groups, then $B = E$.

Proof. From 3.3, it follows that either $E \subseteq B$ or $B \subseteq E$. Let $E \subseteq B$. Hence $B \cap E = E \neq \{0\}$. Since $\boldsymbol{B}$ is an lc-group and (ii) is satisfied, by 4.1, we have $B \subseteq E$. We conclude that $E = B$. In the case $B \subseteq E$, the proof is analogous. □

Let $\boldsymbol{G}$ be a cyclically ordered group. If $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$ implies that $\operatorname{card} A =$ or $\operatorname{card} B = 1$, then $\boldsymbol{G}$ is said to be *lexicographically indecomposable.*

4.3. Lemma. *Let $\boldsymbol{G}$ be an lc-group. Then $\boldsymbol{G}$ is lexicographically indecomposable.*

Proof. By way of contradiction, assume that there exist subgroups A and B of G with $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$, and $\operatorname{card} A > 1$, $\operatorname{card} B > 1$. Then we can find elements $a \in A$, $a \neq 0$, $b \in B$, $b \neq 0$ and thus $a \neq b$. We have $(0,a,b) \in C$ or $(b,a,0) \in C$. Elements 0, a, b are uniquely expressed in the following forms: $0 = 0 + 0$, $a = a + 0$, $b = 0 + b$. Hence $a = 0$, a contradiction. □

From 4.3 and 3.9, it immediately follows

4.4. THEOREM. *Let $\boldsymbol{G}$, $\boldsymbol{H}$ be cyclically ordered groups such that $\boldsymbol{G} \simeq \boldsymbol{H}$, and let all $\boldsymbol{G}_i$, $\boldsymbol{H}_j$ $(i = 1,2,\dots,n;\ j = 1,2,\dots,m)$ be lc-groups. Suppose that*

$$\boldsymbol{G} = \boldsymbol{G}_1 \circ \boldsymbol{G}_2 \circ \dots \circ \boldsymbol{G}_n\,, \qquad \boldsymbol{H} = \boldsymbol{H}_1 \circ \boldsymbol{H}_2 \circ \dots \circ \boldsymbol{H}_m\,.$$

Then $m = n$ and $\boldsymbol{G}_i \simeq \boldsymbol{H}_i$ $(i = 1,2,\dots,n)$.

In the next, we investigate relations between a directed group $(G,+,\leq)$ and a cyclically ordered group $(G,+,C_{\leq})$.

The definitions of a linearly ordered group (directed group) and an lc-group (dc-group) are similar. If $(G,+,\leq)$, $G \neq \{0\}$ is a linearly ordered group, then $(G,+,C_{\leq})$ is an lc-group.

On the other hand, if $(G,+,\leq)$ is a directed group which is not linearly ordered, then a cyclically ordered group $(G,+,C_{\leq})$ fails to be a dc-group.

4.5. LEMMA. *Let $(G,+,\leq)$ be a partially ordered group, and let $\boldsymbol{G} = (G,+,C_{\leq})$ be a cyclically ordered group. If $\boldsymbol{G} = \boldsymbol{A} \circ \boldsymbol{B}$, $\operatorname{card} A > 1$, $\operatorname{card} B > 1$, then either $\boldsymbol{A}$ or $\boldsymbol{B}$ is isolated.*

P r o o f . Assume that $\boldsymbol{A}$ is not isolated. Then there are elements $a_1, a_2, a_3 \in A$ with $(a_1, a_2, a_3) \in C$. Hence $a_1 < a_2 < a_3$ or $a_2 < a_3 < a_1$ or $a_3 < a_1 < a_2$. Therefore there is $a \in A$, $a > 0$.

We have to prove that $\boldsymbol{B}$ is isolated. Suppose (by way of contradiction) that $\boldsymbol{B}$ is not isolated. In an analogous way we prove that there is an element $b \in B$, $b > 0$, and so $0 < a < a + b$. Whence $(0, a, a+b) \in C_{\leq}$. From this we infer that $a = 0$, which is a contradiction. □

4.6. THEOREM. *Let $(G,+,\leq)$ be a directed group and let $\boldsymbol{G}$ be as in 4.5. Then $\boldsymbol{B}$ is isolated.*

P r o o f . With respect to 4.5, it suffices to show that $\boldsymbol{A}$ is not isolated. Since $\operatorname{card} A > 1$, there exists $a \in A$, $a \neq 0$.

Since G is directed, there are $x, y \in G$ with $0 < x < y$, $a < x < y$.

Then

(1) $(0, x, y) \in C_{\leq}$,
(2) $(a, x, y) \in C_{\leq}$.

Let $x = a_1 + b_1$, $y = a_2 + b_2$, $a_i \in A$, $b_i \in B$ $(i = 1,2)$. Suppose that $a = a_1$. By (2), $a = a_1 = a_2$. Then (1) implies that $a_2 = a_1 = 0$, which contradicts the fact that $a \neq 0$. Therefore $a \neq a_1$. Then from (2) it follows that $(a, a_1, a_2) \in C_{\leq}$. □

In 4.5 and 4.6, $\boldsymbol{G}$ is not a dc-group. There exists a cyclically ordered group $\boldsymbol{K}$ which is not a dc-group such that $\boldsymbol{K} = \boldsymbol{G}_1 \circ \boldsymbol{H}_1$, and neither $\boldsymbol{G}_1$ nor $\boldsymbol{H}_1$ is isolated.

It is easy to verify that 4.5 and 4.6 imply the following assertion.

4.7. THEOREM. *Let $(G,+,\leq)$ be a partially ordered (directed) group, and let $\boldsymbol{G}=(G,+,C_{\leq})$ be a cyclically ordered group. If $(G,+,\leq)=A\circ B$, $\operatorname{card} A>1$, $\operatorname{card} B>1$, then $\boldsymbol{G}=\boldsymbol{A}\circ\boldsymbol{B}$ if and only if either $\boldsymbol{A}$ or $\boldsymbol{B}$ is isolated ($\boldsymbol{B}$ is isolated).*

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Received November 13, 1992
Revised April 2, 1993

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