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COMPLETE AND PSEUDOCOMPLETE COLOURINGS OF A GRAPH

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1. Introduction

A vertex colouring of a graph is called complete if it is regular and pseudocomplete (i. e. for any two different colours f and g the graph has two adjacent vertices with colours f and g). The maximal number of colours of a complete colouring of a graph G, called the achromatic number of G, has been studied in [5, 8, 9, 10, 11, 12]. The present paper is mainly devoted to the study of an analogous notion for edge colourings — the achromatic index (called also the line-achromatic number [5]) of a graph. There are considered colourings of infinite graphs and an interesting feature appears: the results for edge colourings are in general simpler than those for vertex colourings.

2. Complete colourings

Let a graph G (loops and multiple edges are admissible) and a set F of colours be given. By a *vertex* [an *edge*] *colouring* of G by F we mean a mapping φ of the vertex set $V(G)$ [edge set $E(G)$] of G into F. If x is a vertex [an edge] of G, then $\varphi(x)$ is called the *colour* of x under the colouring φ. Let s denote the number (the cardinality of the set) of the elements of F that are colours of a vertex [an edge] of G under φ. Then φ is called a vertex [an edge] *s-colouring* of G.

A vertex [an edge] colouring φ of G is said to be

(a) *regular* if any two adjacent vertices [edges] of G have different colours (two vertices [edges] are called *adjacent* if they are different and incident with at least one common edge [vertex]);

(b) *pseudocomplete* if for any two different colours f and g from the image of φ there exist in G two adjacent vertices [edges] with colours f and g;

(c) *complete* if it is regular and pseudocomplete.

A regular (but not pseudocomplete), a pseudocomplete (but not regular) and a complete edge colouring of a graph by $\{1, 2, 3, 4\}$ are shown in Fig. 1. The first two of them are 4-colourings, the third is a 3-colouring.

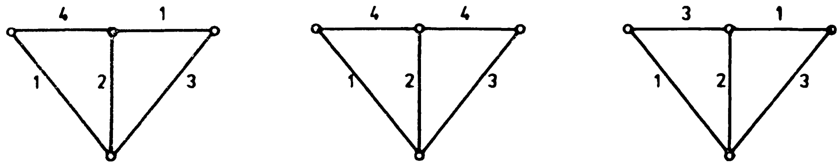

Fig. 1. A regular, a pseudocomplete and a complete edge colouring of a graph.

Fig. 2 shows three complete edge colourings of the octagon (a 2-colouring, a 3-colouring and a 4-colouring).

Note that these notions are closely related to special homomorphisms (cf., e. g., [10, Chapter 12]).

Let G be a graph. By the *derivative* (called also "line graph", "interchange graph", "derived graph" etc.) of G we mean the graph G' without loops or multiple edges whose vertex set is the edge set of G (i. e. $V(G')=E(G)$); two vertices x and y of G' are adjacent if and only if they are adjacent as edges of G.

Let a graph G and a cardinal number s be given. Our main aim is to find conditions for the existence of a complete vertex [edge] s-colouring of G.

The following lemma relates vertex and edge colourings.

Lemma 1. *Every regular [pseudocomplete, complete] edge s-colouring of a graph G is a regular [pseudocomplete, complete] vertex s-colouring of the derivative G′ of G*

Proof. The lemma immediately follows from the definitions of the derivative, vertex and edge colourings.

Thus the above examples provide examples of vertex colourings, too.

One of the aims of this paper is to show that Lemma 1 is not of much use. Thus we give first results for vertex colourings and then results for edge colourings which will be sometimes different.

We shall need several characteristics of a given graph G that are cardinal numbers: By $v(G)$ we denote the *order* (number of vertices) of G, by $e(G)$ the *size* (number of edges) of G by $d(G)$ the *degree* of G (i. e., the supremum of the

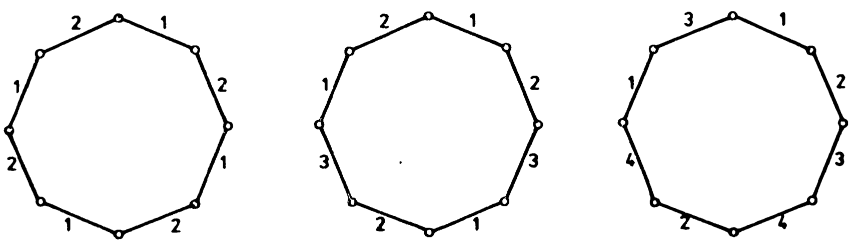

Fig 2. Complete edge colourings of the octagon.

set of degrees of vertices of G; the *degree of a vertex* is the (cardinal) number of the edges incident with v; here the loops are not counted twice) and by $c(G)$ the *dispersion* of G (the number of the components of G with at least two edges)

The subgraph of G formed by the components of G containing at least two edges will be denoted by $J(G)$. An edge of G is called *essential* if it is not *isolated*, i. e., if it is adjacent to at least one edge of G. Evidently, the subgraph of G generated by the set of essential edges of G is $J(G)$ and $c(G)=c(J(G))$.

Denote by C_G $[C'_G]$ the class of cardinal numbers s such that there exists a regular vertex [edge] s-colouring of G. Similarly, denote by P_G $[P'_G]$ the class of cardinal numbers s such that there is a pseudocomplete vertex [edge] s-colouring of G. Finally, denote by A_G $[A'_G]$ the class of cardinal numbers s such that there exists a complete vertex [edge] s-colouring of G.

Evidently, if there exists a vertex [edge] s-colouring of G, then $s \leq v(G)[s \leq e(G)]$.Therefore the classes C_G, P_G and A_G $[C'_G$, P'_G and $A'_G]$ are sets

Obviously, for every graph G we have:

(1) $$A_G \subseteq C_G, \qquad A'_G \subseteq C'_G,$$

(2) $$A_G \subseteq P_G, \qquad A'_G \subseteq P'_G,$$

(3) $$P_G \neq \emptyset, \qquad P'_G \neq \emptyset,$$

(4) $$C_G \neq \emptyset, \qquad C'_G \neq \emptyset.$$

Later (Proposition 1) we shall show that we always have

(5) $$A_G \neq \emptyset, \qquad A'_G \neq \emptyset$$

as well. From Lemma 1 it follows that

(6) $$P_{G'} = P'_G, \qquad C_{G'} = C'_G, \qquad A_{G'} = A'_G$$

for every graph G.

Let c be a vertex [an edge] colouring of G by F and let e be an equivalence on F. For $a \in F$ denote by $[a]_e$ the equivalence class containing a. Define a vertex [an edge] colouring c/e of G thus: for any vertex [edge] x of G put $(c/e)(x)=[c(x)]_e$. We shall call c/e the *quotient colouring.* Quotient colourings may be used for generating complete colourings from regular ones.

Lemma 2. *Let c be a regular vertex [edge] colouring of a graph G. Then there exists a quotient vertex [edge] colouring c/e of G which is complete.*

Proof. According to Lemma 1 it is sufficient to prove the assertion for vertex colourings. Let F be the set of colours. Define a reflexive symmetric relation r on F by

$(a, b) \in r$ if and only if $c(x)=a$, $c(y)=b$ for no adjacent vertices x, y of G.

Let e be a maximal equivalence on F which satisfies $e \subseteq r$ (such an equivalence exists by Zorn's lemma). By the definition of r, c/e is a regular colouring. Further,

c/e is a pseudocomplete colouring, because if $[a]_e \neq [b]_e$ and $(c/e)(x) \in [a]_e$, $(c/e)(y) \in [b]_e$ for no two adjacent vertices x, y, then

$$e' = e \cup ([a]_e \times [b]_e) \cup ([b]_e \times [a]_e)$$

is also an equivalence on F contained in r, which is a contradiction with the maximality of e.

Lemma 3. *Let H be a subgraph of G. Let c be a regular vertex [edge] r-colouring of G and let its restriction $c|_{V(H)}$ [$c|_{E(H)}$] be a complete vertex [edge] s-colouring of H. Then for every complete quotient t-colouring c/e of G we have: $s \leq t \leq r$.*

Proof is immediate.

Now we define six important characteristics of a given graph G:

The *chromatic number* $\chi(G)$ [*chromatic index* $\chi'(G)$] of G is the least number s of colours such that there exists a regular vertex [edge] s-colouring of G. (Instead of "chromatic index" also the terms "chromatic class", "edge chromatic number" and "line chromatic number" are used.)

The *pseudoachromatic number* $\psi(G)$ [*pseudoachromatic index* $\psi'(G)$] of G is the supremum of the set of cardinal numbers s for which there exists a pseudocomplete vertex [edge] s-colouring of G.

The *achromatic number* $\alpha(G)$ [*achromatic index* $\alpha'(G)$] is the supremum of the set of cardinal numbers s for which there exists a complete vertex [edge] s-colouring of G.

Briefly: $\chi(G) = \min C_G$, $\chi'(G) = \min C'_G$, $\psi(G) = \sup P_G$, $\psi'(G) = \sup P'_G$, $\alpha(G) = \sup A_G$, $\alpha'(G) = \sup A'_G$.

It will be seen that these are the only non-trivial extrema defined by the sets C_G, P_G, A_G, C'_G, P'_G and A'_G.

From (6) it follows that

(7) $$\chi'(G) = \chi(G'), \quad \psi'(G) = \psi(G'), \quad \alpha'(G) = \alpha(G').$$

Moreover, we evidently have

(8) $$\chi(G) \leq \alpha(G) \leq \psi(G) \leq v(G),$$

(9) $$d(G) \leq \chi'(G) \leq \alpha'(G) \leq \psi'(G) \leq e(G).$$

If H is a subgraph of G, then

(10) $$\begin{aligned} \chi(H) \leq \chi(G), \quad & \chi'(H) \leq \chi'(G), \\ \psi(H) \leq \psi(G), \quad & \psi'(H) \leq \psi'(G), \\ & \alpha'(H) \leq \alpha'(G). \end{aligned}$$

(The last inequality follows from Lemmas 2 and 3.)

Simple examples (e. g. the circuit G on 4 vertices and its subgraph H with 3 edges) show that the inequality

(11) $$\alpha(H) \leq \alpha(G)$$

does not hold in general. However, it follows from Lemmas 2 and 3 that if H is an induced subgraph of G, then (11) holds.

Later (Theorem 2) we shall show that for every graph G there exist max P_G [max P'_G] and max A_G [max A'_G]. Hence the pseudoachromatic number [index] of G can be defined also as the greatest number s of colours in a pseudocomplete vertex [edge] s-colouring of G and the achromatic number [index] of G as the greatest number s of colours in a complete vertex [edge] s-colouring of G.

The chromatic index — which is the edge analogy to the chromatic number — has been treated in many papers. A survey of main results can be found in [3, 6, 13, 16, 17]. The achromatic index (cf. [5]) is the edge analogy to the achromatic number studied in [5, 8, 9, 10, 11, 12]. The pseudoachromatic index is the analogy to the concept of the pseudoachromatic number introduced and studied in [9].

Example. It is easy to show that for the graph Q_3 of the cube we have $\psi'(Q_3) = \alpha'(Q_3) = 6$. A complete edge 6-colouring of Q_3 is given in Fig. 3.

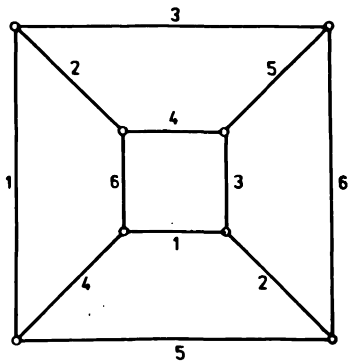

Fig. 3. A complete edge 6-colouring of the cube.

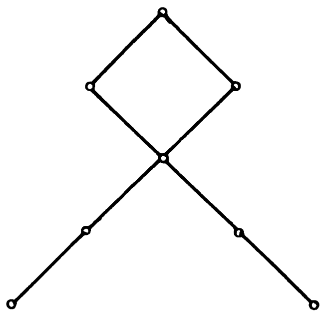

Fig. 4. A graph G with $\alpha'(G) \neq \psi'(G)$.

Obviously, if $\chi(G)$ [$\chi'(G)$] is finite, then every regular vertex $\chi(G)$-colouring [regular edge $\chi'(G)$-colouring] is complete. Thus, every graph G with a finite chromatic number [index] has a complete vertex [edge] colouring. This is true in general:

Proposition 1. *Every graph G has a complete vertex $\chi(G)$-colouring [complete edge $\chi'(G)$-colouring].*

Proof. Let c be a regular vertex $\chi(G)$-colouring [edge $\chi'(G)$-colouring]. By Lemma 2 there exists a quotient vertex [edge] colouring c/e of G that is complete. Evidently, c/e is a $\chi(G)$-colouring [$\chi'(G)$-colouring].

Proposition 2. *Let H be an induced subgraph of G. Then every complete vertex s-colouring of H can be extended into a complete vertex r-colouring of G such that*

$$(12)\qquad r \leqslant s + t,$$

where $t = \chi(G)$. If $t \geqslant 2$, then the estimate (12) *is sharp.*

Proof. Let c be a regular vertex t-colouring of G by A and let d be a complete vertex s-colouring of H by B (assume $A \cap B = \emptyset$). Define a regular vertex colouring g thus: $g|_{V(H)} = d$, $g|_{V(G)-V(H)} = c$. By Lemma 2 there exists a quotient vertex colouring g/e of G which is complete. Evidently, g/e is an extension of d and it is an r-colouring such that (12) holds.

Now we prove that (12) is sharp in the sense that for every s, t ($t \geqslant 2$) there exists a graph G with $\chi(G) = t$, an induced subgraph H of G and a complete vertex s-colouring d of H which cannot be extended into a complete r-colouring of G such that $r < s + t$.

Let S, T be sets such that $|S| = s$, $|T| = t$ and choose $x \notin S$. Put $S^* = S \cup \{x\}$. Let G be a graph without loops or multiple edges with $V(G) = S^* \times T$ such that two vertices (σ_1, τ_1), (σ_2, τ_2) are adjacent if and only if $\tau_1 \neq \tau_2$ and either $\sigma_1 \neq \sigma_2$, or $\sigma_1 = \sigma_2 = x$. Evidently, $\chi(G) = t$. Let H be a subgraph of G induced by the set $V(H) = S \times T$. A complete vertex s-colouring d of H is defined by $d(\sigma, \tau) = \sigma$. It is easy to check that d has the desired properties.

3. Vertex colourings

If s and r are cardinal numbers, denote by $\langle s, r \rangle$ the set of all cardinals t such that $s \leqslant t \leqslant r$. The cardinality of a set I will be denoted by $|I|$.

Here we shall determine C_G, P_G and A_G for any graph G (Theorem 2). We also prove that $\alpha(G)$ and $\psi(G)$ are in general independent (except $\alpha(G) \leqslant \psi(G)$; see Theorem 3) and we determine $\psi(G)$ provided that it is infinite (Theorem 1).

We need the notion of the *independence number* $\beta(G)$ of a graph G. Let G be a graph; denote by $I(G)$ the set of all independent subsets of $V(G)$. Put $\beta(G) = \sup\{|I| \mid I \in I(G)\}$. We prove first:

Lemma 4. *Let G be a graph. Then*

$$\beta(G') = \max\{|I| \mid I \in I(G')\}.$$

Proof. It suffices to consider the case when $\beta(G')$ is a limit cardinal. Let $X \subset V(G)$ satisfy $|X| < \beta(G')$. Consider the graph $G - X = (V(G) - X, E(G) - E)$, where E is the set of all edges of G incident with a vertex of X. Then $\beta((G - X)') = \beta(G')$ as there are at most $|X|$ independent edges which do not belong to $G - X$. Let $\sup\{\beta_\iota \mid \iota < \gamma\} = \beta(G')$ and let $\beta_\iota < \beta_\lambda$ whenever $\iota < \lambda$. Now we can construct by the induction on ι a family $\{I_\iota \mid \iota < \gamma\}$ that satisfies:

$$I_\iota \in I(G'),$$
$$|I_\iota| = \beta_\iota,$$
$$I_\iota \subset E(G - V_\iota),$$

where V_ι is the set of all vertices of G incident with an edge of $\cup\{I_\lambda | \lambda < \iota\}$. But then $\cup\{I'_\iota | \iota < \gamma\} = I$ is an independent set of edges of G of cardinality $\beta(G')$.

Remark. G' in Lemma 4 cannot be replaced by G. A simple counterexample is the complement of the disjoint union of the complete graphs of all orders $< \alpha$ for a limit cardinal α.

Theorem 1. *Let G be a graph with an infinite pseudoachromatic number $\psi(G)$. Then $\psi(G) = \beta(G')$.*

(In other words: *If the pseudoachromatic number is infinite, then it equals the maximal number of independent edges* — cf. Lemma 4.)

Proof. I. Suppose that $\psi(G) < \beta(G')$. By Lemma 4 there exist $\beta(G')$ independent edges in G. As $\psi(G)$ is infinite, $\beta(G')$ is infinite as well so that there exists in G a (vertex) pseudocomplete $\beta(G')$-colouring. It follows that $\beta(G') \leq \psi(G)$, a contradiction.

II. Suppose that $\psi(G) > \beta(G')$. Distinguish two cases:

a) $\psi(G) > \aleph_0$. Put $A = \max\{\beta(G'), \aleph_0\}$. As $A < \psi(G)$, there exists an infinite cardinal number s such that $A < s \leq \psi(G)$ and G has a pseudocomplete vertex s-colouring. Form s disjoint pairs of colours and choose for each of these pairs $\{f, g\}$ an edge joining vertices with colours f and g. We get s independent edges, which is impossible, since $s > A \geq \beta(G')$.

b) $\psi(G) = \aleph_0$. Then $\beta(G')$ is finite. Evidently there exists a natural number s such that $2(\beta(G') + 1) < s \leq \aleph_0$ and G has a pseudocomplete vertex s-colouring. Form $\beta(G') + 1$ disjoint pairs of colours and choose for each of them, say, $\{f, g\}$, an edge joining vertices with colours f and g. We get $\beta(G') + 1$ independent edges, which is impossible.

As I and II have led to contradictions, we have $\psi(G) = \beta(G')$.

Theorem 2. *For any graph G we have:*

$$C_G = \langle \chi(G), v(G) \rangle. \tag{13}$$

$$P_G = \begin{cases} \{0\}, \textit{ if } v(G) = 0; \\ \langle 1, \psi(G) \rangle, \textit{ if } v(G) \neq 0. \end{cases} \tag{14}$$

$$A_G = \langle \chi(G), \alpha(G) \rangle. \tag{15}$$

Proof. (13) Let S be a set of cardinality $|S| = s$, $s \in \langle \chi(G), v(G) \rangle$ and let A and B be disjoint subsets of S such that $A \cup B = S$, $|A| = \chi(G)$. Let f be a regular vertex $\chi(G)$-colouring of G with the set A of colours. Obviously there exist injections

$$g\colon A \to V(G),$$
$$h\colon B \to V(G) - g(A).$$

Then the mapping i given by

$$i(v) = \begin{cases} h^{-1}(v) \text{ if } v \in h(B), \\ f(v) \text{ if } v \in V(G) - h(B) \end{cases}$$

is a regular vertex s-colouring of G.

(14) As the case $v(G) = 0$ is trivial, suppose $v(G) \neq 0$. Let $1 \leqslant s < t \leqslant \psi(G)$ and let there exist a pseudocomplete vertex t-colouring of G. Then identifying an appropriate number of colours we get easily a pseudocomplete vertex s-colouring of G. It remains to prove $\psi(G) \in P_G$. If $\psi(G)$ is finite, this is evident. However, if $\psi(G)$ is infinite, according to Theorem 1 we have $\psi(G) = \beta(G')$, so that there exist $\psi(G)$ independent edges in G and it is easy to find a pseudocomplete vertex $\psi(G)$-colouring of G.

(15) From the definitions of $\chi(G)$ and $\alpha(G)$ it follows that $A_G \subseteq \langle \chi(G), \alpha(G) \rangle$ and $\chi(G) \in A_G$. Suppose that $\chi(G) < s \leqslant \alpha(G)$. To prove that G has a complete vertex s-colouring (and thus to generalize the Homomorphism Interpolation Theorem; see [10, 12] and in more general form [4]), we distinguish three cases.

(A) $s = \alpha(G)$. We may assume that $\alpha(G)$ is a limit cardinal. Let $\sup \{\alpha_\iota \mid \iota < \gamma\} = \alpha(G)$ and suppose $\alpha_\iota < \alpha_\lambda$ for $\iota < \lambda$. Now we can construct by the induction a family of graphs $\{G_\iota \mid \iota < \gamma\}$ such that

1° G_ι is a subgraph of G, $\alpha(G_\iota) \geqslant \alpha_\iota$ for $\iota < \gamma$;
2° $V(G_\iota) \cap V(G_\lambda) = \emptyset$ for $\iota \neq \lambda$;
3° $\alpha(G - \cup(V(G_\lambda) \mid \lambda < \iota)) = \alpha(G)$ for $\iota < \gamma$.

Let $c_\iota\colon V(G_\iota) \to A_\iota$ be a complete vertex β_ι-colouring, where $\beta_\iota \geqslant \alpha_\iota$; assume $A_\iota \cap A_\lambda = \emptyset$ for $\iota \neq \lambda$. Then

$$d\colon \quad \cup(V(G_\iota) \mid \iota < \gamma) \to \cup(A_\iota \mid \iota < \gamma)$$

is a regular β-colouring, where $\beta \geqslant \sup \beta_\iota \geqslant \sup \alpha_\iota = \alpha(G)$ so that $\beta = \alpha(G)$. Let c be any regular vertex colouring of G that is an extension of d. Then by Lemma 2 there exists a quotient vertex colouring c/e that is complete. But c/e has to be an $\alpha(G)$-colouring by Lemma 3.

(B) $\chi(G) < s < \alpha(G)$, where s is finite. We use the method of M. Boguszak, S. Poljak and J. Tůma [4]. Evidently there is a complete vertex s'-colouring τ of G for some s', where $s < s' \leqslant \alpha(G)$. Choose s colours $c_1, c_2, \ldots, c_s$ from these s' colours and for every pair (c_i, c_j) of these colours $(i \neq j)$ choose in G an edge whose endvertices have colours c_i and c_j. Let M be the set of all endvertices of these edges and let H be a subgraph of G induced by M. Obviously there is a complete vertex s-colouring of H. Let e be the equivalence on $V(G)$ defined thus $(u, v \in V(G))$:

$$(u, v) \in e \quad \text{if and only if} \quad u = v \quad \text{or}$$
$$u, v \in V(H), \tau(u) = \tau(v).$$

Evidently $\chi(G/e) \geq s$. However, the graph G/e can be obtained from G by a finite number of identifications of two vertices. Thus we get a finite sequence of graphs

$$P = \{G = G_0, G_1, G_2, G_3, \ldots, G/e = G_n\}.$$

As with each identification the chromatic number increases by one or remains unaltered, in P there is a graph G_k with chromatic number $\chi(G_k) = s$. Let G_k arise from G by an equivalence e', i. e. $G_k = G/e'$. Colour G by s colours $c_1, c_2, \ldots, c_s$ as follows. If a vertex v belongs to the class of e' whose vertices correspond to a vertex of G_k coloured by c_i, we colour v by c_i. Evidently we obtain a complete vertex s-colouring of G.

(C) $\chi(G) < s < \alpha(G)$, where s is infinite. As in (B), there is a complete vertex s'-colouring of G for some s', where $s < s' \leq \alpha(G)$. Let H be a subgraph of G induced by the vertices coloured by some of fixed s colours. According to Proposition 2 the complete vertex s-colouring of H can be extended into a complete vertex r-colouring c of G such that $r \leq s + \chi(G)$. By Lemma 2 there is an equivalence e such that c/e is a complete vertex colouring of G. By Lemma 3 c/e is a t-colouring, where $s \leq t \leq r \leq s + \chi(G) = s$, because s is infinite and $\chi(G) < s$. Therefore $t = s$ and c/e is a complete vertex s-colouring of G.

Theorem 3. *Let s and t be cardinal numbers such that $2 \leq s \leq t$. Then there exists a graph G with $\alpha(G) = s$ and $\psi(G) = t$.*

Proof. Let G be a complete s-partite graph. If t is infinite, suppose that every part of G has t vertices. If t is finite, let two parts of G have $t - s + 1$ vertices each and each of the remaining $s - 2$ parts of G consist of one vertex. In both cases it is easy to check that $\alpha(G) = s$ and $\psi(G) = t$.

Remarks. 1. In Theorem 3 we must suppose $s \geq 2$, because if $\alpha(G) < 2$, then $\psi(G) = \alpha(G)$.

2. A (vertex) colouring of a graph G can be considered as a special case of a partition of a set, namely of the vertex set of G. In this way many notions and results concerning the graph colourings including the Homomorphism Interpolation Theorem may be generalized [1, 4, 7, 15].

4. Edge colourings

Theorem 4. *For every graph G we have:*

$$C'_G = \langle \chi'(G), e(G) \rangle;$$

$$P'_G = \begin{cases} \{0\} \text{ if } e(G)=0, \\ \langle 1, \psi'(G)\rangle \text{ if } e(G)\neq 0; \end{cases}$$

$$A'_G = \langle \chi'(G), \alpha'(G)\rangle .$$

Proof. These results easily follow from (6), Theorem 2, (7) and the fact that $v(G')=e(G)$.

To make Theorem 4 more applicable, we need some results concerning $\chi'(G)$, $\psi'(G)$ and $\alpha'(G)$ — see Propositions 3 and 4 below.

Proposition 3. *Let G be a graph with degree $d(G)$ and chromatic index $\chi'(G)$. If $d(G)$ is finite, then*

(16) $$d(G)\leq \chi'(G)\leq [\tfrac{3}{2}d(G)].$$

If $d(G)$ is infinite, then

(17) $$\chi'(G)=d(G).$$

Proof. (16) has been proved in [6, Theorem 3]; for finite graphs also in [3, 14, 16, 17]. (17) has been established in [6, Theorem 1]. Evidently, the fact that in [6] loops were excluded, is not essential.

Lemma 5. *Let G be a connected graph with a finite chromatic index $\chi'(G)$ and an infinite size (number of edges). Let $\chi'(G)\leq s\leq \aleph_0$. Then there exists a complete edge s-colouring of G.*

Proof. There exists a complete edge $\chi'(G)$-colouring φ of G. Suppose that the colours are 1, 2, 3, ..., $\chi'(G)\geq 2$. Choose to every pair of colours f and f^* two adjacent edges e and e^* such that $\varphi(e)=f$, $\varphi(e^*)=f^*$. As $\chi'(G)$ is finite, all the edges chosen in such a way generate a finite subgraph H of G. According to Proposition 3 $d(G)$ is also finite. As G is connected, there exists an infinite path in G. But H is a finite graph, therefore in $G-H$ there is also an infinite path $(e_1, e_2, e_3, \ldots)$. Change the colours of the edges $e_1, e_2, e_3, \ldots$ into the colours $f_1=1, f_2=2, f_3=1, f_4=3, f_5=2, f_6=3, f_7=1, f_8=4, f_9=2, f_{10}=4, f_{11}=3, f_{12}=4, f_{13}=1, f_{14}=5$, ... in such a way that the edges $e_1, e_2, e_3, \ldots$ are successively coloured by the colours $f_1, f_2, f_3, \ldots$; however if in colouring an edge e such a colour f_i should be used that the regularity of the colouring is destroyed, then delete f_i from the sequence and colour e by the next colour f_j $(j>i)$ that does not spoil the regularity of colouring. If s is finite, then the sequence $(f_1, f_2, f_3, \ldots)$ ends with the colours $s-1$ and s. After exhausting the members of the sequence (possibly with deleting some members) the process of changing the colours ends and the colours of the other edges of the path $(e_1, e_2, e_3, \ldots)$ remain unaltered. If $s=\aleph_0$, then the sequence of colours $(f_1, f_2, f_3, \ldots)$ is infinite and (possibly after deleting some members) by it all the edges of the path $(e_1, e_2, e_3, \ldots)$ can be coloured. It is evident that a complete edge s-colouring of G will arise.

Proposition 4. *Let G be a graph with the dispersion $c(G)$, degree $d(G)$ and size $e(G)$. Then we have:*

(i) *If G has no essential edges, then*

$$\alpha'(G)=\psi'(G)=\begin{cases}0 \text{ if } e(G)=0,\\ 1 \text{ if } e(G)\neq 0.\end{cases}$$

(ii) *If G has a non-zero finite number of essential edges, then $J(G)$ is a finite graph and*

$$\alpha'(G)=\alpha'(J(G)),\quad \psi'(G)=\psi'(J(G)).$$

(iii) *If G has an infinite number of essential edges, then*

$$\alpha'(G)=\psi'(G)=\max\{c(G), d(G), \aleph_0\}.$$

Proof. (i) and (ii) are trivial. Therefore we prove only (iii).

If either $c(G)$ or $d(G)$ is infinite, then

$$\max\{c(G), d(G), \aleph_0\}=c(G)d(G).$$

As there are at most $c(G)d(G)$ pairs of adjacent vertices it suffices to show that there exists a complete $c(G)d(G)$-colouring of G. However, this fact for $c(G)\leqslant d(G)$ follows from Proposition 3 and for $c(G)>d(G)$ it is evident.

If both $c(G)$ and $d(G)$ are finite, we may apply Proposition 3 and Lemma 5 and we obtain that there exists a complete edge $\aleph_0$-colouring of G. As $e(G)=\aleph_0$, we have

$$\alpha'(G)=\psi'(G)=\aleph_0.$$

Theorem 5. *Let G be a graph with an infinite degree $d(G)$. A complete edge s-colouring of G exists if and only if*

$$d(G)\leqslant s\leqslant\max\{c(G), d(G)\},$$

where $c(G)$ is the dispersion of G.

Proof. If $d(G)$ is infinite, then G has an infinite number of essential edges and the result follows from Theorem 4, Proposition 3 and Proposition 4.

Corollary. *A complete edge s-colouring of a connected graph with an infinite degree $d(G)$ exists if and only if $s=d(G)$.*

Proof. Put $c(G)=1$ in Theorem 5.

5. Results concerning finite graphs

Theorem 4 reduces the problem of existence of a complete, a pseudocomplete, and a regular edge s-colouring of a graph G to determining the characteristics

$e(G)$, $\chi'(G)$, $\psi'(G)$ and $\alpha'(G)$. Proposition 4 for every graph G either determines $\psi'(G)$ and $\alpha'(G)$ or reduces the problem of finding them to the case of finite graphs. Therefore in the sequel we shall study the achromatic index and the pseudoachromatic index of a finite graph. The chromatic index of a finite graph has been studied in [2, 3, 16, 17] and many other works; the case of infinite graphs has been treated in [6].

Basic properties of $\alpha(G)$, $\alpha'(G)$, $\psi(G)$ and $\psi'(G)$ for finite graphs G (especially, for paths P_m and circuits C_m on m vertices) were presented by the first author at the summer school on Combinatorial Structures and Graph Theory in Zlatá Idka (Czechoslovakia) in May 1971 and at the Seminar on Discrete Mathematics in Odessa (USSR) in September 1972. Meanwhile a considerable part of these results has been independently discovered by F. Bories and J. L. Jolivet [5] and by D. Geller and H. Kronk [8]. Therefore we mention them only briefly.

Theorem 6. *Let m be a positive integer. Put*

$$A=\left[\sqrt{\frac{m}{2}}\right].$$

Then we have:

$$\alpha'(C_m)=\alpha(C_m)=\begin{cases}2A \text{ if } m<2A^2+A \text{ or}\\ \text{if } m=2A^2+A+1>1;\\ 2A+1, \text{ otherwise.}\end{cases} \tag{18}$$

$$\psi'(C_m)=\psi(C_m)=\begin{cases}2A & \text{if } m<2A^2+A;\\ 2A+1 & \text{if } m\geqslant 2A^2+A.\end{cases} \tag{19}$$

$$\psi'(P_{m+1})=\psi(P_m)=\alpha'(P_{m+1})=\alpha(P_m)= \tag{20}$$

$$=\begin{cases}2A & \text{if } m\leqslant 2A^2+A,\\ 2A+1 & \text{if } m>2A^2+A.\end{cases}$$

Proof. The results (18) and (20) concerning α and α' were (in an other form) proved in [5] and [8]. It is easy to check that all the proofs can be used for ψ and ψ' as well and we get the same results except the case $m=2A^2+A+1$, when $\psi'(C_m)=\psi(C_m)=2A+1$. The theorem follows.

Let G be a finite graph containing exactly $z(G)$ (unordered) pairs of adjacent edges. The condition

$$\binom{\psi'(G)}{2}\leqslant z(G)$$

implies that

(21) $$\alpha'(G) \leqslant \psi'(G) \leqslant [\tfrac{1}{2}(1+\sqrt{8z(G)+1})].$$

The estimate (21) is attained, e. g., for any star (tree of diameter 1 or 2). From (21) it is possible to deduce further conditions, for instance, for any finite graph G with degree $d(G)$ and order $v(G)$ we have:

(22) $$\alpha'(G) \leqslant \psi'(G) \leqslant [\tfrac{1}{2}(1+\sqrt{4v(G)d(G)(d(G)-1)+1})].$$

However, many basic problems concerning the achromatic [pseudoachromatic] index of a graph remain open. We state only two of them.

Problem 1. Determine the achromatic [pseudoachromatic] index of the finite complete graph of a given order (cf. [5]).

It is easy to find these values for "small" n:

n	1	2	3	4	5	6	7
$\alpha'(K_n)$	0	1	3	3	7	8	11
$\psi'(K_n)$	0	1	3	4	7	8	11

It may be interesting to study graphs G such that $\alpha'(G) \neq \psi'(G)$. From Proposition 4 it follows that if such a graph G has no isolated vertices or edges, then G must be finite. Simple examples are three 4-vertex graphs: the graph of Fig. 1, the circuit C_4 and the complete graph K_4. The graph of Fig. 4 has achromatic index 4 and pseudoachromatic index 5. The circuit C_{2A^2+A+1} ($A=1, 2, 3, \ldots$) has achromatic index $2A$ and pseudoachromatic index $2A+1$ (see Theorem 6).

Problem 2. Find all ordered pairs (s, t) of cardinal numbers such that there exists a graph G with $\alpha'(G)=s$ and $\psi'(G)=t$.

Theorem 3 and Remark 1 show that the corresponding ordered pairs for vertex colourings are (0, 0), (1, 1) and all (s, t) with $2 \leqslant s \leqslant t$.

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ПОЛНЫЕ И ПСЕВДОПОЛНЫЕ РАСКРАСКИ ГРАФА

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Резюме

Раскраска вершин [ребер] графа называетвя псевдополной, если для любых двух различных цветов в графе найдутся две смежные вершины [ребра] окрашенные в эти цвета. Раскраска называется полной, если она псевдополна и правильна. Проблема, при каких условиях граф имеет полную (псевдополную) раскраску вершин [ребер] с данным количеством цветов ведет к изучению ахроматического числа [класса] графа, то есть, максимального количества цветов таких раскрасок. Показано, что эти инварианты графа всегда определены (включая случай бесконечных графов) и что справедливы результаты аналогичны известной теореме об интерполяции гомоморфизмов. Кроме того, для любых двух кардинальных чисел s и t таких, что $2 \leq s \leq t$ существует граф с ахроматическим числом s и псевдоахроматическим числом t. Если псевдоахроматическое число графа бесконечно, то оно равно максимальному числу его независимых ребер. Для раскрасок ребер выводятся более простые результаты. Последняя часть статьи посвящена случаю конечных графов.