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ON A NECESSARY CONDITION IN THE CALCULUS OF VARIATIONS IN ORLICZ-SOBOLEV SPACES

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ABSTRACT. In this paper, we prove an approximation theorem in Orlicz Sobolev spaces and we give an application of this approximation result to a necessary condition in the calculus of variations.

1. Introduction

On a bounded domain Ω of $\mathbb{R}^N$, we consider functionals of the kind

$$J(u) = \int_\Omega f(x, u, \nabla u)\, \mathrm{d}x ,$$

for functions u in some Orlicz-Sobolev spaces $W^1L_M(\Omega)$ corresponding to an N-function M. In the L^p case (when $M(t) = \frac{|t|^p}{p}$) the search of sufficient conditions to secure that those functionals attain an extreme value has a long history (see [1]). The most important problem is to verify the weak lower semicontinuity of those functionals with respect to the space involved. Usually this involves hypothesis that the integrand f is convex with respect to the gradient. More recently R. Landes in [5] has studied the reverse problem at a fixed level set and in many situations he has showed that if J is weakly lower semicontinuous at one fixed (nonvoid) level set, then this particular level set is an extreme value of J or the defining function f is convex in the gradient. The above statement for f as function of u (or of x and u) is not hard to prove (see [5]) but when $f = f(x, \nabla u)$ or $f = f(x, u, \nabla u)$ this is due to an approximation result in Sobolev spaces.

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Since this approximation is important for possible application in nonlinear partial differential equations and in calculus of variations, the main purpose in the first part of this paper is to give the same approximation in the more general settings of the Orlicz-Sobolev space $W^1L_M(\Omega)$ and the second part of this article is devoted to the application of this approximation in the calculus of variations, however, we prove when $f = f(x, \nabla u)$ that if J is weakly lower semicontinuous at one fixed level set in the space $W^1L_M(\Omega)$, then this particular level set is an extreme value of J or the function f is convex with respect to the gradient.

2. Preliminaries

In this section we list briefly some definitions and well-known about, N-functions and Orlicz-Sobolev spaces. Standard references are [3], [4], [7].

2.1. Let $M: \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation:

$$M(t) = \int_0^t m(\tau)\, \mathrm{d}\tau\,,$$

where $m: \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by

$$\overline{M}(t) = \int_0^t \overline{m}(\tau)\, \mathrm{d}\tau\,,$$

where $\overline{m}: \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{m}(t) = \sup\{s : \ m(s) \le t\}$. Clearly $\overline{\overline{M}} = M$ and one has Young's inequality

$$t \cdot s \le M(t) + \overline{M}(s) \qquad \text{for all} \quad s, t \ge 0\,.$$

It is well known that we can assume that m and $\overline{m}$ are continuous and strictly increasing. We will extend the N-functions into even functions on all $\mathbb{R}$.

The N-function M is said to satisfy the Δ_2-condition everywhere (resp. near infinity) if there exist $k > 0$ (resp. $t_0 > 0$) such that

$$M(2t) \le kM(t)$$

for all $t \ge 0$ (resp. $t \ge t_0$).

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M(|u(x)|)\,\mathrm{d}x < +\infty$$

$$(\text{resp. } \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)\mathrm{d}x < +\infty \quad \text{for some } \lambda > 0).$$

$L_M(\Omega)$ is a Banach space under the norm:

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)\mathrm{d}x \le 1\right\}.$$

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)\,\mathrm{d}x$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2-condition for all t or for t large, according to whether Ω has infinite measure or not.

2.3. We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm:

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega}\,.$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we have the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$, and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $L_M(\Omega)$, (denoted by $u_n \to u$ (mod) in $L_M(\Omega)$) if for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{|u_n - u|}{\lambda}\right)\mathrm{d}x \to 0 \qquad \text{as} \quad n \to \infty\,.$$

If M satisfies the Δ_2 condition (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence (see [3]).

3. Approximation result

THEOREM 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let M be an N-function. If u is a function in $W^1L_M(\Omega)$, then for almost every $x_0 \in \Omega$, there is a ball $B(x_0,\alpha)$, $\alpha > 0$, a constant $C(\alpha, x_0)$ and a function $u_\alpha \in W^1L_M(\Omega)$ satisfying:*

i) $u_\alpha \to u$ (mod) *in* $W^1L_M(\Omega)$ *as* $\alpha \to 0$;

ii) $u_\alpha \equiv C(\alpha, x_0)$ *in* $B(x_0,\alpha)$.

Proof. Let Ψ_α be a C_0^∞ cut-off function with support in $B(0,2\alpha)$ such that
$$\Psi_\alpha \equiv 1 \quad \text{in } B(0,\alpha) \qquad \text{and} \qquad |\nabla\Psi_\alpha| \le \frac{2}{\alpha},$$
and let x_0 be a Lebesgue point of the function u in Ω, hence we can take $C(\alpha,x_0) = u(x_0)$. We define in Ω the function u_α by
$$u_\alpha(x) = u(x)\big(1 - \Psi_\alpha(x - x_0)\big) + u(x_0)\Psi_\alpha(x - x_0). \tag{3.1}$$
First, by virtue of Lebesgue theorem, we certainly have
$$u_\alpha \to u \quad (\text{mod}) \ \text{in } L_M(\Omega) \qquad \text{as} \quad \alpha \to 0. \tag{3.2}$$
Next, we will prove that
$$\frac{\partial u_{\alpha_k}}{\partial x_i} \to \frac{\partial u}{\partial x_i} \ (\text{mod}) \ \text{in } L_M(\Omega) \qquad \text{for all} \quad 1 \le i \le N. \tag{3.3}$$
for a sequence α_k with $\alpha_k \to 0$ as $k \to \infty$. For that, we have
$$\frac{\partial}{\partial x_i}\big(u(x) - u_\alpha(x)\big) = \frac{\partial u(x)}{\partial x_i}\Psi_\alpha(x - x_0) + \frac{\partial}{\partial x_i}\Psi_\alpha(x - x_0)\big(u(x) - u(x_0)\big)$$
and the convexity of M allows to have
$$\begin{aligned}&\int_\Omega M\left(\left|\lambda\frac{\partial}{\partial x_i}\big(u(x) - u_\alpha(x)\big)\right|\right)\,dx\\ &\le \frac{1}{2}\int_\Omega M\left(2\lambda\left|\frac{\partial u(x)}{\partial x_i}\Psi_\alpha(x - x_0)\right|\right)\,dx\\ &\qquad + \frac{1}{2}\int_\Omega M\left(2\lambda\left|\frac{\partial}{\partial x_i}\Psi_\alpha(x - x_0)\big(u(x) - u(x_0)\big)\right|\right)\,dx.\end{aligned}$$
By using the Lebesgue theorem, the first term of the right hand-side in the last inequality converges to zero as $\alpha \to 0$, then it remains to show that
$$\int_\Omega M\left(2\lambda\left|\frac{\partial}{\partial x_i}\Psi_\alpha(x - x_0)\big(u(x) - u(x_0)\big)\right|\right)\,dx \to 0 \qquad \text{as} \quad \alpha \to 0, \tag{3.4}$$
which is a direct deduction of the following lemma:

Lemma 2. *For almost every $x_0 \in \Omega$ there exists a sequence α_k with $\alpha_k \to 0$ as $k \to \infty$ such that*

$$\int\limits_{B(x_0,2\alpha_k)} M\left(\frac{\lambda\,|u(x)-u(x_0)|}{\alpha_k}\right) \mathrm{d}x \to 0 \qquad \textit{as} \quad k \to \infty\,.$$

Proof of Lemma 2. Let $x_0 \in \Omega$. For each $t > 0$, we define the set

$$\Omega_t = \{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) > t\}\,.$$

Let $\alpha_0 > 0$. For $\alpha < \alpha_0$, we consider the function $\Phi_\alpha \colon \Omega_{2\alpha_0} \to \mathbb{R}$ defined by

$$\Phi_\alpha(y) = \int\limits_{B(y,2\alpha)} M\left(\frac{\lambda\,|u(x)-u(y)|}{\alpha}\right) \mathrm{d}x\,.$$

Since $\Phi_\alpha(y) = \int\limits_\Omega M\left(\lambda\frac{|u(x)-u(y)|}{\alpha}\right)\cdot\chi_{B(0,2\alpha)}\,\mathrm{d}x$, the function $\Phi_\alpha \colon \Omega_{2\alpha_0} \to \mathbb{R}$ is measurable; χ_E, as usual, denotes the characteristic function of the set E. For all $\alpha_0 > 0$, we shall show that

$$|\Phi_\alpha(y)| \to 0 \quad \text{in } L^1(\Omega_{2\alpha_0}) \qquad \text{as} \quad \alpha \to 0\,,\ \alpha < \alpha_0\,. \tag{3.5}$$

This obviously implies the statement of Lemma 2. To verify (3.5), we denote $u_\delta = u * \varphi_\delta$ the mollification of u, where $\varphi_\delta \in \mathcal{D}(\mathbb{R}^N)$, $\varphi_\delta \equiv 1$ for $|x| \geq \delta$, $\varphi_\delta \geq 0$ and $\int\limits_{\mathbb{R}^N} \varphi_\delta(x)\,\mathrm{d}x = 1$.

Hence, φ_δ is well defined in $\Omega_{2\alpha_0}$ for $\delta < \alpha_0$ and we have

$$\begin{aligned}
\int\limits_{\Omega_{2\alpha_0}} |\Phi_\alpha(y)|\,\mathrm{d}y &= \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(y,2\alpha)} M\left(\frac{\lambda\,|u(x)-u(y)|}{\alpha}\right) \mathrm{d}x\,\mathrm{d}y \\
&\leq \lim_{\delta\to 0} \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(0,2\alpha)} M\left(\frac{\lambda\,|u_\delta(y-x)-u_\delta(y)|}{\alpha}\right) \mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Since u_δ is continuously differentiable, we may estimate

$$I_\alpha = \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(0,2\alpha)} M\left(\frac{\lambda\,|u_\delta(y-x)-u_\delta(y)|}{\alpha}\right) \mathrm{d}x\,\mathrm{d}y\,.$$

In fact, we have

$$\begin{aligned}
I_\alpha &\leq \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(0,2\alpha)} M\left(\lambda \int\limits_0^1 \frac{|\nabla u_\delta(y-tx)|\,|x|}{\alpha}\,\mathrm{d}t\right) \mathrm{d}x\,\mathrm{d}y \\
&\leq \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(0,2\alpha)} M\left(\lambda \int\limits_0^1 2\,|\nabla u_\delta(y-tx)|\,\mathrm{d}t\right) \mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Then, it follows by Jensen's inequality that

$$
\begin{aligned}
I_\alpha &\leq \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \int_0^1 M(2\lambda\, |\nabla u_\delta(y - tx)|)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y \\
&\underset{(*)}{=} \int_0^1 \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} M\left(\lambda \left| \int_{B(0,\delta)} 2\nabla u(y - tx - z)\varphi_\delta(z)\, \mathrm{d}z \right|\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \\
&\leq K_2 \int_0^1 \int_{B(0,2\alpha)} \int_{B(0,\delta)} \left(\int_{\Omega_{2\alpha_0}} M(\lambda K_1\, |\nabla u(y - tx - z)|)\, \mathrm{d}y \right) \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}t \\
&= K_2 \int_0^1 \int_{B(0,2\alpha)} \int_{B(0,\delta)} \left\| M(\lambda K_1\, |\nabla u|) \right\|_1 \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}t \\
&\leq K_3 \left\| M(\lambda K_1\, |\nabla u|) \right\|_1 \left(\frac{\sigma_N}{N}\right)^2 (2\alpha)^N
\end{aligned}
$$

for some positive constants K_1, K_2 and K_3 (δ_N denotes the measure of the unit sphere in $\mathbb{R}^N$). So, we obtain

$$
I_\alpha \to 0 \qquad \text{as} \quad \alpha \to 0 .
$$

It then follows for $\alpha_0 > 0$ that

$$
\int_{\Omega_{2\alpha_0}} |\Phi_\alpha(y)|\, \mathrm{d}y \to 0 \qquad \text{as} \quad \alpha \to 0 , \quad \alpha < \alpha_0 ,
$$

which allows to conclude that for almost every $x_0 \in \Omega$, we have

$$
\Phi_{\alpha_k}(x_0) \to 0 \qquad \text{as} \quad k \to \infty ,
$$

for a subsequence α_k with $\alpha_k \to 0$ as $k \to \infty$. To justify $(*)$ we recall that in $\Omega_{2\alpha_0}$ the differentiation and the mollification commute for $\delta < \alpha_0$ which proves the statement of Lemma 2. □

Remark 3. In the particular case when $M(t) = \frac{|t|^p}{p}$, $1 < p < \infty$, we obtain the statement of [5; Lemma 2.1] (with $C(x_0) = u(x_0)$).

4. Functional depending on x and ∇u

On a bounded domain $\Omega \subset \mathbb{R}^N$, we consider functional of the kind

$$J(u) = \int_{\Omega} f\big(x, \nabla u(x)\big)\, \mathrm{d}x\,, \tag{4.1}$$

where $J\colon W^1L_M(\Omega) \to \mathbb{R}$ is continuous and where $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x,\xi)| \le l(x)k\big(|\xi|\big) \tag{4.2}$$

for some nondecreasing function $k\colon \mathbb{R} \to \mathbb{R}$ and some $l(x) \in L^1(\Omega)$. For each real μ, we write N_μ for the level set of the functional J, i.e.

$$N_\mu = \{u \in W^1L_M(\Omega):\ J(u) = \mu\}$$

and $\overline{N}_\mu^{\omega}$ for the closure of N_μ in $W^1L_M(\Omega)$ for the weak topology $\big(\Pi L_M, \Pi E_{\overline{M}}\big)$.

Definition 4. A functional $J\colon W^1L_M(\Omega) \to \mathbb{R}$ is called *weakly lower semicontinuous at a level set* N_μ if $J(u) \le \mu$ for all $u \in \overline{N}_\mu^{\omega}$.

Remark 5. Note that this definition does not imply that $J|_{\overline{N}_\mu^{\omega}}$ is weakly lower semicontinuous.

Theorem 6. *Let $J\colon W^1L_M(\Omega) \to \mathbb{R}$ be a continuous functional defined as in* (4.1) *with the Carathéodory function f satisfying* (4.2). *If J is weakly lower semicontinuous at a (non-empty) level set N_μ and if μ is not an extreme value of J, then $f(x,\xi)$ is convex in ξ for almost all $x \in \Omega$.*

Proof. Let us assume that the real μ is not an extreme value of J, then we shall show that

$$f\big(x, \lambda\xi + (1-\lambda)\xi^*\big) \le \lambda f(x,\xi) + (1-\lambda) f(x,\xi^*)$$

for all $\lambda \in [0,1]$, all ξ, $\xi^* \in \mathbb{R}^N$ and for a.e. $x \in \Omega$. We can assume that $\mu = 0$ and that in $W^1L_M(\Omega)$ there are two functions $\hat{a}_1$ and $\hat{a}_2$, say, such that

$$J(\hat{a}_1) < -\varepsilon_0 \qquad \text{and} \qquad J(\hat{a}_2) > \varepsilon_0$$

for some $\varepsilon_0 > 0$. Let x_0 be a Lebesgue point of $f(x,\xi)$ for all $\xi \in \mathbb{Q}^N$. We can assume that $x_0 = 0$. Using the continuity of the functional J and Theorem 1, there is a ball $B(0,R_0) \subset \Omega$ and there are functions $\bar{b}$, $\bar{b}_1$ and $\bar{b}_2$ (see [5]) such that

$$\nabla\bar{b} = \nabla\bar{b}_1 = \nabla\bar{b}_2 \equiv 0 \qquad \text{on} \quad B(0,R_0)\,, \tag{4.3}$$

$$J(\bar{b}_1) < \tfrac{7}{8}\varepsilon_0\,, \quad J(\bar{b}_2) > \tfrac{7}{8}\varepsilon_0 \quad \text{and} \quad \big|J(\bar{b})\big| < \tfrac{1}{8}\varepsilon_0\,. \tag{4.4}$$

Furthermore, for all function $\overline{a}$ satisfying $|J(\overline{a})| < \frac{7}{8}\varepsilon_0$, there is a number $t_i \in [0,1]$ with $i = i(\overline{a}) \in \{1,2\}$ such that the function $\overline{c} = \overline{a} + t_i(\overline{b}_i - \overline{a})$ lies in the level set N_0, i.e.

$$J(\overline{c}) = \mu = 0 .$$

Let us now fix $\lambda \in [0,1] \cap \mathbb{Q}$ and $\xi, \xi^* \in \mathbb{Q}^N$. We define the sequence of functions

$$\widehat{c}_n(x) = \langle \xi^*, x\rangle + \int_0^{\langle \xi - \xi^*, x\rangle} g_\lambda(nt)\, \mathrm{d}t ,$$

where $\langle\, ,\, \rangle$ denotes the usual inner product and where

$$g_\lambda(t) = \begin{cases} 1 & \text{if } 0 < t < \lambda , \\ 0 & \text{if } \lambda < t < 1 . \end{cases}$$

We recall the fact that (see [5])

$$g_n(x) = g_\lambda(nx_1) \to \lambda \qquad \text{in} \quad L^\infty(\Omega) \text{ weak star}$$

and

$$1 - g_n(x) \to 1 - \lambda \qquad \text{in} \quad L^\infty(\Omega) \text{ weak star} .$$

The sequence $\widehat{c}_n(x)$ has the properties

$$\nabla \widehat{c}_n(x) = \xi^* + (\xi - \xi^*) g_\lambda\big(n\langle \xi - \xi^*, x\rangle\big) ;$$

$$\widehat{c}_n(x) \to \widehat{c}_0 \quad \text{in } W^1 L_M(\Omega) \qquad \text{for} \quad \sigma\big(\Pi L_M, \Pi E_{\overline{M}}\big) ,$$

where

$$\widehat{c}_0(x) = \big\langle \lambda\xi + (1-\lambda)\xi^*, x\big\rangle .$$

Let $\psi \colon \mathbb{R} \to \mathbb{R}$ be a C^∞-function with support in the interval (-1.1) and $\psi(t) = 1$ for $|t| < \frac{1}{2}$. Defining $\overline{c}_R(x) = \psi\big(\frac{|x|}{R}\big)\widehat{c}_0(x)$, for $R > 0$, we calculate

$$\nabla \overline{c}_R(x) = \psi'\left(\frac{|x|}{R}\right) \frac{|x|}{R} \widehat{c}_0(x) + \psi\left(\frac{|x|}{R}\right) \nabla \widehat{c}_0(x) .$$

Moreover, the function $\overline{c}_R(x)$ satisfies the properties (see [5; Proposition 3.1]):

$$|\nabla \overline{c}_R(x)| \le C \qquad \text{in} \quad \Omega , \tag{4.5}$$

$$\int_{B(0,R)} f\big(x, \nabla \overline{c}_R(x)\big)\, \mathrm{d}x \to 0 \qquad \text{as} \quad R \to 0 . \tag{4.6}$$

Note that (4.2) is used for to prove (4.6). Next, we consider the sequence $\widehat{c}_n(x)$ in a ball $B(0,r)$, say. We shall show that it is possible to alter each element of the sequence $\widehat{c}_n(x)$ in such a manner that it coincides with the limit $\widehat{c}_0(x)$ at the boundary.

The following lemma is a generalization of [5; Proposition 3.2] in Orlicz-Sobolev spaces.

LEMMA 7. *There is a sequence $a_n(x)$ in $W^1L_M(\Omega)$ satisfying*

i) $a_n(x) = \widehat{c}_0(x) = \langle \lambda\xi + (1-\lambda)\xi^*, x\rangle$ *in* $\partial B(0,r)$,

ii) $a_n - \widehat{c}_n \to 0$ (mod) *in* $W^1L_M(\Omega)$ *as* $n \to \infty$,

iii) $a_n \to \widehat{c}_0$ *in* $W^1L_M(\Omega)$ *for* $\sigma(\Pi L_M, \Pi E_{\overline{M}})$,

iv) $\|\nabla a_n\|_\infty + \|\nabla \overline{c}_n\|_\infty \le C$,

v) $\left| \int_{B(0,r)} f(x, \nabla \widehat{c}_n(x))\, dx - \int_{B(0,r)} f(x, \nabla a_n(x))\, dx \right| \to 0$ *as* $n \to \infty$,

vi) $\int_{B(0,r)} f(x, \nabla a_n(x))\, dx \to 0$ *as* $r \to 0$ *uniformly in* n.

Now, we are in a position to complete the proof of Theorem 6. For $R \le R_0$ and $r = \frac{R}{2}$, we define the sequence:

$$\widehat{b}_n(x) = \begin{cases} \overline{b}(x) & \text{for } x \in \Omega \setminus B(0,R), \\ \overline{b}(x) + \overline{c}_R(x) & \text{for } x \in B(0,R) \setminus B(0,r), \\ \overline{b}(x) + a_n(x) & \text{for } x \in B(0,r), \end{cases}$$

which converges in $W^1L_M(\Omega)$ for the weak topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ to

$$b_0(x) = \begin{cases} \overline{b}(x) & \text{for } x \in \Omega \setminus B(0,R), \\ \overline{b}(x) + \overline{c}_R(x) & \text{for } x \in B(0,R). \end{cases}$$

On account of (4.5), (4.6) and Lemma 7 (as in [5]), we have for $R > 0$ small enough,

$$|J(\widehat{b}_n)| < \frac{7}{8}\varepsilon_0 \qquad \text{for all} \quad n.$$

Hence, for any n, we find numbers $t_n \in [0,1]$ and $i_n \in \{1,2\}$ such that for

$$b_n := \widehat{b}_n + t_n(\overline{b}_{i_n} - \widehat{b}_n),$$

we have

$$J(b_n) = 0.$$

Now, choosing a subsequence such that $t_n \to t_0$ and $i_n = i$, $i \in \{1,2\}$, we have

$$b_n \to b_0 \quad \text{in} \quad W^1L_M(\Omega) \qquad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}).$$

Because, of the continuity of J with strong topology of $W^1L_M(\Omega)$, we have

$$\lim_{n\to\infty} J(\overline{b} + t_n(\overline{b}_{i_n} - \overline{b})) = J(\overline{b} + t_0(\overline{b}_i - \overline{b})),$$

and by construction

$$f(x, \nabla(\overline{b} + t_n(\overline{b}_i - \overline{b}))) = f(x, 0) \qquad \text{in} \quad B(0,R).$$

yielding,

$$\lim_{n\to\infty} \int\limits_{B(0,R)} f\bigl(x,\nabla(b_n)(x)\bigr)\,\mathrm{d}x \ge \int\limits_{B(0,R)} f\bigl(x,\nabla b_0(x)\bigr)\,\mathrm{d}x\,.$$

Since $b_n = b_0$ in $B(0,R)\setminus B(0,r)$ and $r=\frac{R}{2}$, we finally get

$$\begin{aligned}\int\limits_{B(0,r)} f\bigl(x,\lambda\xi+(1-\lambda)\xi^*\bigr)\,\mathrm{d}x &= \int\limits_{B(0,r)} f\bigl(x,\nabla b_0(x)\bigr)\,\mathrm{d}x\\ &\le \lim_{n\to\infty}\int\limits_{B(0,r)} f\bigl(x,\nabla b_n(x)\bigr)\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int\limits_{B(0,R)} f\bigl(x,\nabla a_n(x)\bigr)\,\mathrm{d}x\\ &= \lambda\int\limits_{B(0,r)} f(x,\xi)\,\mathrm{d}x + (1-\lambda)\int\limits_{B(0,r)} f(x,\xi^*)\,\mathrm{d}x\,.\end{aligned}$$

Since the above inequality can be obtained for all ball $B(0,r)$ with radius $r<\frac{R}{2}$, we conclude that

$$f\bigl(x_0,\lambda\xi+(1-\lambda)\xi^*\bigr) \le \lambda f(x_0,\xi)+(1-\lambda)f(x_0,\xi^*)\,,$$

for all $\lambda\in[0,1]\cap\mathbb{Q}$ and all $\xi,\xi^*\in\mathbb{Q}^N$. It then follows by the continuity of $f(x,\xi)$ with respect to ξ, that the above inequality holds for all $\lambda\in[0,1]$ and all $\xi,\xi^*\in\mathbb{R}^N$.

Proof of Lemma 7. Let $\widetilde{\omega}_\delta$ be a C^∞-function with support in $[-1,1]$ such that $\widetilde{\omega}_\delta=1$ for $|t|<1-\delta$ and $|\widetilde{\omega}'_\delta(t)|<\frac{2}{\delta}$ for all t. Defining the functions

$$\omega_\delta(x)=\widetilde{\omega}_\delta\left(\frac{|x|}{r}\right)$$

and

$$a_{n,\delta}(x)=\widehat{c}_0(x)+\omega_\delta(x)\bigl(\widehat{c}_n(x)-\widehat{c}_0(x)\bigr)\,,$$

we have the following inequalities

$$\bigl|\nabla\bigl(\widehat{c}_n(x)-\widehat{c}_0(x)\bigr)\bigr|\bigl(1-\omega_\delta(x)\bigr) \le C'r\bigl(|\xi^*|+|\xi|\bigr)\bigl(1-\omega_\delta(x)\bigr)\,, \tag{4.7}$$

$$|\widehat{c}_n(x)-\widehat{c}_0(x)|\,|\nabla\omega_\delta(x)| \le O(n^{-1})\frac{1}{\delta}\chi_{\mathrm{supp}}(\nabla\omega_\delta)\,. \tag{4.8}$$

$$\begin{aligned}\int\limits_\Omega M\bigl(|a_{n,\delta}-\widehat{c}_n|\bigr)\,\mathrm{d}x &+ \int\limits_\Omega M\bigl(|\nabla(a_{n,\delta}-\widehat{c}_n)|\bigr)\,\mathrm{d}x\\ &\le O(\delta)+C\int\limits_{B(0,r)} M\Bigl(\nabla\bigl(\widehat{c}_n(x)-\widehat{c}_0(x)\bigr)\bigl(1-\omega_\delta(x)\bigr)\Bigr)\,\mathrm{d}x \end{aligned} \tag{4.9}$$

for some positive constants C' and C.

For (4.7) and (4.8) see the proof of [5; Proposition 3.2]. Assuming now that (4.9) is true, thus we get

$$\omega_\delta(x) = \begin{cases} 0 & \text{on } \Omega \setminus \overline{B}(0,r)\,, \\ 1 & \text{on } B\big(0,(1-\delta)r\big)\,, \\ \widetilde{\omega}_\delta\left(\frac{|x|}{r}\right) & \text{on } \overline{B}(0,r) \setminus B\big(0,(1-\delta)r\big)\,, \end{cases}$$

this implies that

$$a_{n,\delta}(x) - \widehat{c}_n(x) = \begin{cases} \widehat{c}_0(x) - \widehat{c}_n(x) & \text{on } \Omega \setminus \overline{B}(0,r)\,, \\ 0 & \text{on } B\big(0,(1-\delta)r\big)\,, \\ \left(1 - \widetilde{\omega}_\delta\left(\frac{|x|}{r}\right)\right)\left(\widehat{c}_0(x) - \widehat{c}_n(x)\right) & \text{on } \overline{B}(0,r) \setminus B\big(0,(1-\delta)r\big) \end{cases}$$

and

$$\begin{aligned} &\nabla\left(a_{n,\delta}(x) - \widehat{c}_n(x)\right) \\ &= \begin{cases} \nabla\left(\widehat{c}_0(x) - \widehat{c}_n(x)\right) & \text{on } \Omega \setminus \overline{B}(0,r)\,, \\ 0 & \text{on } B\big(0,(1-\delta)r\big)\,, \\ \nabla\left(\widetilde{\omega}_\delta\left(\frac{|x|}{r}\right)\right)\left(\widehat{c}_0(x) - \widehat{c}_n(x)\right) & \\ \quad + \left(1 - \widetilde{\omega}_\delta\left(\frac{|x|}{r}\right)\right)\nabla\left(\left(\widehat{c}_0(x) - \widehat{c}_n(x)\right)\right) & \text{on } \overline{B}(0,r) \setminus B\big(0,(1-\delta)r\big)\,. \end{cases} \end{aligned}$$

Hence, we have the estimate

$$\begin{aligned} &\int_\Omega M\left(|a_{n,\delta} - \widehat{c}_n|\right)\,\mathrm{d}x + \int_\Omega M\left(|\nabla(a_{n,\delta} - \widehat{c}_n)|\right)\,\mathrm{d}x \\ &\qquad \le O(\delta) + C \int_{B(0,r)\setminus B(0,(1-\delta)r)} M\left(\left|\nabla\left((\widehat{c}_n - \widehat{c}_0)(1-\omega_\delta)\right)\right|\right)\,\mathrm{d}x \\ &\qquad \le O(\delta) + CM\Big(C_1 O(n^{-1})\frac{1}{\delta}\Big)\left|\overline{B}(0,r) \setminus B\big(0,(1-\delta)r\big)\right|\,. \end{aligned}$$

Selecting numbers δ_n such that $O\left(n^{-1}\right)\frac{1}{\delta_n} = 1$, this implies that

$$O(\delta_n) = O\big(n^{-1}\big) \qquad \text{and} \qquad \delta_n \to 0 \quad \text{as } n \to \infty\,.$$

Then, we conclude that

$$\begin{aligned} &\int_\Omega M\left(|a_{n,\delta} - \widehat{c}_n|\right)\,\mathrm{d}x + \int_\Omega M\left(|\nabla(a_{n,\delta} - \widehat{c}_n)|\right)\,\mathrm{d}x \\ &\qquad\qquad \le O\big(n^{-1}\big) + CM\Big(C_1 O\big(n^{-1}\big)\frac{1}{\delta}\Big)\left|\overline{B}(0,r) \setminus B\big(0,(1-\delta)r\big)\right|\,, \end{aligned}$$

which converges to 0 as $n \to \infty$. We define the functions $a_n = a_{n,\delta}$ and we have

$$a_{n,\delta} - \widehat{c}_n \to 0 \ (\mathrm{mod}) \quad \text{in } W^1L_M(\Omega) \qquad \text{as} \quad n \to \infty ,$$

which gives (ii) in Lemma 7 and

$$a_n - \widehat{c}_0 = (a_n - \widehat{c}_n) + (a_n - \widehat{c}_0) \to 0 \quad \text{in } W^1L_M(\Omega) \qquad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}) .$$

The properties i), iv) and vi) are satisfied by the definition of a_n. Now, we return to show the inequality (4.9). In fact we can write

$$\begin{aligned}
&\int_\Omega M(|a_{n,\delta} - \widehat{c}_n|)\, \mathrm{d}x + \int_\Omega M(|\nabla(a_{n,\delta} - \widehat{c}_n)|)\, \mathrm{d}x \\
&= \int_{\Omega\setminus\overline{B}(0,r)} M(|\widehat{c}_n - \widehat{c}_0|)\, \mathrm{d}x + \int_{\overline{B}(0,r)} M(|\widehat{c}_n - \widehat{c}_0|(1-\omega_\delta))\, \mathrm{d}x \\
&\quad + \int_{\Omega\setminus\overline{B}(0,r)} M(|\nabla(\widehat{c}_n - \widehat{c}_0)t|)\, \mathrm{d}x + \int_{\overline{B}(0,r)} M\big(|\nabla((\widehat{c}_n - \widehat{c}_0)(1-\omega_\delta))|\big)\, \mathrm{d}x .
\end{aligned}$$

Since

$$\big(1 - \omega_\delta(x)\big) \to 0 \qquad \text{a.e.} \quad x \in \overline{B}(0,r)$$

and

$$\int_{\Omega\setminus\overline{B}(0,r)} M(|\widehat{c}_n - \widehat{c}_0|)\, \mathrm{d}x + \int_{\Omega\setminus\overline{B}(0,r)} M(|\nabla(\widehat{c}_n - \widehat{c}_0)|)\, \mathrm{d}x \to 0 \qquad \text{as} \quad n \to \infty ,$$

then, we conclude that

$$\begin{aligned}
&\int_\Omega M(|a_{n,\delta} - \widehat{c}_n|)\, \mathrm{d}x + \int_\Omega M(|\nabla(a_{n,\delta} - \widehat{c}_n)|)\, \mathrm{d}x \\
&\qquad\qquad \le 0\,(\delta) + C \int_{B(0,r)} M\,(|\nabla\,((\widehat{c}_n - \widehat{c}_0)(1-\omega_\delta))|)\, \mathrm{d}x ,
\end{aligned}$$

which implies the inequality (4.9). □

COROLLARY 8. *Under the same assumptions as in Theorem 6 suppose that there is a (non-empty) weakly closed level set N_μ. If μ is not an extreme value of J, then the function $f(x, \nabla u(x))$ is affine in the gradient.*

Remark 9. It's not clear how to extend the previous argument to the situation where $f = f(x, u, \nabla u)$.

Remark 10.

1) If $M(t) = \frac{|t|^p}{p}$, $1 < p < \infty$, we obtain the statement of [5; Theorem 3.1].

2) Note that when $f = f(x, u, \nabla u)$, the same result holds for $1 < p < N$, but it seems to be an open problem in the general case when $N \leq p < \infty$ (see [5]).

Remark 11. Note that when $f = f(u)$ or $f = f(x, u)$, we can easily adapt the same argument of [5].

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