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ON A CERTAIN THREE-POINT BOUNDARY VALUE PROBLEM

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In this paper there are proved theorems of existence and uniqueness of solutions of the equation

$$u'' = f(t, u, u'), \tag{0.1}$$

satisfying the conditions

$$u'(a) = A, \; u(b) - u(t_0) = B \tag{0.2}$$

or

$$u'(a) = 0, \; u(b) - u(t_0) = 0, \tag{0.3}$$

where $-\infty < a < t_0 < b < +\infty$, $A, B \in (-\infty, +\infty)$. We use the method of lower and upper functions here.

The question of existence and uniqueness of solutions of the problem (0.1), (0.2) were studied by V. Šeda ([13]) by means of a method different from that used here and the results obtained in this paper are different. A similar three-point problem, with the boundary condition $u(a) = A$, $u(b) - u(t_0) = B$, was solved by I. Kiguradze and A. Lomtatidze in [7, 8]. Further, in the works [9—11] there were proved existence and uniqueness theorems for four-point boundary value problems with the boundary condition $u(c) - u(a) = A$, $u(b) - u(d) = B$, where $-\infty < a < c < d < b < +\infty$.

1. The main results

We will use the following notations:
$\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{D} = [a, b] \times \mathbb{R}^2$, $\mathbb{N}$ — the set of all natural numbers, $\alpha = \max\{1, |A|\}$,

$$p_i, q_i \in [1, +\infty], \; 1/p_i + 1/q_i = 1, \; i = 1, \ldots, n, \; n \in \mathbb{N}, \; g_0(t) = \alpha_0 t^2 + \beta_0 t,$$

where

$\alpha_0 = [B/(b - t_0) - A](b + t_0 - 2a)^{-1}$, $\beta_0 = [A(b + t_0) - 2aB/(b - t_0)](b + t_0 -$

$-2a)^{-1}$, $AC^1(a, b)$ is the set of all real functions having absolutely continuous first derivatives on $[a, b]$,
$\mathrm{Car}_{\mathrm{loc}}(\mathcal{D})$ is the set of all real functions satisfying the local Carathéodory conditions on $\mathcal{D}$,
a.e. = "almost every".
We say that some property is satisfied on $\mathcal{D}$ if it is satisfied for a.e. $t \in [a, b]$ and every $(x, y) \in \mathbb{R}^2$. Let $d_1, d_2 \in C(a, b)$, $d_1(t) \leqq d_2(t)$ for $t \in [a, b]$. We say that some property is satisfied on $D(d_1(t), d_2(t))$ if it is satisfied for a.e. $t \in [a, b]$ and for every $x \in [d_1(t), d_2(t)]$, $|y| \geqq \alpha$.

Definition. *A function $u \in AC^1(a, b)$ which fulfils* (0.1) *for a.e. $t \in [a, b]$ will be called a solution of the equation* (0.1). *Each solution of* (0.1) *which satisfies* (0.2) *will be called a solution of the problem* (0.1), (0.2).

Definition. *A function $\sigma_1 \in AC^1(a, b)$ will be called a lower function of the problem* (0.1), (0.2) *if*

$$\sigma_1''(t) \geqq f(t, \sigma_1, \sigma_1') \quad \text{for a.e. } t \in [a, b], \tag{1.1}$$

$$\sigma_1'(a) \geqq A, \qquad \sigma_1(b) - \sigma_1(t_0) \leqq B. \tag{1.2}$$

A function $\sigma_2 \in AC^1(a, b)$ will be called an upper function of the problem (0.1), (0.2) *if*

$$\sigma_2''(t) \leqq f(t, \sigma_2, \sigma_2') \text{ for a.e. } t \in [a, b], \tag{1.3}$$

$$\sigma_2'(a) \leqq A, \qquad \sigma_2(b) - \sigma_2(t_0) \geqq B. \tag{1.4}$$

In the whole paper we suppose that $f \in \mathrm{Car}_{\mathrm{loc}}(\mathcal{D})$ and denote

$$r_i = \max\{|\sigma_1^{(i)}(t)| + |\sigma_2^{(i)}(t)| : a \leqq t \leqq b\}, \qquad i = 0, 1.$$

Theorem 1. *Let σ_1 be a lower and σ_2 an upper function of* (0.1), (0.2) *and $\sigma_1(t) \leqq \sigma_2(t)$ for $a \leqq t \leqq b$. Further, let on the $D(\sigma_1(t), \sigma_2(t))$ the inequality*

$$f(t, x, y) \operatorname{sgn} y \leqq \omega(y) \sum_{i=1}^{n} g_i(t) h_i(x) (1 + |y|)^{1/q_i} \tag{1.5}$$

be satisfied, where $g_i \in L^{p_i}(a, b)$, $h_i \in L^{q_i}(-r_0, r_0)$, $i = 1, \ldots, n$, and $\omega \in C(R)$ is a positive function such that

$$\int_a^{+\infty} \frac{\mathrm{d}s}{\omega(s)} = \int_a^{+\infty} \frac{\mathrm{d}s}{\omega(-s)} = +\infty. \tag{1.6}$$

Then the problem (0.1), (0.2) *has a solution.*

Theorem 2. *Let g_i, h_i, $i = 1, \ldots, n$, and ω be the functions from Theorem* 1 *and let there exist $r \in (0, +\infty)$ such that the condition* (1.5) *is satisfied on $D(g_0(t) - r, g_0(t) + r)$ and the condition*

(1.7) $$(f(t, x + g_0(t), g_0'(t)) - 2\alpha_0)\operatorname{sgn} x \geqq 0 \qquad \textit{for } |x| \geqq r$$

is satisfied on $\mathbb{D}$.
Then the problem (0.1), (0.2) *has a solution.*

Corollary 1. *Let* g_i, h_i, $i = 1, \ldots, n$, *and* ω *be the functions from Theorem* 1 *and let there exist* $r \in (0, +\infty)$ *such that* (1.5) *is satisfied on* $D(-r, r)$ *and the condition*

(1.8) $$f(t, x, 0)\operatorname{sgn} x \geqq 0 \qquad \textit{for } |x| \geqq r$$

is fulfilled on $\mathbb{D}$.
Then the problem (0.1), (0.3) *has a solution.*

Theorem 3. *Let thee exist a non-negative function* $h \in L(a, b)$ *such that on the set* $\mathbb{D}$ *the inequality*

(1.9) $$f(t, x_1, y_1) - f(t, x_2, y_2) + h(t)|y_1 - y_2| > 0 \qquad \text{for } x_1 > x_2$$

is satisfied.
Then the problem (0.1), (0.2) *has not more then one solution.*

Corollary 2. *Let all assumptions of Theorem* 3 *be satisfied and let there exist* $r \in (0, +\infty)$ *such that*

(1.10) $$f(t, (-1)^i r, 0)(-1)^i \geqq 0 \qquad \textit{for a.e. } t \in (a, b),\ i = 1, 2\,.$$

Then the problem (0.1), (0.3) *has just one solution.*

Corollary 3. *Let all assumptions of Theorem* 3 *be satisfied and let thee exist* $r \in (0, +\infty)$ *such that*

(1.11) $$[f(t, (-1)^i r + g_0(t), g_0'(t)) - 2\alpha_0](-1)^i \geqq 0$$
for a.e. $t \in (a, b)$ *and* $i = 1, 2$.

Then the problem (0.1), (0.2) *has just one solution.*

2. Auxiliary statements

Lemma 1. *Let* $k \in (0, +\infty)$. *Then the problem*

(2.1) $$v'' = k^2 v\,,$$

(2.2) $$v'(a) = 0\,, \qquad v(b) - v(t_0) = 0$$

has only the trivial solution and there exists $c_k \in (0, +\infty)$ *such that*

(2.3) $$\left|\frac{\partial G(t, s)}{\partial t}\right| + |G(t, s)| \leqq c_k \qquad \text{for } a \leqq t, s \leqq b\,,$$

where G *is the Green function for the problem* (2.1), (2.2).

Proof. Let us suppose that the solution of (2.1) $v(t) = \alpha_1 e^{kt} + \alpha_2 e^{-kt}$, α_1, $\alpha_2 \in \mathbb{R}$, satisfies (2.2). We obtain the system

(2.4) $$\alpha \mathbf{M} = 0$$

where $\alpha = (\alpha_1, \alpha_2)$, and

(2.5) $$\det \mathbf{M} = k(e^{k(b+t_0)} - e^{2ak})(e^{kb} - e^{kt_0})e^{-k(a+b+t_0)} > 0.$$

Therefore the system (2.4) has only the trivial solution $\alpha_1 = \alpha_2 = 0$. Let

$$G(t, s) = \begin{cases} a_1(s)e^{kt} + a_2(s)e^{-kt} & \text{for } a \leqq s \leqq t \leqq b \\ b_1(s)e^{kt} + b_2(s)e^{-kt} & \text{for } a \leqq t \leqq s \leqq b, \end{cases}$$

and

(2.6) $$\lim_{t\to s+} G(t, s) - \lim_{t\to s-} G(t, s) = 0, \qquad \lim_{t\to s+} \frac{\partial G(t, s)}{\partial t} - \lim_{t\to s-} \frac{\partial G(t, s)}{\partial t} = 1,$$

(2.7) $$\frac{\partial G(a, s)}{\partial t} = 0, \qquad G(b, s) - G(t_0, s) = 0.$$

From (2.6) we get

(2.8) $$a_1(s) - b_1(s) = e^{-ks}/2k, \qquad a_2(s) - b_2(s) = -e^{ks}/2k.$$

Since (2.7), (2.8),

(2.9) $$\begin{cases} b_1(s)k e^{ka} - b_2(s)k e^{-ka} = 0 \\ b_1(s)(e^{kb} - e^{kt_0}) + b_2(s)(e^{-kb} - e^{-kt_0}) = \\ = -(e^{k(b-s)} - e^{k(t_0-s)} - e^{k(s-b)} + e^{k(s-t_0)})/2k \qquad \text{for } s \in [a, t_2] \end{cases}$$

and

(2.10) $$\begin{cases} b_1(s)k e^{ka} - b_2(s)k e^{-ka} = 0 \\ b_1(s)(e^{kb} - e^{kt_0}) + b_2(s)(e^{-kb} - e^{-kt_0}) = \\ = -(e^{k(b-s)} - e^{k(s-b)})/2k \qquad \text{for } s \in [t_0, b]. \end{cases}$$

The systems (2.9), (2.10) have the same matrix $\mathbf{M}$ as the system (2.4), and so, by (2.5), the functions b_1, b_2, a_1, a_2 are uniquely determined on $[a, b]$. It is not difficult to show that the constant

$$c_k = (1/k + 1)e^{kb_0}(1 + (e^{kb_0} + 1)/\det \mathbf{M}),$$

where $b_0 = \max\{|a|, |b|\}$, satisfies (2.3).

Lemma 2. *Let there exist $h \in L(a, b)$ such that*

(2.11) $$|f(t, x, y)| \leqq h(t) \text{ on } \mathbb{D}.$$

Then for any $k \in (0, +\infty)$ the problem

$$u'' = k^2 u + f(t, u, u'), \tag{2.12}$$

$$u'(a) = A, \; u(b) - u(t_0) = B \tag{2.13}$$

has a solution.

Proof. The proof is analogous to the proof of Lemma 2 in [12] and so it is omitted.

3. A priori estimates

Lemma 3. *Let $r \in (0, +\infty)$, $g_i \in L^{p_i}(a, b)$, $h_i \in L^{q_i}(-r, r)$ $i = 1, \ldots, n$, and $\omega \in C(\mathbb{R})$ be a positive function satisfying the condition (1.6). Then there exists $r^* \in (\alpha, +\infty)$ such that for any function $u \in AC^1(a, b)$ the conditions*

$$u'(a) = A, \; |u(t)| \leqq r \qquad \text{for } a \leqq t \leqq b \tag{3.1}$$

and

$$u''(t) \operatorname{sgn} u'(t) \leqq \omega(u'(t)) \sum_{i=1}^{n} g_i(t) h_i(u(t)) (1 + |u'(t)|)^{1/q_i} \tag{3.2}$$

$$\text{for a.e. } t \in [a, b] \text{ and } |u'(t)| \geqq \alpha$$

imply the estimate

$$|u'(t)| \leqq r^* \qquad \text{for } a \leqq t \leqq b. \tag{3.3}$$

Proof. We will write $\|g\|_{L^p(a,b)} = \left(\int_a^b |g(t)|^p \, \mathrm{d}t \right)^{1/p}$ for $1 \leqq p < +\infty$ and $\|g\|_{L^\infty(a,b)} = \operatorname{ess\,sup} \{ |g(t)| : a < t < b \}$. Put $c_0 = 2 \sum_{i=1}^{n} \|g_i\|_{L^{p_i}(a,b)} \|h_i\|_{L^{q_i}(-r,r)}$. From (1.6) it follows that there exists $r^* \in (\alpha, +\infty)$ such that

$$\int_\alpha^{r^*} \frac{\mathrm{d}s}{\omega(s)} > c_0 \qquad \text{and} \qquad \int_\alpha^{r^*} \frac{\mathrm{d}s}{\omega(-s)} > c_0. \tag{3.4}$$

Let $u \in AC^1(a, b)$ satisfy (3.1) and (3.2) and let there exist $t_0 \in (a, b]$ such that

$$|u'(t_0)| > r^*. \tag{3.5}$$

Let $[t_1, t_2] \subset [a, b]$ be the maximal interval containing t_0 in which $|u'(t)| \geqq \alpha$ and let $t^* \in (t_1, t_2]$ be such point that

$$|u'(t^*)| = c_1 = \max \{ |u'(t)| : t_1 \leqq t \leqq t_2 \}. \tag{3.6}$$

Then, from (3.2), it follows

$$\int_{t_1}^{t^*} \frac{u''(t)\,\mathrm{sgn}\, u'(t)}{\omega(u'(t))}\,\mathrm{d}t = \int_{t_1}^{t^*} \sum_{i=1}^{n} g_i(t)\, h_i(u(t))\,(1 + |u'(t)|)^{1/q_i}\,\mathrm{d}t\,.$$

If $u'(t) \geqq \alpha$ on $[t_1, t^*]$, then by the Hölder inequality, we can obtain from the last inequality

$$\int_{\alpha}^{c_1} \frac{\mathrm{d}s}{\omega(s)} \leqq c_0\,. \tag{3.7}$$

According to (3.4) and (3.7), $c_1 < r^*$, which contradicts (3.5). If $u'(t) \leqq -\alpha$ on $[t_1, t^*]$, then we get a similar contradiction. Therefore the estimate (3.3) is valid.

4. Existence proposition

Proposition: *Let σ_2 be a lower function and σ_2 an upper function of the problem* (0.1), (0.2) *and $\sigma_1(t) \leqq \sigma_2(t)$ for $a \leqq t \leqq b$. Further, let on the set $D(\sigma_1(t), \sigma_2(t))$ the inequality*

$$|f(t, x, y)| \leqq g(t) \tag{4.1}$$

be valid, where $g \in L(a, b)$.
Then the problem (0.1), (0.2) *has a solution u fulfilling the condition*

$$\sigma_1(t) \leqq u(t) \leqq \sigma_2(t) \qquad \textit{for } a \leqq t \leqq b\,. \tag{4.2}$$

Proof. Similarly as in [12] put

$$w_i(t, x, y) = (-1)^i m(x - \sigma_i)(f(t, \sigma_i, \sigma_i') - f(t, \sigma_i, y) + (-1)^i r_0/m],\ i = 1, 2,$$

and

$$f_m(t, x, y) = \begin{cases} f(t, \sigma_1, \sigma_1') - r_0/m & \text{for } x \leqq \sigma_1(t) - 1/m \\ f(t, \sigma_1, y) + w_1(t, x, y) & \text{for } \sigma_1(t) - 1/m < x < \sigma_1(t) \\ f(t, x, y) & \text{for } \sigma_1(t) \leqq x \leqq \sigma_2(t) \\ f(t, \sigma_2, y) + w_2(t, x, y) & \text{for } \sigma_2(t) < x < \sigma_2(t) + 1/m \\ f(t, \sigma_2, \sigma_2') + r_0/m & \text{for } x \geqq \sigma_2(t) + 1/m, \end{cases} \tag{4.3}$$

where $m \in \mathbb{N}$, $(t, x, y) \in [a, b] \times \mathbb{R}^2$. Then, by (4.1),

$$|f_m(t, x, y)| \leqq r_0 + g(t) \text{ on } \mathbb{D}. \tag{4.4}$$

Let us consider the equation

$$(4.5) \qquad u''(t) = u/m + f_m(t, u, u'), \qquad m \in \mathbb{N}.$$

According to Lemma 2, the problem (4.5), (0.2) has a solution u_m. We shall show that u_m satisfies the inequalities

$$(4.6) \qquad \sigma_1(t) - 1/m \leqq u_m(t) \leqq \sigma_2(t) + 1/m \qquad \text{for } a \leqq t \leqq b.$$

By (1.1) and (1.3),

$$(4.7) \qquad (-1)^i(f_m(t, x, y) - \sigma_i''(t)) \geqq r_0/m$$
$$\text{for } (-1)^i(x - \sigma_i(t)) \geqq 1/m,\ i = 1, 2,\ m \in \mathbb{N}.$$

Put $v(t) = (-1)^i(u_m(t) - \sigma_i(t)) - 1/m$ for $a \leqq t \leqq b$, $i \in \{1, 2\}$.
Then from (0.2), (1.2), (1.4) it follows

$$(4.8) \qquad v'(a) \geqq 0, \qquad v(b) - v(t_0) \leqq 0.$$

This means that there exists $b_1 \in (t_0, b)$ such that

$$(4.9) \qquad v'(b_1) \leqq 0.$$

Let us suppose that (4.6) is not satisfied on (a,b_1). Then for certain $i \in \{1, 2\}$ and $t_0 \in (a, b_1)$

$$v(t_0) > 0.$$

In view of (4.8) there exists $t_* \in [a, t_0)$ such that

$$v(t_*) \geqq 0,\ v'(t_*) \geqq 0 \qquad \text{and} \qquad v(t) > 0 \text{ on } (t_*, t_0].$$

Let $t^* \in (t_0, b_1]$ be such that

$$(4.10) \qquad v(t^*) = 0 \qquad \text{and} \qquad v(t) > 0 \text{ on } [t_0, t^*).$$

In view of (4.7) there is satisfied $v''(t) \geqq (r_0 + (-1)^i u_m(t))/m \geqq 1/m^2$ for $t \in [t_*, t^*]$. Integrating the latter from t_* to t, where $t \in (t_*, t^*]$, we get $v'(t) > 0$ for $t \in (t_*, t^*]$, which contradicts (4.10). Therefore $v(t) > 0$, $v'(t) > 0$ on $(t_*, b_1]$. But in view of (4.9) this is impossible. Hence, we have proved $v(t) \leqq 0$ for $a \leqq t \leqq b_1$. Moreover, by (4.8), $v(b) \leqq 0$. Supposing that (4.6) is not satisfied on (b_1, b), we get a similar contradiction as for (a, b_1). Consequently u_m satisfies (4.6) on $[a, b]$.

From (0.2), (4.5), (4.6) it follows that the sequences $(u_m)_{m=1}^{\infty}$ and $(u_m')_{m=1}^{\infty}$ are uniformly bounded and equi-continuous on $[a, b]$ and thus, by the Arzelà-Ascoli lemma, without loss of generality we can suppose that they are uniformly converging on $[a, b]$. By (4.3)—(4.6), the function $u(t) = \lim_{m \leftrightarrow \infty} u_m(t)$ on $[a, b]$ satisfies (4.2) and is a solution of the problem (0.1), (0.2). Proposition is proved.

5. Proofs of Theorems

Proof of Theorem 1. Let r^* be the constant found by Lemma 3 for $r = r_0$. Put $\varrho_0 = r^* + r_0 + r_1$,

$$\chi(\varrho_0, s) = \begin{cases} 1 & \text{for } 0 \leqq s \leqq \varrho_0 \\ 2 - s/\varrho_0 & \text{for } \varrho_0 < s < 2\varrho_0 \\ 0 & \text{for } s \geqq 2\varrho_0 \end{cases}$$

(5.1) $$\tilde{f}(t, x, y) = \chi(\varrho_0, |x| + |y|) f(t, x, y) \text{ on } \mathbb{D}$$

and consider the equation

(5.2) $$u'' = \tilde{f}(t, u, u').$$

Since $\max\{|\sigma_i(t)| + |\sigma_i'(t)| : a \leqq t \leqq b\} < \varrho_0$, $i = 1, 2$, σ_1 is a lower and σ_2 an upper function of the problem (5.2), (0.2). Further $|\tilde{f}(t, x, y)| \leqq g(t)$ on $\mathbb{D}$, where $g(t) = \sup\{|f(t, x, y)| : |x| + |y| \leqq 2\varrho_0\} \in L(a, b)$. Thus, by Proposition, the problem (5.2), (0.2) has a solution u satisfying (4.2). Consequently u fulfils (3.1) for $r = r_0$. Further, according to (1.5) $u'' \operatorname{sgn} u' \leqq \omega(u') \sum_{i=1}^{n} g_i(t) h_i(u(t)) (1 + |u'(t)|)^{1/q_i}$ for a.e. $t \in [a, b]$ and $|u'(t)| \geqq \alpha$. Therefore, by Lemma 3, u satisfies the inequality (3.3). Consequently, by (4.2), we get

(5.3) $$|u(t)| + |u'(t)| \leqq \varrho_0 \qquad \text{for } a \leqq t \leqq b.$$

In view of (5.1), (5.2), (5.3), u is a solution of the problem (0.1), (0.2).

Proof of Theorem 2. Let us put $\sigma_1(t) = g_0(t) - r$, $\sigma_2(t) = g_0(t) + r$ for $a \leqq t \leqq b$. Then $\sigma_1''(t) = \sigma_2''(t) = 2\alpha_0$ and, by (1.7), $f(t, \sigma_1, \sigma_1') = f(t, g_0 - r, g_0') \leqq 2\alpha_0$ and $f(t, \sigma_2, \sigma_2') = f(t, g_0 + r, g_0') \geqq 2\alpha_0$ for a.e. $t \in [a, b]$. Further $\sigma_1'(a) = \sigma_2'(a) = g_0'(a) = A$ and $\sigma_1(b) - \sigma_1(t_0) = \sigma_2(b) - \sigma_2(t_0) = g_0(b) - g_0(t_0) = B$. Therefore σ_1 is a lower and σ_2 is an upper function of the problem (0.1), (0.2) and $\sigma_1(t) \leqq \sigma_2(t)$ on $[a, b]$. Thus, by Theorem 1, the problem (0.1), (0.2) has a solution. Theorem 2 is proved.

Proof of Theorem 3. Let us assume that the problem (0.1), (0.2) has two solutions u_1, u_2. Put $v = u_1 - u_2$ on $[a, b]$. Then

(5.4) $$v'(a) = 0, \; v(b) - v(t_0) = 0$$

and there exists $t_1 \in (t_0, b)$ such that

(5.5) $$v'(t_1) = 0.$$

First let us suppose that $v(s_0) \neq 0$ for some $s_0 \in (a, t_1)$. Then there exist $t_*, t^* \in [a, t_1]$ such that

$$v(t) > 0 \qquad \text{for } t \in (t_*, t^*),\ v'(t_*) \geqq 0,\ v'(t^*) \leqq 0. \tag{5.6}$$

From (1.9) we get $v''(t) + \tilde{h}(t) v'(t) > 0$ on $[t_*, t^*]$, where $\tilde{h}(t) = h(t) \operatorname{sgn} v'(t)$. Thus

$$\left(\exp\left(\int_a^t \tilde{h}(\tau)\,d\tau\right) v'(t)\right)' > 0 \qquad \text{on } [t_*, t^*]. \tag{5.7}$$

Integrating (5.7) from t_* to t^*, we get by (5.6),

$$0 \geqq \exp\left(\int_a^{t^*} \tilde{h}(\tau)\,d\tau\right) v'(t^*) - \exp\left(\int_a^{t_*} \tilde{h}(\tau)\,d\tau\right) v'(t_*) > 0. \tag{5.8}$$

The contradiction (5.8) implies $v(t) = 0$ for $t \in [a, t_1]$. From (5.4) it follows that

$$v(b) = 0. \tag{5.9}$$

Now, let us suppose that $v(s_0) > 0$ for $s_0 \in (t_1, b)$. Then there exist $t_*, t^* \in [t_1, b]$ such that (5.6) is fulfilled. Therefore, by (5.7), we get the contradiction (5.8). Thus $v(t) = 0$ for $t \in [a, b]$. This completes the proof.

Proof of Corollary 2. The uniqueness is clear. Let us prove the existence. Let $x > r$. Then, by (1.9), (1.10), $f(t, x, 0) - f(t, r, 0) > 0$ and thus $f(t, x, 0) \geqq 0$ for $x \geqq r$. If $x < -r$, then $f(t, -r, 0) - f(t, x, 0) > 0$ and so $f(t, x, 0) \leqq 0$ for $x \leqq -r$. Therefore f satisfies (1.8) on $\mathbb{D}$.

Further, according to (1.9), (1.10), if $y \geqq \alpha$, $x \in [-r, r)$, then $f(t, x, y) < f(t, r, 0) + h(t)|y|$ and if $y \leqq -\alpha$, $x \in (-r, r]$, then $-f(t, x, y) < -f(t, -r, 0) + h(t)|y|$. Thus $f(t, x, y) \operatorname{sgn} y \leqq h_1(t) + h_2(t)|y|$ on $D(-r, r)$, where

$$h_1(t) = \begin{cases} f(t, r, 0) & \text{for } y \geqq \alpha \\ -f(t, -r, 0) & \text{for } y \leqq -\alpha, \end{cases} \qquad h_2(t) = h(t),\ t \in (a, b).$$

Consequently f satisfies (1.5) on $D(-r, r)$ and by Corollary 1 the problem (0.1), (0.3) has a solution. This completes the proof.

Proof of Corollary 3. Put $g(t, x, y) = f(t, x + g_0, y + g_0') - g_0''$. Then f satisfies (1.9) exactly just g satisfies (1.9). Further, if f satisfies (1.11), then g satisfies (1.10) and so, by Corollary 2, the problem

$$v'' = g(t, v, v'), \qquad v'(a) = 0, \qquad v(b) - v(t_0) = 0$$

has just one solution v. Then $u = v + g_0$ is the unique solution of (0.1), (0.2).

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ОБ ОДНОЙ ТРЕХТОЧЕЧНОЙ ЗАДАЧЕ

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Резюме

В статье доказаны теоремы существования и единственности решений задачи

$$u'' = f(t, u, u'), \; u'(a) = A, \; u(b) - u(t_0) = B,$$

где

$$-\infty < a < t_0 < b < +\infty, \; A, \; B \in (-\infty, +\infty).$$