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NIKODÝM CONVERGENCE THEOREM FOR UNIFORM SPACE VALUED FUNCTIONS DEFINED ON D–POSETS[1]

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ABSTRACT. Nikodým convergence type theorem with necessary and sufficient conditions for a sequence of functions defined on a D-poset and with values in a uniform space is proved.

1. Introduction

The classical Nikodým convergence theorem says that the limit of a sequence of countable additive measures is again a countable additive measure. This important theorem of the measure theory has many generalizations in different directions, even for non-additive set functions. E. P a p [24], [25] has investigated set functions with values in an arbitrary uniform space Y, without considering any algebraic operation on Y.

On the other hand, by the need of mathematical foundations of propositional calculus of quantum mechanics there were developed many structures as quantum logic (= orthomodular poset) [5], [6], [7], [8], [9], [12], [16], [26], [29], orthoalgebra [15] and very recently D-poset [18] , [19].

In this paper, we obtain necessary and sufficient conditions for Nikodým convergence theorem to be true for a sequence of functions defined on a D-poset and with values in a uniform space.

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Key words: D-poset, $\sigma(\oplus)$-D-poset, uniform space, orthomodular poset, MV-algebra, orthoalgebra.

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2. $\sigma(\oplus)$-D-poset

We have by [18], [19], [14]

DEFINITION 2.1. A *D-poset* (*difference poset*) is a partially ordered set $\boldsymbol{L}$ with a partial ordering $\leq$, maximal element $\mathbf{1}$, and with a partial binary operation $\ominus\colon \boldsymbol{L}\times\boldsymbol{L}\to\boldsymbol{L}$, called *difference*, such that, for $a,b\in\boldsymbol{L}$, $b\ominus a$ is defined if and only if $a\leq b$, for that the following axioms hold for $a,b,c\in\boldsymbol{L}$:

(DP$_1$) $b\ominus a\leq b$;
(DP$_2$) $b\ominus(b\ominus a)=a$;
(DP$_3$) $a\leq b\leq c \implies c\ominus b\leq c\ominus a$ and $(c\ominus a)\ominus(c\ominus b)=b\ominus a$.

Then there exists also a minimal element $\mathbf{0}$ $(=\mathbf{1}\ominus\mathbf{1})$.

The following properties of the operation $\ominus$ have been proved in [19]:

(a) $a\ominus\mathbf{0}=a$.
(b) $a\ominus a=\mathbf{0}$.
(c) $a\leq b \implies b\ominus a=\mathbf{0} \iff b=a$.
(d) $a\leq b \implies b\ominus a=b \iff a=\mathbf{0}$.
(e) $a\leq b\leq c \implies b\ominus a\leq c\ominus a$ and $(c\ominus a)\ominus(b\ominus a)=c\ominus b$.
(f) $b\leq c,\ a\leq c\ominus b \implies b\leq c\ominus a$ and $(c\ominus b)\ominus a=(c\ominus a)\ominus b$.
(g) $a\leq b\leq c \implies a\leq c\ominus(b\ominus a)$ and $\big(c\ominus(b\ominus a)\big)\ominus a=c\ominus b$.

For an arbitrary but fixed element $a\in\boldsymbol{L}$ we define

$$a^{\perp}:=\mathbf{1}\ominus a\,.$$

We have:

(i) $a^{\perp\perp}=a$;
(ii) $a\leq b \implies b^{\perp}\leq a^{\perp}$.

The elements a and b from $\boldsymbol{L}$ are *orthogonal*, denoted by $a\perp b$, if and only if $a\leq b^{\perp}$ (or $b\leq a^{\perp}$).

We define a partial binary operation $\oplus\colon \boldsymbol{L}\times\boldsymbol{L}\to\boldsymbol{L}$ for orthogonal elements a and b such that

$$b\leq a\oplus b \quad\text{and}\quad a=(a\oplus b)\ominus b\,.$$

This operation $\oplus$ is commutative and associative ([14]).

The notion of D-poset covers many important examples.

Example 2.2. ([6], [7], [8], [10], [11], [26]) An *orthomodular poset* is a partially ordered set $\boldsymbol{O}$ with an ordering $\leq$, the least and greatest elements $\mathbf{0}$ and $\mathbf{1}$, respectively, and an orthocomplementation $\perp\colon \boldsymbol{O}\to\boldsymbol{O}$ such that:

(OM$_1$) $a^{\perp\perp}=a$ $(a\in\boldsymbol{O})$;
(OM$_2$) $a\vee a^{\perp}=\mathbf{1}$ $(a\in\boldsymbol{O})$;

(OM_3) if $a \le b$, then $b^\perp \le a^\perp$;
(OM_4) if $a \le b^\perp$, then $a \vee b \in \boldsymbol{O}$;
(OM_5) if $a \le b$, then $b = a \vee (a \vee b^\perp)^\perp$.

Taking for $a \le b$

$$b \ominus a := (a \vee b^\perp)^\perp ,$$

we obtain that the orthomodular poset $\boldsymbol{O}$ is a D-poset.

Example 2.3. ([2], [21]) An *MV-algebra* is a set $\boldsymbol{M}$ endowed with two binary operations $\oplus$ and $\odot$, an unary operation $\star$ and two elements 0 and 1 such that, for all $a, b, c \in \boldsymbol{M}$,

(MV_1) $a \oplus b = b \oplus a$;
(MV_2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(MV_3) $a \oplus 0 = a$;
(MV_4) $a \oplus 1 = 1$;
(MV_5) $(a^\star)^\star = a$;
(MV_6) $0^\star = 1$;
(MV_7) $a \oplus a^\star = 1$;
(MV_8) $(a^\star \oplus b)^\star \oplus b = (a \oplus b^\star)^\star \oplus a$;
(MV_9) $a \odot b = (a^\star \oplus b^\star)^\star$.

Taking

$$a \le b \iff (a \odot b^\star) \oplus b = b$$

and for $a \le b$

$$b \ominus a := (a \oplus b^\star)^\star ,$$

we obtain that the MV-algebra $\boldsymbol{M}$ is a D-poset.

Example 2.4. ([15], [14]) An *orthoalgebra* is a set $\boldsymbol{A}$ with two particular elements $\boldsymbol{0}$, $\boldsymbol{1}$, and with a partial binary operation $\oplus\colon \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$ such that for all $a, b, c \in \boldsymbol{A}$,

(OA_1) if $a \oplus b \in \boldsymbol{A}$, then $b \oplus a \in \boldsymbol{A}$ and $a \oplus b = b \oplus a$;
(OA_2) if $b \oplus c \in \boldsymbol{A}$ and $a \oplus (b \oplus c) \in \boldsymbol{A}$, then $a \oplus b \in \boldsymbol{A}$ and $(a \oplus b) \ominus c \in \boldsymbol{A}$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;
(OA_3) for any $a \in \boldsymbol{A}$ there is a unique $b \in \boldsymbol{A}$ such that $a \oplus b$ is defined, and $a \oplus b = \boldsymbol{1}$;
(OA_4) if $a \oplus a$ is defined, then $a = \boldsymbol{0}$.

We have $a \le b$ if and only if there exists an element $c \in \boldsymbol{A}$ such that $a \oplus c$ is defined in $\boldsymbol{A}$ and $a \oplus c = b$. An element b is the orthocomplement of a (denoted by $a^\perp$) if and only if b is a (unique) element of $\boldsymbol{A}$ such that $b \oplus a$ is defined in $\boldsymbol{A}$ and $a \oplus b = \boldsymbol{1}$.

Taking for $a \le b$

$$b \ominus a := (a \oplus b^\perp)^\perp ,$$

we obtain that the orthoalgebra $\boldsymbol{A}$ is a D-poset. We remark that each orthomodular poset (Example 2.2) is an orthoalgebra, but the opposite is not true (see example of R. Wright in [15]).

Example 2.5. ([19], [14]) Let $\mathcal{E}(H)$ be the set of all Hermitian operators T on a Hilbert space H with $O \leq T \leq I$, where O and I are the zero and identity operators, respectively, on H. The set $\mathcal{E}(H)$ is a D-poset, which is not an orthoalgebra.

Example 2.6. ([18]) Let Ω be a nonempty set and $\mathcal{F}$ the family of all fuzzy sets on Ω, i.e., $\mathcal{F} = [0,1]^{\Omega}$. We have for $f, g \in \mathcal{F}$

$$f \leq g \iff f(\omega) \leq g(\omega) \qquad (\omega \in \Omega).$$

Let $\Phi\colon [0,1] \to [0,\infty)$ be an injective increasing continuous function such that $\Phi(0) = 0$. Taking for $f \leq g$

$$(g \ominus f)(\omega) = \Phi^{-1}\Big(\Phi\big(g(\omega)\big) - \Phi\big(f(\omega)\big)\Big) \qquad (\omega \in \Omega)$$

we obtain that $\mathcal{F}$ is a D-poset.

$\boldsymbol{L}$ will always denote a D-poset. Let $\{a_1, \dots, a_n\} \subset \boldsymbol{L}$. We define

$$a_1 \oplus \dots \oplus a_n = \begin{cases} \mathbf{0} & \text{for } n = 0, \\ a_1 & \text{for } n = 1, \\ (a_1 \oplus \dots \oplus a_{n-1}) \oplus a_n & \text{for } n \geq 3, \end{cases}$$

supposing that $a_1 \oplus \dots \oplus a_{n-1}$ and $a_1 \oplus \dots \oplus a_n$ exist in $\boldsymbol{L}$. We have by [14]

DEFINITION 2.7. A finite subset $\{a_1, \dots, a_n\}$ of $\boldsymbol{L}$ is *$\oplus$-orthogonal* if $a_1 \oplus \dots \oplus a_n$ exists in $\boldsymbol{L}$.

We say that an $\oplus$-orthogonal subset $\{a_1, \dots, a_n\}$ of $\boldsymbol{L}$ has a *$\oplus$-sum*, $\bigoplus\limits_{i=1}^{n} a_i$, defined by

$$\bigoplus_{i=1}^{n} a_i := a_1 \oplus \dots \oplus a_n .$$

We remark that the preceding $\oplus$-sum is independent of any permutation of elements.

DEFINITION 2.8. A subset G of $\boldsymbol{L}$ is *$\oplus$-orthogonal* if every finite subset F of G is $\oplus$-orthogonal.

We say that an $\oplus$-orthogonal subset $G = \{a_i : i \in I\}$ of $\boldsymbol{L}$ has an *$\oplus$-sum* in $\boldsymbol{L}$, $\bigoplus\limits_{i \in I} a_i$, if in $\boldsymbol{L}$ there exists the join

$$\bigoplus_{i \in I} a_i := \sup\Big\{ \bigoplus_{i \in F} a_i : F \text{ finite subset of } I \Big\}.$$

Any subset of a $\oplus$-orthogonal set is again $\oplus$-orthogonal.

Definition 2.9. A D-poset $\boldsymbol{L}$ is a *complete* D-poset ($\sigma(\oplus)$-D-poset) if, for every $\oplus$-orthogonal subset (every countable $\oplus$-orthogonal subset) G of $\boldsymbol{L}$, there exists the $\oplus$-sum in $\boldsymbol{L}$.

Definition 2.10. A D-poset L is *quasi-σ-complete* if for every $\oplus$-orthogonal sequence (a_i) in $\boldsymbol{L}$ there exists a subsequence $(a_i)_{i\in M}$ such that $\bigoplus\limits_{i\in I} a_i \in \boldsymbol{L}$ for each $I \subset M$.

Remark 2.11. The notion of quasi-σ-ring is introduced by C. Constantinescu [4], [3].

We shall give now an example of a $\sigma(\oplus)$-D-poset.

Example 2.12. Let S be any set of real numbers between 0 and 1, where S satisfies the following conditions

(i) $0 \in S$ and $1 \in S$;
(ii) if $x, y \in S$, then $\min(1, x+y) \in S$;
(iii) if $x, y \in S$, then $\max(0, x+y-1) \in S$;
(iv) if $x \in S$, then $1 - x \in S$.

The operations $\oplus$, $\odot$ and * are defined as follows:

$$\begin{aligned} x \oplus y &:= \min(1, x+y)\,, \\ x \odot y &:= \max(0, x+y-1)\,, \\ x^* &:= 1 - x\,. \end{aligned}$$

The system $(S, \oplus, \odot, ^*, 0, 1)$ is an MV-algebra. If we take $S = [0,1]$, we obtain a σ-MV-algebra with respect to the operation $\oplus$ and, in this way, also a $\sigma(\oplus)$-D-poset, since for $x \le y$ we have that the operation $\ominus$ defined by

$$x \ominus y := (x \oplus y^*)^*$$

gives a σ-D-poset with respect to the operation $\oplus_D$ defined by

$$x \oplus_D y = (y^* \ominus x)^*\,,$$

which coincides with the operation $\oplus$, i.e.,

$$x \oplus_D y = (y^* \ominus x)^* = \big((x \oplus (y^*)^*)^*\big)^* = x \oplus y\,.$$

We remark that for $S = \{0,1\}$ we trivially obtain also a $\sigma(\oplus)$-D-poset. But if $S =$ the set of all rational numbers between 0 and 1, then this is a MV-algebra, and so also a D-poset, which is not $\sigma(\oplus)$-MV-algebra, and so also not a $\sigma(\oplus)$-D-poset.

3. Nikodým convergence theorem

Let Y be a uniform space with the uniformity $\mathcal{U}$. We denote by $\mathcal{D}$ the family of all uniformly continuous pseudometrics defined on $(Y,\mathcal{U})$.

Let $\boldsymbol{L}$ be a D-poset.

DEFINITION 3.1. For $d \in \mathcal{D}$ the *d-semivariation of a function* $\mu : \boldsymbol{L} \to Y$ with respect to a point $x_0 \in Y$ is

$$\tilde{\mu}_d^{x_0}(b) := \sup\{d(\mu(c), x_0) : c \le b, \ c \in \boldsymbol{L}\} \qquad (b \in \boldsymbol{L}).$$

We define for $d \in \mathcal{D}$, $x_0 \in Y$ and a function $\mu : \boldsymbol{L} \to Y$

$$\alpha_d^{x_0}(a, \mu) := \limsup_{n\to\infty}\left\{d(\mu(a \oplus b), x_0) : \tilde{\mu}_d^{x_0}(b) < \frac{1}{n}, \ b \in \boldsymbol{L}\right\} \qquad (a \in \boldsymbol{L}).$$

Remark 3.2. For a set function μ defined on a quasi-σ-ring Σ, the previous definition of $\alpha_d^{x_0}$ coincides with that given in the paper of E. Pap [23].

We shall need, in the proof of the main theorem, the following:

DEFINITION 3.3. A function μ: $\boldsymbol{L} \to Y$ is said to be *x_0-exhaustive for $x_0 \in Y$* if for each $d \in \mathcal{D}$

$$\lim_{n\to\infty} d(\mu(a_n), x_0) = 0$$

for each $\oplus$-orthogonal sequence (a_n) of elements from $\boldsymbol{L}$.

LEMMA 3.4. *Let Y be a uniform space and $\boldsymbol{L}$ a quasi-σ-complete D-poset. If $\mu : \boldsymbol{L} \to Y$ is an x_0-exhaustive function and (a_n) a sequence of $\oplus$-orthogonal elements from $\boldsymbol{L}$, then, for each $d \in \mathcal{D}$ and each $\varepsilon > 0$, there exists a $\oplus$-orthogonal subsequence (a_{n_i}) of (a_n) such that*

$$\tilde{\mu}_d^{x_0}\left(\bigoplus_{i\in I} a_{n_i}\right) < \varepsilon$$

for any $I \subset \mathbb{N}$.

The proof goes taking to a contradiction with the x_0-exhaustivity of the function μ.

THEOREM 3.5. *Let Y be a uniform space and $\boldsymbol{L}$ a quasi-σ-complete D-poset. Let (μ_n) be a sequence of functions μ_n, μ_n: $\boldsymbol{L} \to Y$, such that each μ_n is x_0-exhaustive, and they satisfy the following conditions for the element x_0 from Y:*

(i) *for each $d \in \mathcal{D}$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(\mu_n(a), x_0) < \delta \quad \textit{and} \quad d(\mu_n(b), x_0) < \delta$$

for $a \le b,\ a, b \in \boldsymbol{L}$ $(n \in \mathbb{N})$ *implies*

$$d\big(\mu_n(b \ominus a), x_0\big) < \varepsilon\,;$$

(ii) *for each* $d \in \mathcal{D}$ *and for each* $\delta > 0$, *there exists* $\gamma > 0$ *such that*

$$d\big(\mu_n(a), x_0\big) < \gamma,\ a \in \boldsymbol{L}\ (n \in \mathbb{N}) \implies \alpha_d^{x_0}(a, \mu_n) < \delta\ (n \in \mathbb{N})\,;$$

(iii) *for each* $d \in \mathcal{D}$

$$\lim_{n\to\infty} d\big(\mu_n(a), \mu(a)\big) = 0$$

for each $a \in \boldsymbol{L}$.

Then μ *is* x_0*-exhaustive if and only if* μ_n $(n \in \mathbb{N})$ *are uniformly* x_0*-exhaustive.*

P r o o f . Let us suppose that μ is x_0-exhaustive, but (μ_n) is not uniformly x_0-exhaustive. Hence there exist $\varepsilon > 0$, d from $\mathcal{D}$ and a $\oplus$-orthogonal sequence (a_k) of elements from $\boldsymbol{L}$ and a subsequence (μ_{n_k}) such that

$$d\big(\mu_{n_k}(a_k), x_0\big) > \varepsilon \tag{1}$$

for each $k \in \mathbb{N}$. By (i), we choose $\delta > 0$ corresponding to $\varepsilon > 0$. By (ii), we choose $\gamma > 0$ corresponding to $\delta > 0$. Since μ is x_0-exhaustive, by Lemma 3.4, there exists a $\oplus$-orthogonal subsequence (a_{k_i}) of (a_k) such that

$$\tilde{\mu}_d^{x_0}\Big(\bigoplus_{i\in I} a_{k_i}\Big) < \frac{\gamma}{2} \tag{2}$$

for each $I \subset \mathbb{N}$. Now, let us denote $m_i := \mu_{n_{k_i}}$ and $b_i := a_{k_i}$ $(i \in \mathbb{N})$ and $i_1 = 1$. By (iii), there exists an index i_2 such that

$$d\big(m_{i_2}(b_{i_1}), \mu(b_{i_1})\big) < \frac{\gamma}{2}\,. \tag{3}$$

The inequality

$$d\big(m_{i_2}(b_{i_1}), \mu(b_{i_1})\big) \ge d\big(m_{i_2}(b_{i_1}), x_0\big) - d\big(\mu(b_{i_1}), x_0\big)\,,$$

by (2) and (3), implies

$$d\big(m_{i_2}(b_{i_1}), x_0\big) < \gamma\,. \tag{4}$$

Since m_{i_2} is x_0-exhaustive, we have by Lemma 3.4 that there exists a $\oplus$-orthogonal subsequence (b_i^2) of $(b_i)_{i=i_1+1}$ such that

$$(\tilde{m}_{i_2})_d^{x_0}\Big(\bigoplus_{i\in I} b_i^2\Big) < \frac{\gamma}{2}$$

for each $I \subset \mathbb{N}$. This implies by (4) and (ii)

$$\alpha_d^{x_0}\Big(b_{i_1} \oplus \bigoplus_{i\in I} b_i^2,\ m_{i_2}\Big) < \delta$$

for each $I \subset \mathbb{N}$. Using (2) we obtain

$$d\big(\mu(b_{i_1} \oplus b_k^2),\, x_0\big) < \frac{\gamma}{2}$$

for each $k \in \mathbb{N}$, and, by (iii), there exists an index i_3 such that

$$d\big(m_{i_3}(b_{i_1} \oplus b_k^2),\, \mu(b_{i_1} \oplus b_k^2)\big) < \frac{\gamma}{2}\,.$$

Hence, by the inequality

$$d\big(m_{i_3}(b_{i_1} \oplus b_k^2),\, \mu(b_{i_1} \oplus b_k^2)\big) \ge d\big(m_{i_3}(b_{i_1} \oplus b_k^2),\, x_0\big) - d\big(\mu(b_{i_1} \oplus b_k^2),\, x_0\big)\,,$$

we obtain

$$d\big(m_{i_3}(b_{i_1} \oplus b_{i_2}),\, x_0\big) < \gamma\,, \tag{5}$$

where b_{i_2} is chosen from the sequence (b_k^2). Since m_{i_3} is x_0-exhaustive, by Lemma 3.4, there exists a $\oplus$-orthogonal subsequence (b_i^3) of $(b_i^2)_{i=i_2+1}$ such that

$$(\tilde{m}_{i_3})_d^{x_0}\Big(\bigoplus_{i\in I} b_i^3\Big) < \frac{\gamma}{2}$$

for each $I \subset \mathbb{N}$. This implies by (5) and (ii)

$$d\Big(m_{i_3}\Big(b_{i_1} \oplus b_{i_2} \oplus \bigoplus_{i\in I} b_i^3\Big),\, x_0\Big) < \delta$$

for each $I \subset \mathbb{N}$. Continuing the preceding procedure we obtain two sequences (m_{i_k}) and (b_{i_k}). Taking $b = \bigoplus_{k=1}^{\infty} b_{i_k}$, we choose by (iii) an index k_0 such that

$$d\big(m_{i_{k_0}}(b),\, x_0\big) < \eta < \delta\,. \tag{6}$$

This follows by (2) and the inequality

$$d\big(m_{i_{k_0}}(b),\, \mu(b)\big) \ge d\big(m_{i_{k_0}}(b),\, x_0\big) - d\big(\mu(b), x_0\big)\,.$$

Since, by (DP_1),

$$b \ominus a \le b\,,$$

we obtain by the preceding procedure that

$$d\big(m_{i_{k_0}}(b \ominus a),\, x_0\big) < \delta\,.$$

This, together with (i) and (6), implies by (DP_2)

$$\varepsilon > d\big(m_{i_{k_0}}\big(b \ominus (b \ominus b_{i_{k_o}})\big),\, x_0\big) = d\big(m_{i_{k_0}}(b_{i_{k_0}}),\, x_0\big)$$

and gives a contradiction with (1).

The opposite statement follows by (iii). □

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