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Mathematica Slovaca, Vol. 53 (2003), No. 4, 393--405

Persistent URL: http://dml.cz/dmlcz/130110

Math. Slovaca, **53** (2003), No. 4, 393–405

Mathematica Slovaca

STABILITY AND INVARIANCE OF MULTIVALUED ITERATED FUNCTION SYSTEMS

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(*Communicated by Milan Medveď*)

ABSTRACT. We provide a definition of an attractor to a multivalued iterated function system (IFS) modelled on previous ones existing in the literature (e.g. [Hale, J. K.: *Asymptotic Behavior of Dissipative Systems*. Math. Surveys Monographs 25, Amer. Math. Soc., Providence, RI, 1988]). Such an attractor expressing asymptotic behaviour of a system does not need to be invariant. Then, as a remedy there serves the uniform Hausdorff upper semicontinuity. It was recently shown that condensing multifunctions possess a maximal invariant set which is compact. The theorem ensuring the existence of attractors considered here also exploits compactness-like hypothesis slightly stronger than condensity, namely contractivity with respect to measure of noncompactness. Hence contractivity in measure and uniform Hausdorff upper semicontinuity together do guarantee existence of a compact attractor which is maximal invariant and unique. We also supply examples (e.g. unbounded attractor) and state further questions.

1. Introduction

The phenomenon of a fractal, a "jagged set", has a long history (e.g. the Cantor discontinuum, the Julia set). Although well settled in mathematics, it is still somehow unprecised. After B. Mandelbrot's definition (fractal = set with a fractional dimension), there was proposed a special class of processes — iterated function systems — under which the so-called self-similar sets can be produced ([Hu]). An iterated function system (briefly: IFS) is a finite family of similitudes $\{f_1, f_2, \dots, f_k \colon X \to X\}$ with scales less than 1 (more general — contractions). Such a system appears as some kind of a discrete dynamical system (cascade), since by an action on sets we understand the repeated operation

$$A \overset{F}{\mapsto} f_1(A) \cup f_2(A) \cup \dots \cup f_k(A)\,.$$

2000 Mathematics Subject Classification: Primary 54H25, 47H10, 47H09, 37B99.
Keywords: iterated function system, attractor, Barnsley-Hutchinson's operator, uniformly Hausdorff upper semicontinuous multifunction, μ-contraction.

A fractal is the unique limit of all possible iterations: $A, F(A), F^2(A), \ldots$; the Banach Principle guarantees that such phenomenon occurs. The products of iterations (orbits) are getting closer and closer to a fractal, hence the second name for this "final effect" — an attractor, and the term describing the behaviour of the system — the asymptotic stability (comp. [H], [LM]).

In some instances the operation[1]

$$X \supset A \overset{F}{\mapsto} \overline{f_1(A) \cup \cdots \cup f_k(A)} \subset X$$

behaves more accurately and is called Barnsley-Hutchinson's operator associated with IFS $\{f_1, \ldots, f_k\}$. This is easily carried over a finite family of multifunctions $\varphi_1, \ldots, \varphi_k \colon X \multimap X$, what results in multivalued IFS ([AG], [AF], [L]). Many good conditions (e.g. contractivity, compactness, upper semicontinuity, condensity) are preserved under the set-theoretic sum $\varphi = \bigcup_{i=1}^{k} \varphi_i \colon X \multimap X$, $\varphi(x) = \varphi_1(x) \cup \cdots \cup \varphi_k(x)$. When a finite system of multifunctions is replaced with their sum, the resulting Barnsley-Hutchinson operator is the same in both situations (because of the additivity of closure). Therefore we only speak about IFS consisting of one multifunction[2].

The definition of an attractor considered here is essentially based upon [H]. Our approach in many points agrees with several earlier discussed situations: [Hu], [Ha], [H], [LM], [AG]. Fairly general theorems on existence are presented in [Ha], [JGP], [AF], [Ki] and [L]. Unfortunately, they appear to be weak in the context of asymptotic stability. The aim of this work is to shed some light on connections between stability, attractivity and invariance with special emphasis on the first.

2. Uniform continuity. Set-convergence

By a space we shall always mean a metric space X with a metric d. On this space there acts a map (multifunction) $\varphi \colon X \multimap X$. For the theory of multifunctions (Hausdorff metric, semicontinuity, etc.) we refer to [HP], [D] or [CV]. Some useful designations are collected below:

- *closure* of set A: $\overline{A}$,
- *distance* of point x to set A: $d(x, A) = \inf_{a \in A} d(x, a)$,
- ε*-ball* (ε*-neighbourhood*) around A: $\mathcal{O}_\varepsilon A = \{x \in X : \ d(x, A) < \varepsilon\}$,
- *excess* of set A over set B:
  $e(A, B) = \sup_{a \in A} d(a, B) = \inf\{\varepsilon > 0 : \ A \subset \mathcal{O}_\varepsilon B\}$,

[1] Overlining stands for closure.

[2] Infinite families are permitted in [Ki].

- *Hausdorff metric*:

$$h(A,B) = \max\{e(A,B), e(B,A)\} = \inf\{\varepsilon > 0 : \ A \subset \mathcal{O}_\varepsilon B \,, \ \ B \subset \mathcal{O}_\varepsilon A\} \,.$$

From the variety of continuity concepts we choose the most common and appropriate for our investigations. A multifunction $\varphi \colon X \multimap X$ (with closed values) is

- *graph-closed* (or Gr*-closed*) if and only if

$$\operatorname{Gr}(\varphi) = \{(x,y) \in X \times X : \ x \in X \ \ \& \ \ y \in \varphi(x)\}$$

  is closed in the product $X \times X$;

- *Hausdorff upper semicontinuous* (*h-u.s.c.* for brevity) if and only if

$$(\forall\, x_0)(\forall\, \varepsilon > 0)(\exists\, \delta > 0)(\forall\, x)\big(d(x,x_0) < \delta \implies e[\varphi(x), \varphi(x_0)] < \varepsilon\big) \,;$$

- *upper semicontinuous* (*u.s.c.* for brevity) if and only if

$$(\forall\, x_0)(\forall\, V \ \text{ open})$$
$$\Big(\varphi(x_0) \subset V \implies (\exists\, U \ \text{ open})\big[x_0 \in U \ \ \& \ \ (\forall\, x \in U)\big(\varphi(x) \subset V\big)\big]\Big) \,;$$

- *lower semicontinuous* (*l.s.c.* for brevity) if and only if

$$(\forall\, x_0)(\forall\, V \ \text{ open})$$
$$\Big(\varphi(x_0) \cap V \neq \emptyset \implies (\exists\, U \ \text{ open})\big[x_0 \in U \ \ \& \ \ (\forall\, x \in U)\big(\varphi(x) \cap V \neq \emptyset\big)\big]\Big) \,;$$

- *Hausdorff continuous* (or *h-continuous*) if and only if

$$(\forall\, x_0)(\forall\, \varepsilon > 0)(\exists\, \delta > 0)(\forall\, x)\Big(d(x,x_0) < \delta \implies h[\varphi(x), \varphi(x_0)] < \varepsilon\Big) \,;$$

- *uniformly Hausdorff upper semicontinuous* (*u.h-u.s.c.* for brevity) if and only if

$$(\forall\, A \ \text{ closed})(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\big(\varphi[\mathcal{O}_\delta A] \subset \mathcal{O}_\varepsilon \varphi(A)\big) \,;$$

- *uniformly Hausdorff continuous* (*u.h-continuous* for brevity) if and only if

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)(\forall\, x_1, x_2)\Big(d(x_1,x_2) < \delta \implies h[\varphi(x_1), \varphi(x_2)] < \varepsilon\Big) \,;$$

- *Lipschitz continuous* with constant L if and only if

$$(\forall\, x_1, x_2)\Big(h[\varphi(x_1), \varphi(x_2)] \leqslant L \cdot d(x_1,x_2)\Big) \,.$$

Some novelty seems to be the uniform Hausdorff upper semicontinuity. To enlighten its connection with the uniform Hausdorff continuity we present:

PROPOSITION 1. *The following conditions are equivalent:*

a) φ *is u.h-continuous;*

b) $(\forall\, \varepsilon > 0)(\exists\, \delta > 0)(\forall\, x_1, x_2)\big(d(x_1, x_2) < \delta \implies e[\varphi(x_1), \varphi(x_2)] < \varepsilon\big)$;

c) $(\forall\, \varepsilon > 0)(\exists\, \delta > 0)(\forall\, A)\big(\varphi(\mathcal{O}_\delta A) \subset \mathcal{O}_\varepsilon \varphi(A)\big)$.

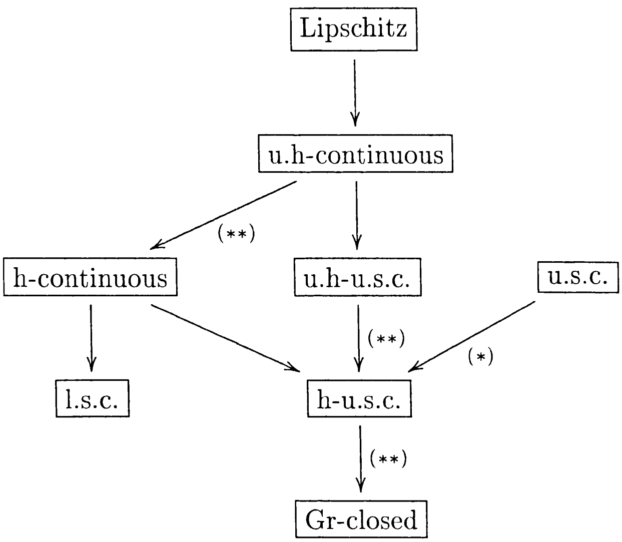
P r o o f. An easy reasoning can go through the scheme c) $\implies$ b) $\implies$ a) $\implies$ c). We only prove the last implication. Note, for $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi\big(\mathcal{O}_\delta\{a\}\big) \subset \mathcal{O}_\varepsilon \varphi(a)$. Summing up this inclusion along all $a \in A$ gives the desired formula. $\square$

Condition c) of the above proposition is not just a displacement of quantifier when confronted with the expression describing the uniform Hausdorff upper semicontinuity. We also omit the requirement on the set A to be closed. So, what would happen if in the definition of a uniformly Hausdorff upper semicontinuous multifunction one would abandon the closedness of the set A? Then the resulting type of continuity,

$$(\forall\, A)(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\big(\varphi[\mathcal{O}_\delta A] \subset \mathcal{O}_\varepsilon \varphi(A)\big)$$

(A is an arbitrary set, not necessarily closed), is stronger than the lower semicontinuity, and does not have to be satisfied by an upper semicontinuous multifunction on a compact space (see Example 1 and Proposition 2 to follow).

The hierarchy of introduced continuity concepts for multifunction $\varphi\colon X \multimap X$ with closed values is shown on the diagram:



$(*)$ — reverse implication true when φ has compact values,
$(**)$ — reverse implication true when φ is with a compact domain and with compact values.

One of the striking properties admitted by u.h-u.s.c. multifunctions provides:

Proposition 2. *Let X be a compact metric space and $\varphi\colon X \multimap X$ be h-u.s.c.. Then φ is u.h-u.s.c..*

Proof. Fix $\varepsilon > 0$ and A — closed (thus compact). Thanks to the Hausdorff upper semicontinuity, for any $x \in A$ there exists $\delta(x) > 0$ such that

$$\varphi[\mathcal{O}_{\delta(x)}\{x\}] \subset \mathcal{O}_\varepsilon \varphi(x).$$

By compactness of A the covering $\bigcup_{a\in A} \mathcal{O}_{\delta(a)}\{a\}$ has the associated Lebesgue number $\delta > 0$ (see [K], [E]), i.e.

$$(\forall\, a \in A)(\exists\, x \in A)\big(\mathcal{O}_\delta\{a\} \subset \mathcal{O}_{\delta(x)}\{x\}\big).$$

Hence

$$\begin{aligned}\varphi[\mathcal{O}_\delta A] &= \bigcup_{a\in A} \varphi[\mathcal{O}_\delta\{a\}] \subset \bigcup_{x\in A} \varphi[\mathcal{O}_{\delta(x)}\{x\}] \\ &\subset \bigcup_{x\in A} \mathcal{O}_\varepsilon \varphi(x) = \mathcal{O}_\varepsilon \varphi(A).\end{aligned}$$

This means the uniform Hausdorff upper semicontinuity of φ. □

Let us hint this is an analogue to the classical theorem: a continuous (multi)function on compactum is uniformly continuous. Additionally we offer an example of an u.h-u.s.c. multifunction.

Example 1. Let $X = [0,1]$ (with the Euclidean metric), $\varphi\colon X \multimap X$,

$$\varphi(x) = \begin{cases} \{0\} & \text{for } x < 1/2, \\ [0,1] & \text{for } x = 1/2, \\ \{1\} & \text{for } x > 1/2. \end{cases}$$

It is an u.s.c. map (hence h-u.s.c.) and by Proposition 2, it is even u.h-u.s.c.. But it is not l.s.c. (thus discontinuous). This shows the essentiality of closedness in the definition of an u.h-u.s.c. multifunction.

Let us list modes of convergence we shall have in mind. With a sequence of sets $\{A_n\}_{n=1}^\infty$ one associates ([K], [HP], [BG]):

- $\operatorname{Lim\,sup} A_n = \bigcap_m \bigcup_{n\geqslant m} A_n$ (*set-theoretic upper limit*);
- $\operatorname{Ls} A_n = \bigcap_m \overline{\bigcup_{n\geqslant m} A_n}$ (*topological upper limit*);
- $A_n \xrightarrow[n\to\infty]{\mathrm{e}} A \iff (\forall\,\varepsilon > 0)(\exists\, n_0)(\forall\, n \geqslant n_0)(A_n \subset \mathcal{O}_\varepsilon A)$ (*e-convergence*).

Convergence with respect to Hausdorff metric will be denoted by $A_n \xrightarrow[n\to\infty]{\mathrm{h}} A$ (*h-convergence*). Observe that e-convergence to a compact set coincides with convergence in upper Vietoris hypertopology (see [HP], [Ki]), for we have:

$$(\forall K \text{ compact})(\forall U \text{ open})\big(K \subset U \implies (\exists \varepsilon > 0)(K \subset \mathcal{O}_\varepsilon K \subset U)\big).$$

Also note that the e-convergence is weaker than the h-convergence and does not give a unique limit[3].

PROPOSITION 3. *There holds:*

(i) $\operatorname{Lim\,sup} A_n \subset \overline{\operatorname{Lim\,sup} A_n} \subset \operatorname{Ls} A_n = \operatorname{Ls} \overline{A_n} = \overline{\operatorname{Ls} A_n}$;

(ii) $\big(A_n \xrightarrow[n\to\infty]{\mathrm{e}} A$ *& A closed*$\big) \implies \operatorname{Ls} A_n \subset A$;

(iii) $A_n \xrightarrow[n\to\infty]{\mathrm{e}} A \subset B \implies A_n \xrightarrow[n\to\infty]{\mathrm{e}} B$;

(iv) $B_n \subset A_n \xrightarrow[n\to\infty]{\mathrm{e}} A \implies B_n \xrightarrow[n\to\infty]{\mathrm{e}} A$;

(v) $\big(A_n \xrightarrow[n\to\infty]{\mathrm{h}} A$ *&* $A_n \xrightarrow[n\to\infty]{\mathrm{e}} B$ *& A closed*$\big) \implies A \subset B$;

(vi) $\Big(\{A_n\}_{n=1}^{\infty}$ *decreasing sequence &* $A = \bigcap_n A_n$ *&* $A_n \xrightarrow[n\to\infty]{\mathrm{e}} A\Big)$
$\implies A_n \xrightarrow[n\to\infty]{\mathrm{h}} A$;

(vii) $\Big(\{A_n\}_{n=1}^{\infty}$ *decreasing sequence of nonempty compacts &* $A = \bigcap_n A_n\Big)$
$\implies A_n \xrightarrow[n\to\infty]{\mathrm{h}} A$;

(viii) X *— complete,* $\{A_n\}_{n=1}^{\infty}$ *— decreasing sequence of nonempty closed subsets of X with* $\mu(A_n) \xrightarrow[n\to\infty]{} 0$ *(μ denotes measure of noncompactness: Kuratowski's or Hausdorff's, see* [D], [HP]*),* $A = \bigcap_n A_n$ *imply* $A_n \xrightarrow[n\to\infty]{\mathrm{h}} A$.

P r o o f. For the first statement one can consult [HP] or [K]. For the second, fix $\varepsilon > 0$ and take n_0 such that $A_n \subset \mathcal{O}_\varepsilon A$ for all $n \geqslant n_0$. Hence

$$\bigcap_m \overline{\bigcup_{n\geqslant m} A_n} \subset \overline{\bigcup_{n\geqslant n_0} A_n} \subset \overline{\mathcal{O}_\varepsilon A} \subset \mathcal{O}_{2\varepsilon} A.$$

The ε being arbitrary causes $\operatorname{Ls} A_n \subset \bigcap_{\varepsilon>0} \mathcal{O}_{2\varepsilon} A = \overline{A} = A$.

[3]The upper Vietoris hypertopology is never Hausdorff, although it often is T_1.

The third, the fourth, the fifth and the sixth statements are obvious. The last two we prove together[4]. Either by the Riesz or by the Kuratowski Theorem (see [E], [K], [D]) we immediately see the nonemptiness and compactness of $A = \bigcap_n A_n$. Assume, on contrary, that for some open $U \supset A$, there exist infinitely many sets A_{n_j} for which $A_{n_j} \not\subset U$, i.e. $A_{n_j} \cap (X \setminus U) \neq \emptyset$ (closed). Defining $B_{n_j} = A_{n_j} \cap (X \setminus U)$ one again obtains a decreasing sequence of nonempty closed sets satisfying the hypothesis of Riesz's or Kuratowski's Theorem. Therefore $\bigcap_j B_{n_j} \neq \emptyset$. On the other hand:

$$\bigcap_j B_{n_j} = \bigcap_j A_{n_j} \cap (X \setminus U) = \bigcap_n A_n \cap (X \setminus U) = A \cap (X \setminus U) = \emptyset,$$

a contradiction. □

The e-convergence plays the leading role in our investigations. It is some kind of uniform approximation of an intersection, what illustrates the following example.

EXAMPLE 2. Let $X = (0,1] \times [0,1)$, $A = (0,1] \times \{0\}$, $A_n = \left(0, \frac{1}{n+1}\right] \times [0,1) \cup (0,1] \times \{0\}$, $n = 1, 2, \dots$. All sets $A, A_1, A_2, \dots$ are nonempty, closed and bounded. The family $\{A_n\}_{n=1}^{\infty}$ decreases and $A = \bigcap_n A_n$ holds. However, the sequence $\{A_n\}_{n=1}^{\infty}$ fails to converge with respect to excess e. It is caused by the incompleteness of X, which made possible to keep $e(A_n, A)$ positive and constant.

3. Attractors

The multifunction $\varphi\colon X \multimap X$ acts on sets in a natural way through the operation $A \mapsto \varphi(A)$. However, for some reasons (e.g. when φ does not preserve closedness or, at least, compactness) it is more convenient to consider the Barnsley-Hutchinson operator[5] $F\colon 2^X \setminus \{\emptyset\} \to 2^X \setminus \{\emptyset\}$ (2^X — the family of all subsets of X) generated by φ:

$$F(A) = \overline{\varphi(A)} \qquad \text{for all} \quad A \subset X, \ A \neq \emptyset.$$

To introduce the attractivity notion, being on the one hand strong enough, and on the other one flexible, we consider the following iterative procedure (e.g. [L]):

$$F^n(A) = \begin{cases} \overline{\varphi(A)}, & \text{when } n = 1, \\ \overline{\varphi[F^{n-1}(A)]}, & \text{when } n > 1. \end{cases}$$

[4]The main idea is taken from [Ki; Chap. 1, Theorem 1].

[5]Of course we implicitly assume that a multifunction has nonempty values.

This is nothing else but the usual iteration of the Barnsley-Hutchinson operator F. Henceforth, F will always mean the Barnsley-Hutchinson operator associated with φ.

The main object of our study — an attractor — can be given now an appropriate definition. We say that the set M *attracts* $A \subset X$ under $\varphi : X \multimap X$, whenever $F^n(A) \xrightarrow[n\to\infty]{\mathrm{e}} M$ (comp. [H]). We call a closed set M an *attractor*, if M attracts all sets i.e.

$$(\forall\, A \subset X)\big(F^n(A) \xrightarrow[n\to\infty]{\mathrm{e}} M\big) \tag{At}$$

and M is minimal with respect to the property (At).

Recall (comp. [L], [LM]) a set A is *invariant* (resp. *subinvariant*, *superinvariant*) if the relation $\varphi(A) = A$ (resp. $\varphi(A) \subset A$, $\varphi(A) \supset A$) holds. By a set *superinvariant in closure* we shall understand a set A for which $A \subset \overline{\varphi(A)}$. Superinvariance naturally implies superinvariance in closure. Analogously, A is *invariant in closure* if $A = \overline{\varphi(A)}$.

The ω*-limit set* of A is recognized as follows: $\omega(A) = \operatorname{Ls} \varphi^n(A)$ (comp. [H]); however, we will not invoke this notion explicitly.

We would also like to point out that if additionally the multifunction φ is l.s.c., then instead of $\{F^n(A)\}_{n=1}^{\infty}$ a more natural iteration process $\{\varphi^n(A)\}_{n=1}^{\infty}$ can be used to define attractivity. Indeed, the equality $F^n(A) = \overline{\varphi^n(A)}$ makes this possible (comp. [LM]).

PROPOSITION 4. *An attractor M under φ contains all sets superinvariant in closure: $A^* \subset \overline{\varphi(A^*)} \implies A^* \subset M$.*

P r o o f. Let $A^* \subset \overline{\varphi(A^*)} = F(A^*)$. By induction $A^* \subset F^n(A^*)$ for all n. Since $F^n(A^*) \xrightarrow[n\to\infty]{\mathrm{e}} M$,

$$A^* \subset \operatorname{Ls} F^n(A^*) \subset M\,,$$

thanks to Proposition 3(i),(ii). □

We use the above observation in the next important proposition.

PROPOSITION 5. *An attractor M under u.h-u.s.c. φ is a maximal one among all sets superinvariant in closure. Moreover, an attractor M is (maximal) invariant set in closure, $M = \overline{\varphi(M)}$.*

P r o o f. Fix $\varepsilon > 0$ and take $\delta > 0$ such that $\varphi(\mathcal{O}_\delta M) \subset \mathcal{O}_{\varepsilon/2}\varphi(M)$. Consider any set A. Since M is an attractor,

$$(\exists\, n_0)(\forall\, n \geqslant n_0)\big(F^n(A) \subset \mathcal{O}_\delta M\big)\,.$$

Thus

$$F^{n+1}(A) = \overline{\varphi\big(F^n(A)\big)} \subset \overline{\varphi(\mathcal{O}_\delta M)} \subset \overline{\mathcal{O}_{\varepsilon/2}\varphi(M)} \subset \mathcal{O}_\varepsilon \overline{\varphi(M)}\,,$$

and this means $\overline{\varphi(M)}$ attracts A. By minimality of M with respect to attraction, we get inclusion $M \subset \overline{\varphi(M)}$ (superinvariance in closure).

Containing all sets superinvariant in closure (Proposition 4) M must be maximal. Therefore M is also invariant in closure and consequently maximal (this time among sets invariant in closure). To see this, let M be maximal with the property $M \subset \overline{\varphi(M)}$. Hence $\overline{\varphi(M)} \subset \overline{\varphi\big[\overline{\varphi(M)}\big]}$, i.e. $\overline{\varphi(M)}$ is superinvariant in closure. M being maximal, it has to contain $\overline{\varphi(M)}$. Altogether, $M = \overline{\varphi(M)}$. □

Since the inclusion "$\subset$" is not a total order (there are incomparable sets), it is worth noticing that all sets (super)invariant in closure are inside an attractor. And it could not be the other way, because if $M_i = \overline{\varphi(M_i)}$, $i = 1,2$, both maximal, then $M_1 \cup M_2 = \overline{\varphi(M_1)} \cup \overline{\varphi(M_2)} = \overline{\varphi(M_1 \cup M_2)}$ would be greater, a contradiction. This means the uniqueness of an attractor[6].

If an attractor M is compact and φ preserves compactness (e.g. the Lipschitz map with compact values), then M is invariant, $M = \overline{\varphi(M)} = \varphi(M)$, and maximal amongst invariant sets. Because φ does not have to preserve closedness (recall nonexpansive arctan), we were led to the concept of superinvariance in closure.

After these deliberations, one may ask, whether attractors do exist? The general result on existence exploits a compactness-like assumption on multifunction (comp. [L]) and reads as follows (μ denotes either Kuratowski's or Hausdorff's measure of noncompactness, see [D], [HP]):

THEOREM 1. *Suppose X is complete and a multivalued function $\varphi\colon X \multimap X$ satisfies:*

(i) *φ is a μ-contraction, i.e. for some $L < 1$ and all $A \subset X$*

$$\mu\big[\varphi(A)\big] \leqslant L \cdot \mu(A)\,;$$

(ii) *φ is bounded, i.e. $\mu\big[\varphi(X)\big] < \infty$.*

Let F be the Barnsley-Hutchinson operator corresponding to φ. Then the set $\bigcap\limits_{n=1}^{\infty} F^n(X)$ is the compact attractor to φ.

Proof. Denote $M = \bigcap\limits_{n=1}^{\infty} F^n(X)$. A decreasing family of closed nonempty sets $\{F^n(X)\}_{n=1}^{\infty}$ has the property $\mu\big[F^n(X)\big] \underset{n\to\infty}{\longrightarrow} 0$, because φ is a bounded μ-contraction. By Proposition 3(viii) the set M is nonempty, compact and $F^n(X) \underset{n\to\infty}{\overset{\mathrm{h}}{\longrightarrow}} M$.

[6]Remember that the minimality assumption and global attractivity appear together in the definition of an attractor.

For any $A \subset X$ we have $F^n(A) \subset F^n(X)$, so $F^n(A) \xrightarrow[n\to\infty]{\mathrm{e}} M$ (Proposition 3(iv)); thus M attracts every set. There is no smaller closed set which could attract all sets. Indeed, assume N is a closed attracting set. By Proposition 3(v) we know that $M \subset N$, what completes the proof. □

Remark that the set playing above the role of an attractor can be seen as some variant of the core (= center) to multifunction ([GN]).

In [L] we have proved the existence of (nonempty) maximal invariant set for φ which is bounded condensing only[7]. So here a question arises: what about our attractor from the just presented theorem? Is it a maximal invariant? Undoubtedly, the attractor given here is subinvariant. To assure its invariance one needs to make additional assumption on φ, it shall be u.h-u.s.c. (comp. [Ki; Chap. 1, Theorem 1]). That is the main reason why we have introduced a class of the uniformly Hausdorff upper semicontinuous maps.

4. Examples of attractors. Problems

Here we briefly discuss the relevance of abstract theory on two naturally arisen examples. Then some questions are stated.

EXAMPLE 3. Let $\varphi = \{f\}$, $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{3}x$. We know very well that $\varphi^n(A) \xrightarrow[n\to\infty]{\mathrm{e}} \{0\}$ (even in the Hausdorff metric), however for bounded sets A only. When all sets are considered, the attractor is $\mathbb{R}$. Of course, in the classic Barnsley-Hutchinson approach, $\{0\}$ is the attractor to our IFS. But yet, restricting φ to some bounded subinvariant set containing 0, both concepts (ours and the classic one) coincide.

EXAMPLE 4. Let $g_0, g_1\colon [0,1] \to [0,1]$, $g_0(x) = \frac{1}{3}x$, $g_1(x) = \frac{1}{3}x + \frac{2}{3}$. Denote by $[x]$ the integer part of x (i.e. $[x] = \max\{n \in \mathbb{Z} :\ n \leqslant x\}$) and define $f_i\colon \mathbb{R} \to \mathbb{R}$, $i = 0,1$, as follows:

$$f_i(x) = g_i\big(x - [x]\big) + [x].$$

We know that IFS $\{g_i :\ i = 0,1\}$ produces the ternary Cantor set in $[0,1]$. Taking into account the geometrical meaning of f_i we see the attractor to IFS $\{f_i :\ i = 0,1\}$ in $\mathbb{R}$: it is the multiplied copy of Cantor's set, each one belonging to its own integer interval. We cannot expect to obtain this attractor via an iterative procedure which starts at an arbitrarily chosen point/set. The attractivity (e-convergence) is weaker than the convergence in the Hausdorff metric.

The conclusion from the above examples is threefold.

[7] Each μ-contraction is μ-condensing.

The first: even classical attractors are boundedly asymptotic. After restriction to a bounded subinvariant set everything coincides.

The second: there exist attractors asymptotically stable for all (bounded and unbounded) sets.

The third: there are unbounded attractors (sometimes such trivial as the whole space, and sometimes rather complicated as multiplied Cantor set).

At the end we propose some problems, which seem to be very natural:

(A) Which closed [compact] sets are attractors to an iterated (multi)function system?

(B) Can the "contractivity-compactness-condensity barrier" be crossed out? For example, what about multifunctions with the Lipschitz constant greater than 1?

(C) What (physical) interpretation can so generalized attractors be endowed with?

Very trivial solution to (A) is offered by taking as whole space X the desired set to be the attractor, and IFS $\{f\}$ consisting of exactly one function $f\colon X \to X$, $f(x) = x$ (identity). So one is asked: Which subsets of a fixed space X are attractors? Especially, which are attractors for hyperbolic IFS?

Question (B) has been raised by J. Andres and L. Górniewicz [AG] in context of the so-called Lifshits Theorem. Unfortunately the expectation is that the Lifshits constant equals 1 for most interesting hyperspaces.

Question (C) was indicated to me by T. Schreiber (N. Copernicus Univ., Toruń).

Acknowledgment

I express my gratitude to Professors J. Andres and L. Górniewicz for fruitful discussions. I would also like to thank B. Kieninger, who sent me his thesis.

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Added in proof

1. The uniqueness of an attractor M is implicitly used when proving Proposition 5. Moreover, the attractor — if exists — has always the form $M = \bigcap\limits_n F^n(X)$. This follows from the following lemma.

LEMMA 1. *For $r \geqslant \eta > 0$ and $A, B \subset X$ we have*

$$\mathcal{O}_\eta A \cap \mathcal{O}_\eta B \subset \mathcal{O}_\eta\big(A \cap \mathcal{O}_{2r} B\big).$$

2. As was observed in [COZ], Proposition 3(viii) holds true for any abstract measure of noncompactness μ satisfying e.g.

$$(\textit{regularity}) \qquad \mu(A) = 0 \iff \overline{A} \text{ is compact,}$$

and

$$(\textit{ultra-additivity}) \quad \mu(A \cup B) = \max\{\mu(A), \mu(B)\}.$$

3. Problem (B) from Section 4 has been solved in [L2].

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Received November 22, 2002

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