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STABLE AND NON-STABLE NON-CHAOTIC MAPS OF THE INTERVAL

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ABSTRACT. We have proved that :

1. Every continuous piecewise monotonic non-chaotic function is stable.

2. For any continuous piecewise monotonic function with zero entropy, every infinite ω-limit set is perfect.

1. Introduction

This paper studies *one-dimensional maps of the interval I*. Such a map is often used as a growing model of some biological population. Two questions arise naturally. When the map is *chaotic* and how large can be the *scrambled set* of such a map?

It is known that such a map can be of the positive Lebesque measure ([8] among others) but it cannot be residual on any subinterval of I ([2]).

When the map is *non-chaotic*, it is interesting *how stable such a predictable behaviour under the influence of the small perturbations is*. We will answer this question for a large class of mappings.

Denote by $C(I,I)$ the *class of the continuous mappings of the real compact interval I to itself.* The *trajectory of* x $(\in I)$ is the sequence $\{f^n(x)\}_{n=0}^{\infty}$ denoted also as $\{x(n)\}_{n=0}^{\infty}$ and the ω*-limit set* $\omega_f(x)$ of x is the set of all limit points of this trajectory.

The *set of all periodic points* is denoted by $\mathrm{Per}(f)$ and $\mathrm{h}(f)$ is the *topological entropy of a map* f.

There is the following *classification of functions* $f \in C(I,I)$ ([7], [4]):

For any $f \in C(I,I)$ one of the following conditions holds:

(i) All trajectories are approximable by cycles, i.e. for all x and $\varepsilon > 0$ there exists a periodic point p such that $\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \varepsilon$.

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(ii) For some $\varepsilon > 0$ there exists an ε-scrambled set S (non empty and perfect), i.e. such a set that for all $x, y \in S$, $x \neq y$, and all $p \in \mathrm{Per}\, f$

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \varepsilon$$
$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$$
$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| > \varepsilon.$$

The function f possessing property (ii) is called *ε-chaotic*.

In the present paper we will investigate the stability of continuous functions possessing the property (i). In the whole paper only continuous functions will be considered and if we say "function f" we will in fact mean "continuous function f".

Now we give

1.1. Definition. *A non-chaotic function f is stable if for all $\varepsilon > 0$ all trajectories of every function g sufficiently near to f are ε approximable by cycles, i.e., for every x there is a $p \in \mathrm{Per}(g)$ such that*

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \varepsilon.$$

The main aim of this paper is to prove the following

1.2. Theorem. *Every piecewise monotonic non-chaotic function is stable.*

This result is completed by

1.3. Theorem. *There exists a non-chaotic non-stable map.*

We will also prove a useful and non-trivial

1.4. Theorem. *For any piecewise monotonic function with zero topological entropy, every infinite ω-limit set is perfect.*

In Section 2 we recall several useful theorems known from the literature, then in Section 3 we prove Theorem 1.4 and in Section 4 we prove Theorem 1.2. Finally in Section 5 we show an example of a non-chaotic non stable map.

2. Preliminary notions and constructions

2.1. Theorem. *A non-chaotic function $f \in C(I, I)$ is stable if and only if the following conditions hold:*

(i) *$\mathrm{Per}(f)$ is nowhere dense (or equivalently $\mathrm{Per}(f)$ is of the first Baire category, or $\mathrm{Per}(f)$ does not contain an interval).*

(ii) *For any infinite* ω *-limit set* $\omega_f(x)$ *and any* $n \geq 0$ *there exists a system* $\mathcal{I}_n = \{I_n^i;\ 1 \leq i \leq 2^n\}$ *of periodic intervals such that*

$$\omega_f(x) = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} I_n^i.$$

P r o o f. [6], Theorem 2.2.

2.2. Definition. *Two points* $u, v \in I$ *are* f *-separable if there are disjoint periodic intervals* $J_u, J_v \subset I$ *with* $u \in J_u$, $v \in J_v$. *Otherwise* u, v *are* f *-nonseparable.*

2.3. Theorem. *Let* $f \in C(I, I)$. *The following conditions are equivalent:*

(i) *For some* $\varepsilon > 0$, f *has a nonempty perfect* ε *-scrambled set.*
(ii) f *has an infinite* ω *-limit set* $\omega_f(x)$ *containing two* f *-nonseparable points.*

P r o o f of this theorem is a part of the proof of the Theorem 3 in [4].

2.4. Theorem. *For* $f \in C(I, I)$ *the following conditions are equivalent:*

(i) f *has a cycle of the order not a power of two.*
(ii) f *has an infinite* $\omega_f(x)$ *containing a cycle.*

P r o o f. [9]; a shorter new proof can be found in [5].

2.5. Definition. *An interval* J *is called a wandering interval if* $J, f(J)$, $f^2(J), f^3(J), \ldots$ *are disjoint and no point from* J *is asymptotically periodic.*

2.6. R e m a r k. There are some different definitions of this notion but for our purposes this one is the most suitable, mentioned among others by van S t r i e n ([13]).

2.7. Theorem. *If* $f \in C(I, I)$ *is of the type* 2^∞ *(i.e. possessing cycles of period* 2^n *for all* $n \geq 0$ *and no others) and* $\omega_f(x)$ *is infinite, then there are two possibilities:*

(i) $\omega_f(x)$ *is a Cantor-like set (nowhere dense without isolated points).*
(ii) $\omega_f(x) = C \cup \{a_n\}$, *where* C *is a Cantor-like set and* $\{a_n\}$ *are isolated points of the set* $\omega_f(x)$.

In the second case every interval J *contiguous to* C *contains at most two points* a_i, a_j. *If* J *contains at least one point* a_i, *then* J *contains a periodic point (otherwise* J *is a wandering interval and* $a_i \notin \omega_f(x)$). *For every* a_n *there exists exactly one point* $b_n \in C$ *such that* $U_n = [a_n, b_n]$ *(or* $U_n = [b_n, a_n]$) *is a wandering interval and for any* $y \in U_n$, $\omega_f(y) = C$.

P r o o f. See [10].

2.8. Corollary. *Keeping the notation as above, we have $f^k(x) \notin U_n$ for any $k \geq 0$ and $n \geq 0$.*

P r o o f. Let $f^k(x) \in U_n$. Denote by $y = f^k(x)$. Then $C = \omega_f(y) = \omega_f(f^k(x)) = \omega_f(x) = C \cup \{a_n\}$, which is a contradiction. ◘

2.9. Definition. *We call each U_n an organic wandering interval and let $U(x) = \{U_n\}$ be the set of such intervals for the infinite $\omega_f(x)$. Now let $U(f) = \bigcup U(x)$ be the union of such sets over all x with infinite $\omega_f(x)$.*

2.10. Theorem. *Let f be of the type 2^∞. For every infinite $\omega_f(x)$ there exists a system $\{K_n\}_{n=0}^{\infty}$, $K_n = \bigcup\{J_i^n;\ 0 \leq i \leq 2^n - 1\}$ where J_i^n are closed intervals such that*

(i) *$\omega_f(x) \subset K_n = J_0^n \cup \cdots \cup J_{2^n-1}^n$, and the endpoints of J_i^n are contained in the $\omega_f(x)$.*

(ii) *$K_{n+1} \subseteq K_n$, $\bigcap\limits_{n=0}^{\infty} K_n = \omega_f(x) \cup D$; D consists of the wandering intervals.*

(iii) *for any i, $f^{2^n}(\omega_f(x) \cap J_i^n) \subset J_i^n$*

(iv) *$f(J_i^n) \supset J_{i+1}^n \quad (\bmod 2^n)$*

(v) *if we define $L_n = \{x;\ f^{2^{n-1}}(x) = x\}$, then $L_n \subset (I \setminus K_n)$ and between every neighbouring J_i^n and J_j^n there is at least one $x \in L_n$.*

P r o o f. This theorem is a straightforward consequence of Theorem 3.5 in [7].

In the sequel, we will use the technique of blowing-up the orbits (see [3] among others) introduced by D e n j o y [1].

2.11. Definition. *Let* $\mathrm{orb}(x) = \{y;\ f^n(x) = f^m(y),\ m, n \in \mathbf{N}\}$ *(the orbit of x), and* $\mathrm{borb}(x) = \{y;\ f^n(y) = x,\ n \in \mathbf{N}\}$ *(the backward orbit of x).*

From a given function $f \in C(I, I)$ we can con truct a new function $g \in C(I, I)$ in the following way: If $\mathrm{orb}(x)$ is countable, we replace every $v \in \mathrm{orb}\ x)$ by a compact interval I_v in uch a way that $g(I_v) = I_{f(v)}$, the remaining traj c- tories of other points ar unchanged. In other words we define a nondecreasın function $\tau \in C(I, I)$ such that $\tau(z) = y$ for all $z \in I_y$ (such a function clearly exists). Thus $f \circ \tau = \tau \circ g$.

2.12. R e m a r k If $\mathrm{orb}(x)$ contain an interval, ı e. $f(J)$ y, $y \in \mathrm{orb}\ x)$, then we constru t the function g on J in such a way that $g(J) - I_y$. Thus $\tau / J = \mathrm{Id}$.

With thi modıfication we can blow-up every $\mathrm{orb}(x)$ of a piecewi e monotonıc function.

Now, from Theorem 2.4 and Theorem 2.10 we have the following

2.13. Corollary. *If f is of the type 2^∞ and $\omega_f(x)$ is infinite, then blowing-up a countable orbit of any $y \in \omega_f(x)$ yields a function g with wandering intervals.*

2.14. Definition. *Take any $g \in C(I,I)$ of the type 2^∞. Denote by S the set of all not organic wandering intervals of g. The function $f \in C(I,I)$ of the type 2^∞ is called the basic function for function g if there exists a nondecreasing function $\tau \in C(I,I)$ such that*

(i) $f \circ \tau = \tau \circ g$

(ii) *the sets $\tau(S)$ and $\tau(\mathrm{Per}(g))$ contain no interval*

(iii) *for every $U_i \in U(g)$ there exists $V_j \in U(f)$ such that $\tau(U_i) = V_j$.*

A function of the type 2^∞ is called a basic function if it is the basic function for itself.

2.15. Remark. We will use the technique of blowing-up the orbits only for piecewise monotonic basic functions for which the interval of monotonicity cannot be a wandering non-organic interval or an interval of periodic points.

As we will see only these cases are important for the dynamics of our system.

3. Proof of Theorem 1.4

It is known that a function f with $\mathrm{h}(f) = 0$ is either of the type 2^∞ or of the type 2^n for some n (i.e. f has cycles of order $1, 2, \ldots, 2^n$ and no others). In the last case every ω-limit set is a periodic orbit (cf. [11]).

So it suffices to consider functions of the type 2^∞. Assume that such a function f has the infinite $\omega_f(x)$ with isolated points. Let (in notation of Theorem 2.10) $J_i^n = [z_i^n, y_i^n]$. Comparing Theorem 2.7 (ii) and Theorem 2.10 (i) one can easily prove that there exist n and i, $0 \le i \le 2^n - 1$, such that z_i^n or y_i^n is an isolated point of $\omega_f(x)$. Without loss of generality assume that z_i^n is such a point.

For a fixed n denote by $[b_i, d_i]$ $(= [b_i^n, d_i^n])$ the minimal closed interval which contains $\omega_f(x) \cap J_i^n \setminus \{z_i\ (= z_i^n), y_i\ (= y_i^n)\}$. It is easy to see that $[z_i, b_i]$ is an organic wandering interval. According to Corollary 2.8

$$x(k) \notin [z_i, b_i] \qquad \text{for any} \qquad k \in \mathbf{N}. \tag{1}$$

Since $\omega_f(x) \cap [z_i, b_i] \neq \emptyset$ and z_i is the limit point of $\{x(k)\}$ there must be a sequence of integers $\{k(j)\}_{j=1}^\infty$ such that $x(k(j)) \in J_{i-1}^n$ and $x(k(j+1)) < z_i$ (see(1)). Since $f(y_{i-1})$ and $f(z_{i-1})$ belong to J_i^n (Theorem 2.10 (i),(iv)) there exists a critical point $c \in \mathrm{int}\, J_{i-1}^n$ such that $f(c) < z_i$.

Set $M = \{y \in J_i^n;\ f(y) < z_i\}$. Then $c \in M$ and it is easy to see that $M \cap \omega_f(x) = \emptyset$. If we denote $n(1) := n$, then there exists $n(2) > n(1)$ such that c will be contained in $K_{n(1)} \setminus K_{n(2)}$ and there exists j $(0 \leq j \leq 2^{n(2)} - 1$ such that $z_j^{n(2)}$ or $y_j^{n(2)}$ is an isolated point of $\omega_f(x)$. We can repeat the same argument for $n(2)$.

Set $c(1) := c$. By induction we obtain an increasing sequence $\{n(i)\}_{i=1}^{\infty}$ of integers and an infinite set $\{c(i)\}_{i=1}^{\infty}$ of critical points with $c(i) \notin K_{n(i)} \setminus K_{n(i+1)}$, hence a contradiction. ∎

4. Proof of Theorem 1.2

The proof of Theorem 1.2. is based on some propositions and lemmas.

4.1. Proposition. *Let $f \in C(I, I)$ be of the type 2^{∞}, let $\omega_f(x)$ be an infinite ω-limit set and let $y, z \in \omega_f(x)$. Then the following conditions are equivalent:*

(i) *Points y, z are f-nonseparable (according to Theorem 2.3 f is chaotic).*

(ii) *Interval $[y, z]$ does not contain any periodic point.*

Proof.

(i) ⇒ (ii) Let $y, z \in \omega_f(x)$ be f-nonseparable and assume that there exists a periodic point p of order 2^{n-1} in the interval $[y, z]$. Then by Theorem 2.10 one can easily find intervals $H_i^n \supset J_i^n$, $H_j^n \supset J_j^n$ of period 2^n such that $y \in H_i^n$, $z \in H_j^n$. Since $p \notin H_i^n \cup H_j^n$, we have $H_i^n \cap H_j^n = \emptyset$.

(ii) ⇒ (i) Let $y, z \in \omega_f(x)$ be f-separable; let H_y, H_z be periodic intervals containing y and z, respectively, such that $H_y \cap H_z = \emptyset$. Then for some $k \geq 0$

$$f^k(H_y) = H_y \qquad \text{and} \qquad f^k(H_z) = H_z\,. \tag{2}$$

Assume $y < z$. Denote by u the right endpoint of the interval H_y and v the left endpoint of the interval H_z. From (2) we obtain $f^k(u) < u$, $f^k(v) \geq v$. By the continuity of f there is a point $p \in [u, v]$ with $f^k(p) = p$ and since by Theorem 2.4 $y, z \notin \mathrm{Per}(f)$, we have $p \in (y, z)$. ∎

4.2. Definition. *If $y \in \omega_f(x)$ is both the left and the right-hand limit point of the set $\omega_f(x)$ then y is called a point interior relative to $\omega_f(x)$. Otherwise $y \in \omega_f(x)$ is an endpoint of some interval contiguous to $\omega_f(x)$ (= one sided limit of $\omega_f(x)$) or an isolated point of $\omega_f(x)$ and we will call it a point exterior relative to $\omega_f(x)$.*

4.3. Proposition. *Take an arbitrary basic non-chaotic function f. Let $y \in \omega_f(x)$ and let $\mathrm{orb}_f(y)$ be countable. We blow-up $\mathrm{orb}_f(y)$ and obtain a function g.*

(i) *If $\mathrm{orb}_f(y)$ is such that there exists $z \in \mathrm{orb}_f(y)$ interior relative to $\omega_f(x)$, then g is chaotic.*

(ii) *If g has no infinite ω-limit set with isolated points and if all $z \in \mathrm{orb}_f(y) \cap \omega_f(x)$ are exterior relative to $\omega_f(x)$, then the function g is non-chaotic.*

Proof.

(i) Consider a point $z \in \omega_f(x)$ interior relative to $\omega_f(x)$ and the corresponding interval I_z. Then $I_z = [z(1), z(2)]$ is a wandering interval (Corollary 2.13) of the function g and $z(1), z(2) \in \omega_g(x)$. Since $I_z \cap \mathrm{Per}(g) = \emptyset$, Proposition 4.1 and Theorem 2.3 imply that g is chaotic.

(ii) Denote by $\omega_g(z)$ an infinite ω-limit set of g such that $\tau \circ (\omega_g(z)) = \omega_f(x)$. Since $\omega_g(z)$ is a Cantor-like set and all points of $\mathrm{orb}_f(y) \cap \omega_f(x)$ are exterior relative to $\omega_f(x)$, we see that $\omega_g(z) \cap I_v = \{v\}$ for any $v \in \mathrm{orb}_f(y) \cap \omega_f(x)$. Since f is non-chaotic, by Proposition 4.1 there exists a periodic point p between any two different $u, v \in \omega_f(x)$ and, clearly, this conclusion holds also for every $u, v \in \omega_g(z)$. Thus g is non-chaotic. ■

4.4. Remark. According to Theorem 1.4 g has no organic wandering interval and therefore f has no wandering interval at all. Then Theorem 2.7 implies the Cantor-like structure of any infinite $\omega_f(x)$. In such a case it is easy to see that intervals J_i^n from Theorem 2.10 are such that

$$\omega_f(x) = \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{2^n-1} J_i^n.$$

Theorem 2.10 (v) and the definition of the basic function imply that every interval contiguous to $\omega_f(x)$ contains a periodic point and f has no interval of periodic points.

4.5. Proposition. *Any piecewise monotonic function f with infinite $\omega_f(x)$ without isolated points is non-chaotic stable.*

Proof. In fact, we have proved the stability of f in Remark 4.4; it suffices to consider Theorem 2.1.

Since every contiguous interval contains a periodic point, $[z, y] \cap \mathrm{Per}(f) \neq \emptyset$ for any $z, y \in \omega_f(x)$. Theorem 2.3 and Proposition 4.1 imply that f is non-chaotic. ■

4.6. Definition. *An endpoint of* J_i^n *(notation of Theorem* 2.10) *is of the* m*th level if it is an endpoint of* J_i^m *for some* i, *but is an endpoint of no* J_j^k *for* $k < m$.

4.7. Lemma. *An endpoint of* J_i^n *is mapped by a basic piecewise monotonic function* f *onto an endpoint of some* J_j^m.

Proof. Without loss of generality assume that the left endpoint a of J_i^n is mapped onto some point which is interior relative to $\omega_f(x)$. If f is a homeomorphism on J_i^n, then the endpoints of J_i^n will be mapped to the endpoints of J_{i+1}^n. Thus f cannot be a homeomorphism on J_i^n and there exists a critical point $c \in \operatorname{int} J_i^n$.

Denote by $c(0)$ the critical point nearest to a. (Such a $c(0)$ does exist, because f is piecewise monotonic). Since f is a basic function, $\omega_f(x)$ is a Cantor-like set and it is easy to see that there exists an interval $J_j^m \subset [a, c(0)]$, $a \in J_j^m$ (cf. Remark 4.4). But then f is a homeomorphism on J_j^m and maps its endpoints to endpoints of J_{j+1}^m and thus $f(a)$ is an endpoint of J_{j+1}^m - a contradiction. ∎

4.8. Lemma. *Every endpoint is mapped onto an endpoint of the same or higher level with a finite number of exceptions, when the level of the image is lower than that of the preimage.*

Proof. If a is an endpoint of the level n, $f(a) = b$, and b is of the mth level with $m < n$, then a is the interior point of J_j^m for some j. (Since $n > m$, a is an endpoint of no J_j^m). Since $f(J_j^m) \supset J_{j+1}^m$ and $b = f(a)$ is the endpoint of J_{j+1}^m, there must exist a critical point c such that either $c = a$ or c is contained in the contiguous interval of mth level, one endpoint of which is a.

Take some other $b(1) = f(a(1))$, $b(1)$ of the level $m(1) > m$ and repeat the same construction. Then clearly the point c will get out of our attention and thus one can easily prove that there is a one to one correspondence between the critical points and exceptions from our "rule". Thus the number of exceptions has to be finite. ∎

4.9. Proposition. *Let* f *be a piecewise monotonic basic function and let* $\omega_f(x)$ *be infinite. There is no such a* $y \in \omega_f(x)$, *that* $\operatorname{orb}(y) \cap \omega_f(x)$ *contains only endpoints of the intervals* J_i^n *which cover* $\omega_f(x)$ *according to Theorem* 2.10.

Proof. Suppose $y \in \omega_f(x)$ and $\operatorname{orb}(y) \cap \omega_f(x)$ contains only the endpoints of the intervals J_i^n. Let $\{y_{-n}\}_{n=0}^{\infty} \subset \omega_f(x)$ be such a sequence that $f(y_{-n-1}) = y_{-n}$ and $y_0 = y$. Let k be the number of exceptions for the function f according to Lemma 4.8. If a is of the nth level, and $f(a)$ is of the mth level, $m < n$, then we say that number $n - m$ is the *depth of a jump* (for $a \to f(a)$).

Denote by p the maximal depth of jumps for the function f. Assume that y_{-s} has the level q. Then for m, the level of y_0, we have the following estimate: $m \geq q+(s-k)-kp$, hence $q \leq m+k(p+1)-s$. Thus for every $s > m+k(p+1)$, the level of y_{-s} is less than zero and this is impossible. ∎

P r o o f o f t h e T h e o r e m 1.2. According to Proposition 4.9 for any $y \in \omega_f(x)$ the set $\text{orb}(y) \cap \omega_f(x)$ contains at least one point interior relative to $\omega_f(x)$ and according to Proposition 4.3 by blowing-up the orbit of any such y we obtain the chaotic function. Thus if we take any piecewise monotonic function g of the type 2^∞, g can be non-chaotic stable (if g is the basic function - cf. Proposition 4.5) or chaotic (if g is not the basic function we can obtain it by blowing-up several orbits of the corresponding basic function f). There is no place for non-chaotic non-stable maps in this class. ∎

5. Proof of Theorem 1.3

In the last part of this paper we show an example of a non-chaotic non-stable map.

Define the function $f : [0,1] \to [0,1] = I$

$$f(x) = \begin{cases} x + \frac{2}{3} & x \in [0, \frac{1}{3}] \\ -\frac{7}{3}x + \frac{16}{9} & x \in [\frac{1}{3}, \frac{2}{3}] \end{cases}$$

and for $x \in [\frac{2}{3}, 1]$ in the following way

$$f \mid [\tfrac{2}{3}, 1] = \tau \circ (f \mid [0,1]) \circ \tau^{-1} \qquad \text{where} \qquad \tau(x) = \tfrac{1}{3}x + \tfrac{2}{3}$$

(see Figure 1).

Note that every point has at most three preimages and therefore we can use the technique of blowing-up the orbits.

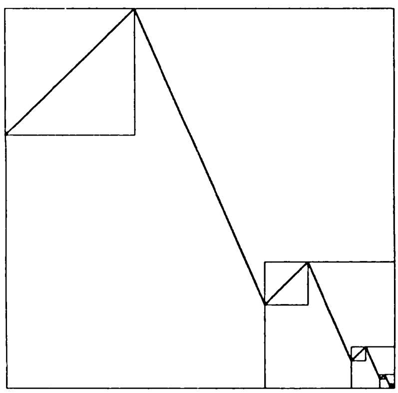

Fig. 1

5.1. Remark. The same map was recently studied also in [12], but from a different point of view and for a different purpose.

We show that f is non-chaotic stable (Proposition 5.2) and that there exists y such that $\mathrm{orb}(y) \cap \omega_f(x)$ contains only points exterior relative to $\omega_f(x)$ (Proposition 5.3). Blowing-up this orbit gives a function g. Proposition 5.4 shows that g is non-stable and Proposition 5.5 shows that g is non-chaotic.

5.2. Proposition. *The function f is non-chaotic stable.*

Proof. Denote $I_n = [1-\frac{1}{3^n}, 1]$, $I_n \cup f(I_n) \cup \cdots \cup f^{2^n-1}(I_n) = K_n$, $I \setminus K_n = N_n$ for all $n \geq 0$ and $\omega(f) = \bigcup_{x \in I} \omega_f(x)$. Then

(i) I_n is a periodic interval with the period 2^n and $f^{2^n}(I_n) = I_n$,

(ii) $\omega(f) \cap N_n = \bigcup_{i \leq n} \mathrm{orb}(p_i)$, p_i is a periodic point of the period 2^{n-1}, and for any $x \in N_n \setminus \mathrm{Per}(f)$ there is s such that $f^s(x) \in K_n$,

(iii) $f^{2^n} \mid I_n, \ldots, f^{2^n} \mid f^{2^n-1}(I_n)$ are linearly conjugate to f,

is clearly true for $n = 1$; for $n > 1$ use induction.

We see that I_n, $f(I_n), \ldots, f^{2^n-1}(I_n)$ are closed intervals of the nth level in the construction of a Cantor set C. Thus

$$C = \bigcap_{n=1}^{\infty} K_n. \tag{3}$$

Now we prove short

Claim. $\omega(f) = C \cup \mathrm{Per}(f)$.

Proof. Denote $W = \omega(f) \setminus \mathrm{Per}(f)$. We show that $W = C$. Take arbitrary points $x \in C$, $y \in I \setminus \mathrm{Per}(f)$ and fix some $\varepsilon > 0$. We show that there is an r such that

$$|f^r(y) - x| < \varepsilon. \tag{4}$$

Take n sufficiently large such that $|I_n| < \varepsilon$ and denote by $L = f^i(I_n) \ni x$. According to (ii) there exists $a(1)$ such that $f^{a(1)}(y) \in K_1$, further there exists $a(2)$ such that $f^{a(2)+a(1)}(y) \in K_2$, $\ldots$, and there exists $a(n)$ such that $f^{a(n)+\cdots+a(1)}(y) \in K_n$. Since I_n has the period 2^n, there is $j < 2^n$ such that $f^{a(n)+\cdots+a(1)+j}(y) \in L$. Take $r = a(n) + \cdots + a(1) + j$. Such an r fulfils (4) and therefore $C \subset W$. The inclusion $W \subset C$ is trivial. ∎

Now take $x, y \in C$, $x \neq y$ and choose m such that $|I_m| < \frac{|x-y|}{2}$. Then by Claim there exist integers i, j, $i \neq j$, such that $x \in f^i(I_m)$ $y \in f^j(I_m)$, $0 \leq i, j \leq 2^m$. According to Theorem 2.3 f is non-chaotic.

Since $\mathrm{Per}(f)$ is countable (and hence of the first category) and property (ii) from the Theorem 2.1 also holds (see (3)), Theorem 2.1 implies stability of f. ∎

5.3. Proposition. *Let f be the above quoted function and C its Cantor ω-limit set. Then the orbit of any point exterior relative to C contains no point interior relative to C.*

P r o o f. Denote by $L(n)$ $(R(n))$ the set of the left (right) endpoints of component intervals of K_n and let $I(n,k) := f^k(I_n)$ for $k \geq 0$. Clearly $L(1) \subset L(2) \subset \dots$, $\quad R(1) \subset R(2) \subset \dots$.

Take some $I(n,k)$. Then $I(n,k)$ is mapped onto $I(n,0)$ after at most $2^n - 1$ iterates. Notice that the right endpoint of $I(n,k)$ is mapped onto the right endpoints of the intervals $I(n,k+1),\dots,I(n,2^{n-1})$ since f is increasing on all intervals of the set $K_n \setminus I(n,0)$.

What can we say about the image of $I(n,0)$?

The right endpoint is mapped onto zero, which is the left endpoint of $I(1,1)$ and the left endpoint is mapped onto the left endpoint of the interval $I(n+1, 2^n+1) \subset K_{n+1}$. Roughly speaking f is such that

$$\dots \to R(n) \to R(n-1) \to \dots \to R(1) \to L(1) \to L(2) \to \dots \to L(n) \to \dots,$$

where $A \to B$ means that any point x from A is mapped onto a point from B after the finite number of iterates.

Consider the backward orbit of an arbitrary point exterior relative to C. Such a point has exactly two preimages, one of which belongs to C and the other to some N_n. Denote the last one by a. Since C is an invariant set, $\operatorname{borb}(a) \cap C = \emptyset$. Thus the orbit of any point exterior relative to C contains no point interior relative to C. ■

5.4. Proposition. *Blowing-up* orb(1) *gives a function g, which is non-stable if it is non-chaotic.*

P r o o f. Define $f : [-\frac{1}{3}, \frac{4}{3}] = I \to I$ such that $f(x) = f(x)$ for all $x \in [0,1]$, f is continuous on I and constant on both intervals $[-\frac{1}{3}, 0]$ and $[1, \frac{4}{3}]$. Let us blow-up the orbit of the point 1 in such a way that all points of the Cantor set $C \subset [0,1]$ (C is the infinite attractor of f) remain unchanged and still $g : I \to I$.

Denote by τ the corresponding semiconjugacy, i.e. $\tau \circ g(x) = f \circ \tau(x)$ for all x. Thus $\tau(x) = x$ for all $x \in C$ and if we take $y = \tau(z)$ where z is such that $\omega_f(z) = C$, then $C = \omega_g(y)$.

This condition can be fulfilled because orb(1) contains only the endpoints of intervals contiguous to C and we can blow-up "into" interiors of these intervals. In other words for any contiguous interval (a_1, a_2) there exist δ_1, δ_2 such that $\tau(a_1, a_1 + \delta_1) = \tau(a_1) = a_1$ and $\tau(a_2 - \delta_2, a_2) = \tau(a_2) = a_2$ (with some modification for the intervals $[-\frac{1}{3}, 0]$, $[1, \frac{4}{3}]$). Recall the notation from Proposition 5.3: $I(n,0) = I_n$, $I(n,k) = f^k(I(n,0))$, and let $I(n,i) = [x(i), y(i)]$. Fix $n > 0$.

Take $P_n = \{P(n,i);\ 0 \le i \le 2^n - 1\}$ an arbitrary system of periodic intervals of period 2^n covering the infinite $\omega_g(y) = C$. Then clearly in a suitable notation, $P(n,i) \supset I(n,i)$ for all n and all i. We show that also $I_{x(i)} \cup I_{y(i)} \subset P(n,i)$ holds.

Take $v = 1 - \frac{2}{3^{n+1}}$. Then $v \in \operatorname{int}(I(n,i))$, $f(v) = y(1)$ is an endpoint of $I(n,1)$ and $f(I(n,0)) = I(n,1)$, $f(P(n,0)) = P(n,1)$, $P(n,0) \supset I(n,0)$. Thus $P(n,1) \supset I(n,1) \cup I_{y(1)}$. The periodicity of $P(n,i)$ and the fact that $y(1)$ after at most 2^{n+1} iterates is mapped onto each endpoint of the set $K_n = \bigcup_{i=0}^{2^n-1} I(n,i)$ implies

$$P(n,i) \supset I(n,i) \cup I_{x(i)} \cup I_{y(i)}. \tag{5}$$

But then $\bigcap_{n=1}^{\infty} P_n \supset \operatorname{orb}(I_1)$. Clearly $\omega_g(y) \subset \bigcap_{n=1}^{\infty} P_n$ and $\omega_g(y) \cap \operatorname{int} \operatorname{orb}(I_1) = \emptyset$ (every interval contained in the $\operatorname{int} \operatorname{orb}(I_1)$ is a wandering interval). Thus (5) implies

$$\bigcap_{n=1}^{\infty} P_n \supset \omega_g(y)$$

and Theorem 2.1 finishes the proof. ■

5.5. Proposition. *Function g is non-chaotic.*

Proof. It is easy to see that (i) and (ii) from the proof of the Proposition 5.2 imply that $\omega_g(y)$ has no isolated points.

Now f is the basic function for g and to finish the proof it suffices to consider Proposition 4.3 (ii). ■

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