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SOME APPLICATIONS OF FRANEL–KLUYVER'S INTEGRAL, II

OTO STRAUCH

1. Introduction

The aim of this paper is to find an expression of $(a, b)^2$, the square of the greatest common divisor of positive integers a, b in a form of a convergent infinite series whose terms are polynomials in $1/a$, $1/b$. We shall also touch upon the question of the rapidity of the convergence of this series. Our main tool to do this will be the theory of uniform distribution and the following integral involving the function fractional part $\{.\}$

$$\int_0^1 \left(\{ax\} - \frac{1}{2}\right)\left(\{bx\} - \frac{1}{2}\right) \mathrm{d}x = \frac{1}{12} \frac{(a, b)^2}{ab}. \tag{1}$$

In 1924 J. Franel [1] used expression (1) to prove the equivalence of the Riemann hypothesis with the estimate

$$\int_0^1 R_N^2(x) \,\mathrm{d}x = O(N^{1/2 + \varepsilon})$$

for the remainder function $R_N(x)$ (for the definition see Part 2) of the discrepancy of the Farey sequence of order n, where $N = \sum_{i=1}^{n} \varphi(i)$ and φ denotes the Euler totient function. In his paper [1] Franel only notes that (1) follows from Fourier's expansion of the function $\{.\}$. In the subsequent paper [2] E. Landau gave an elementary proof of (1) (see also [3, p. 170]). However, twenty years ago J. C. Kluyver [4] proved a more general result. If $B_n(x)$ is the nth Bernoulli polynomial and $B_n = B_n(0)$ the nth Bernoulli number, then he proved (after a slight modification) thah

$$\int_0^1 B_m(\{ax\}) B_n(\{bx\}) \,\mathrm{d}x = (-1)^{m-1} \frac{m!\,n!}{(m+n)!} B_{m+n} \frac{(a, b)^{m+n}}{a^n \cdot b^m}$$

for all positive integers m, n, a, b. In fact Kluyver constructed certain functions $z(x, y)$ of two real variables x and y which for positive integer values of these variables reduce to the greatest common divisor of x and y. Unfortunately, each of his functions $z(x, y)$ involves the function fractional part.

Estimates of sums involving terms on the right-hand side of (1) play a role in the metric theory of the diophantine approximation and in the theory of the uniform distribution. More details can be found in [5—9]. In our opinion the first result on such sums was proved by I. S. Gál [5] who proved that

$$\sum_{i,j=1}^{n} \frac{(q_i, q_j)^2}{q_i q_j} \leqq cn (\log\log n)^2$$

for every finite sequence $\{q_i\}_{i=1}^n$ of distinct positive integers and with c an absolute constant. With this estimate, which does not depend on the sequence $\{q_i\}_{i=1}^n$ and which is the best possible, Gál improved an earlier unpublished result of Erdös with $cn \log n$ on the right-hand side. This result was then extend by T. Dyer and G. Harman [9]. They proved that for any sequences $\{f_i\}_{i=1}^n$, $\{g_i\}_{i=1}^n$ of non-negative real numbers and every sequence $\{q_i\}_{i=1}^n$ of distinct positive integers we have

$$\sum_{i,j=1}^{n} \left(f_i g_j \frac{(q_i, q_j)^2}{q_i q_j}\right)^{1/2} \leqq$$

$$\leqq c\left(\sum_{i=1}^{n} f_i \exp\left(\frac{c' \log i}{\log\log(i+1)}\right)\right)^{1/2} \cdot \left(\sum_{j=1}^{n} g_j \exp\left(\frac{c' \log j}{\log\log(j+1)}\right)\right)^{1/2}$$

with c, c' absolute constants and $c' \leqq 5$.

2. Preliminaries

In this chapter we shall employ the classical part of the theory of uniform distribution. The terminology used is that from the monograph by L. Kuipers and H. Niederreiter [10]. Let in what follows

$$x_1, x_2, \dots, x_N$$

be a finite sequence of real numbers from the interval [0, 1]. Given such a sequence $\{x_n\}_{n=1}^N$ and a subinterval $[0, x)$ of $[0, 1]$ denote

$$\begin{aligned} A([0, x), N, \{x_n\}) &= \operatorname{card}\{n;\ 1 \leqq n \leqq N,\ 0 \leqq x_n < x\}, \\ R_N(x) &= A([0, x), N, x_n) - Nx \quad \text{if} \quad 0 \leqq x < 1, \\ R_N(1) &= 0, \qquad\qquad (2) \\ D_N^* &= \sup_{0 \leqq x \leqq 1} \frac{|R_N(x)|}{N}, \end{aligned}$$

$$D_N^{(2)} = \left(\frac{1}{N^2}\int_0^1 R_N^2(x)\,\mathrm{d}x\right)^{1/2}.$$

The number D_N^* is called the discrepancy, $D_N^{(2)}$ the L^2 discrepancy and $R_N(x)$ the remainder of the sequence $\{x_n\}_{n=1}^N$. Some other representations for D_N^*, $D_N^{(2)}$ or $\int_0^1 R_N^2(x)\,\mathrm{d}x$ are known, e.g.

$$D_N^* = \frac{1}{2N} + \max_{1\leqq n\leqq N}\left|x_n - \frac{2n-1}{2N}\right| \tag{3}$$

$$\int_0^1 R_N^2(x)\,\mathrm{d}x$$

$$= \frac{1}{12} + N\sum_{n=1}^N\left(x_n - \frac{2n-1}{2N}\right)^2 \tag{4}$$

$$= \left(\sum_{n=1}^N\left(x_n - \frac{1}{2}\right)\right)^2 + \frac{1}{2\pi^2}\sum_{h=1}^\infty\frac{1}{h^2}\left|\sum_{n=1}^N \mathrm{e}^{2\pi i h x_n}\right|^2 \tag{5}$$

$$= \left(\sum_{n=1}^N\left(x_n - \frac{1}{2}\right)\right)^2 + \int_0^1\left(\sum_{n=1}^N\left(\{x_n + x\} - \frac{1}{2}\right)\right)^2\mathrm{d}x \tag{6}$$

$$= \sum_{n=1}^N\left(x_n - \frac{1}{2}\right) + N\sum_{n=1}^N\left(x_n - \frac{n}{N}\right)^2 - \frac{1}{6} \tag{7}$$

$$= \frac{N^2}{3} + N\sum_{n=1}^N x_n^2 + \sum_{n=1}^N x_n - 2\sum_{n=1}^N n x_n \tag{8}$$

$$= \frac{N^2}{3} + N\sum_{n=1}^N x_n^2 - \sum_{m,n=1}^N \max(x_m, x_n) \tag{9}$$

$$= \frac{N^2}{3} + N\sum_{n=1}^N x_n^2 - N\sum_{n=1}^N x_n - \frac{1}{2}\sum_{m,n=1}^N |x_m - x_n|. \tag{10}$$

In (4), (7) and (8) should be supposed that the elements of the sequence $\{x_n\}_{n=1}^N$ are ordered according to their magnitude.

These identities were used often in the past and some of them can be found explicitly in [12]. For the proof of (3)—(10) refer to [10], pp. 91, 161, 110, 144, 145. Relation (10) follows immediately from (9).

The next identity is also useful.

Lemma. *Given a finite sequence* $\{x_n\}_{n=1}^N$ *of real numbers from the interval* [0, 1] *we have*

$$\int_0^1 R_N^2(x)\,dx = \int_0^1\int_0^1 -\frac{|x-y|}{2}\,dR_N(x)\,dR_N(y). \tag{11}$$

Proof. Although the identity (11) can be proved directly by integration by parts, we prove it using some other ideas.

It follows from the definition of the Riemann–Stieltjes integral that for all continuous functions $f\colon [0, 1] \to R$ we have

$$\int_0^1 f(x)\,dR_N(x) = \sum_{n=1}^{N} f(x_n) - N\int_0^1 f(x)\,dx. \tag{12}$$

Applying (12) to both variables of a continuous function $f(x, y)\colon [0, 1]^2 \to R$ we receive

$$\sum_{m,n=1}^{N} f(x_m, x_n) - N^2\int_0^1\int_0^1 f(x, y)\,dx\,dy$$

$$= \sum_{n=1}^{N}\left(\sum_{m=1}^{N} f(x_m, x_n) - N\int_0^1 f(x, x_n)\,dx\right) + N\int_0^1 dx\left(\sum_{n=1}^{N} f(x, x_n) - N\int_0^1 f(x, y)\,dy\right)$$

$$= \sum_{n=1}^{N}\int_0^1 f(x, x_n)\,dR_N(x) + N\int_0^1 dx\int_0^1 f(x, y)\,dR_N(y)$$

$$= \int_0^1\left(\sum_{n=1}^{N} f(x, x_n) - N\int_0^1 f(x, y)\,dy\right)dR_N(x) + N\int_0^1\left(\int_0^1 f(x, y)\,dy\right)dR_N(x) +$$

$$+ N\int_0^1 dx\int_0^1 f(x, y)\,dR_N(y).$$

If the function $f(x, y)$ is symmetrical, then we can continue with the equality

$$= \int_0^1\int_0^1 f(x, y)\,dR_N(y)\,dR_N(x) + 2N\int_0^1\int_0^1 f(x, y)\,dy\,dR_N(x). \tag{13}$$

If the function $f(x, y)$ has continuous partial derivatives, then the Riemann–Stieltjes integrals in the above equality can be replaced by the Riemann integrals in the following way

$$\int_0^1\int_0^1 f(x, y)\,dR_N(x)\,dR_N(y) = \int_0^1\int_0^1 R_N(x)\,R_N(y)\,\frac{\partial^2}{\partial x\,\partial y} f(x, y)\,dx\,dy, \tag{14}$$

$$2N\int_0^1\int_0^1 f(x, y)\,dy\,dR_N(x) = -2N\int_0^1\int_0^1 R_N(x)\,\frac{\partial}{\partial x} f(x, y)\,dx\,dy. \tag{15}$$

$$F(x, y) = \frac{1}{3} + \frac{x^2 + y^2}{2} - \frac{x + y}{2} - \frac{|x - y|}{2}, \tag{16}$$

then (10) gives

$$\int_0^1 R_N^2(x)\,\mathrm{d}x = \sum_{m,\,n=1}^{N} F(x_m, x_n) \quad \text{and} \quad \int_0^1 \int_0^1 F(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Using (13)—(15) we obtain

$$\int_0^1 R_N^2(x)\,\mathrm{d}x = \sum_{m,\,n=1}^{N} F(x_m, x_n) - N^2 \int_0^1 \int_0^1 F(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \int_0^1 -\frac{|x - y|}{2}\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)$$

$$+ \int_0^1 \int_0^1 R_N(x)\,R_N(y)\,\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{3} + \frac{x^2 + y^2}{2} - \frac{x + y}{2}\right)\mathrm{d}x\,\mathrm{d}y$$

$$+ 2N \int_0^1 \int_0^1 -\frac{|x - y|}{2}\,\mathrm{d}y\,\mathrm{d}R_N(x)$$

$$- 2N \int_0^1 \int_0^1 R_N(x)\,\frac{\partial}{\partial x}\left(\frac{1}{3} + \frac{x^2 + y^2}{2} - \frac{x + y}{2}\right)\mathrm{d}x\,\mathrm{d}y$$

and (11) follows by routine calculations.

Remark 1. The identity (12) was first used by Koksma in [11] with the left-hand side in the form $-\int_0^1 R_N(x)\,\mathrm{d}f(x)$ and for functions which have a bounded variation and which are continuous at all points from $\{x_n\}_{n=1}^{N}$. By integration by parts of the left-hand side of (12) it can be shown that the Koksma identity is true also for continuous functions with unbounded variation.

Remark 2. We shall apply the Lemma very often in the following version: Let $\{y_n\}_{n=1}^{N_1}$ and $\{z_n\}_{n=1}^{N_2}$ be two sequences of real numbers from the interval [0, 1] with remainders $R_{N_1}(x)$ and $R_{N_2}(x)$, respectively. Let $\{x_n\}_{n=1}^{N}$ be a superposition of $\{y_n\}_{n=1}^{N_1}$ and $\{z_n\}_{n=1}^{N_2}$, that is, a sequence obtained by listing the terms of the $\{y_n\}_{n=1}^{N_1}$ and $\{z_n\}_{n=1}^{N_2}$ in some order. Then

$$R_N(x) = R_{N_1}(x) + R_{N_2}(x), \quad N = N_1 + N_2$$

and repeated applications of (11) on the sequences $\{x_n\}_{n=1}^{N}$, $\{y_n\}_{n=1}^{N_1}$, and $\{z_n\}_{n=1}^{N_2}$ gives

$$\int_0^1 R_{N_1}(x)\,R_{N_2}(x)\,\mathrm{d}x = \int_0^1 \int_0^1 -\frac{|x - y|}{2}\,\mathrm{d}R_{N_1}(x)\,\mathrm{d}R_{N_2}(y). \tag{17}$$

Using these sequences in (10), we have

$$\int_0^1 R_{N_1}(x) R_{N_2}(x)\,\mathrm{d}x = \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} F(y_m, z_n) \tag{18}$$

for every two sequences $\{y_n\}_{n=1}^{N_1}$ and $\{z_n\}_{n=1}^{N_2}$ from [0, 1].

Interesting results can be prove applying (1), (4)—(10), (17) and (18) to some sequences. For example, let

$$q_1, q_2, \ldots, q_n$$

be a finite sequence of positive integers (not necessary distinct). Define

$$\{x_n\}_{n=1}^N = \left\{\frac{1}{q_1}, \frac{2}{q_1}, \ldots, \frac{q_1}{q_1}, \frac{1}{q_2}, \frac{2}{q_2}, \ldots, \frac{q_2}{q_2}, \ldots, \frac{1}{q_n}, \frac{2}{q_n}, \ldots, \frac{q_n}{q_n}\right\}. \tag{19}$$

If $N = \sum_{i=1}^n q_i$, then we have

$$A([0, x], N, \{x_n\}) = \sum_{i=1}^n [q_i x]$$

and Franel – Kluyver's integral (1) gives

$$\int_0^1 R_N^2(x)\,\mathrm{d}x = \int_0^1 \left(-\sum_{i=1}^n \{q_i x\}\right)^2 \mathrm{d}x = \frac{n^2}{4} + \frac{1}{12} \sum_{i,j=1}^n \frac{(q_i, q_j)^2}{q_i q_j}. \tag{20}$$

On the other hand, the right-hand side of (5) with respect to the sequence (19) can be calculated to the form

$$\int_0^1 R_N^2(x)\,\mathrm{d}x = \frac{n^2}{4} + \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left(\sum_{\substack{i=1 \\ q_i \backslash h}}^n q_i\right)^2.$$

Thus together with (20) this gives that for every sequence $\{q_i\}_{i=1}^n$ of positive integers we have

$$\frac{1}{12} \sum_{i,j=1}^n \frac{(q_i, q_j)^2}{q_i q_j} = \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left(\sum_{\substack{i=1 \\ q_i \backslash h}}^n q_i\right)^2. \tag{21}$$

Note that the same result can be proved for the similar sequence

$$\{x_n\}_{n=1}^N = \left\{\frac{0}{q_1}, \frac{1}{q_1}, \ldots, \frac{q_1 - 1}{q_1}, \frac{0}{q_2}, \frac{1}{q_2}, \ldots, \frac{q_2 - 1}{q_2}, \ldots, \frac{0}{q_n}, \frac{1}{q_n}, \ldots, \frac{q_n - 1}{q_n}\right\}. \tag{22}$$

This follows immediately from the fact that by (5) both sequence $\{x_n\}_{n=1}^N$ and $\{1 - x_n\}_{n=1}^N$ have the same L^2 discrepancy.

The result of type (21) can be proved using (17) also for sequence (22). This will be the aim of the next section.

3. Main results

First we express

$$|x| = \sqrt{1 - (1 - x^2)}$$

in the binomial series

$$|x| = 1 - \sum_{k=1}^{\infty} \frac{1}{2k-1} \cdot \frac{(2k)!}{(2^k k!)^2} (1 - x^2)^k. \tag{23}$$

Applying the Stirling formula in the following from

$$k! = \sqrt{2k}\, k^k e^{-k} \left(\sqrt{\pi} + \frac{\omega_k}{\sqrt{2k}}\right), \quad |\omega_k| \leqq 1$$

we obtain

$$\frac{1}{2k-1} \cdot \frac{(2k)!}{(2^k k!)^2} = \frac{1}{2k-1} \cdot \frac{1}{\sqrt{\pi k}} \left(1 + \frac{\theta_k}{\sqrt{4\pi k}}\right), \quad |\theta_k| \leqq 9. \tag{24}$$

An immediate consequence is that the series (23) converges uniformly for $|x| \leqq 1$.

Substituting (23) in (17) we can prove the following result.

Theorem 1. *For every triplet of positive integers a, b and X we have*

$$\frac{1}{12} \cdot \frac{(a, b)^2}{ab}$$

$$= \sum_{k=2}^{\infty} \frac{1}{2(2k-1)} \cdot \frac{(2k)!}{(2^k k!)^2} \cdot \sum_{\substack{r,s=1 \\ 2 \leqq r+s \leqq k}}^{k} \frac{1}{X^{2(r+s)-2}} \binom{2(r+s)}{2r} \cdot \frac{B_{2r}}{a^{2r-1}} \cdot \frac{B_{2s}}{b^{2s-1}} \cdot$$

$$\cdot \frac{-2}{2(r+s)(2(r+s)-1)} \cdot \left[(-1)^{r+s-1} \binom{k}{r+s-1} - \right.$$

$$\left. - (-1)^k . 2^{2k-2(r+s)+2} \binom{k}{2k-2(r+s)+2}\right],$$

where B_r denotes the rth Bernoulli number and for binomial coefficients $\binom{m}{n}$ we put

$$\binom{m}{n} = 0 \quad \text{if} \quad n < 0 \quad \text{or} \quad n > m.$$

Proof. First of all, it follows from (17) and (14) that

$$\int_0^1 R_{N_1}(x)\,R_{N_2}(x)\,\mathrm{d}x = \left(\int_0^1 R_{N_1}(x)\,\mathrm{d}x\right)\left(\int_0^1 R_{N_2}(x)\,\mathrm{d}x\right) +$$

$$+ \int_0^1\int_0^1 \frac{1-|x-y|-(1-(x-y)^2)}{2}\,\mathrm{d}R_{N_1}(x)\,\mathrm{d}R_{N_2}(y). \qquad (25)$$

Since the series (23) converges uniformly, we may integrate term by term and obtain the following

$$\int_0^1 R_{N_1}(x)\,R_{N_2}(x)\,\mathrm{d}x = \left(\int_0^1 R_{N_1}(x)\,\mathrm{d}x\right)\left(\int_0^1 R_{N_2}(x)\,\mathrm{d}x\right)$$

$$+ \sum_{k=2}^{\infty} \frac{1}{2(2k-1)}\cdot\frac{(2k)!}{(2^k k!)^2}\int_0^1\int_0^1 [(1-(x-y)^2)^k - (1-(x-y)^2)]\,\mathrm{d}R_{N_1}(x)\,\mathrm{d}R_{N_2}(y). \qquad (26)$$

For the kth power we have

$$\int_0^1\int_0^1 (1-(x-y)^2)^k\,\mathrm{d}R_{N_1}(x)\,\mathrm{d}R_{N_2}(y)$$

$$= \sum_{\substack{i_1+i_2+i_3+i_4=k\\ i_1,i_2,i_3,i_4\geqq 0}} \frac{k!\,(-1)^{i_1+i_2}2^{i_3}}{i_1!\,i_2!\,i_3!\,i_4!}\int_0^1 x^{2i_1+i_3}\,\mathrm{d}R_{N_1}(x)\int_0^1 y^{2i_2+i_3}\,\mathrm{d}R_{N_2}(y). \qquad (27)$$

Using (12) we get

$$\int_0^1 x^k\,\mathrm{d}R_{N_1}(x) = \sum_{n=1}^{N_1} y_n^k - N_1\cdot\frac{1}{k+1} \qquad (28)$$

and especially

$$\int_0^1 R_{N_1}(x)\,\mathrm{d}x = -\int_0^1 x\,\mathrm{d}R_{N_1}(x) = \frac{N_1}{2} - \sum_{n=1}^{N_1} y_n. \qquad (29)$$

The identities (25)—(29) hold for every couple of the sequences $\{y_n\}_{n=1}^{N_1}$ and $\{z_n\}_{n=1}^{N_2}$ of real numbers from [0, 1] with the remainders $R_{N_1}(x)$ and $R_{N_2}(x)$, respectively.

We are now in the position to prove our theorem. Using (20) and (29) we obtain for the sequences

$$\{y_n\}_{n=1}^{N_1} = \left\{\frac{0}{a}, \frac{1}{a}, \ldots, \frac{a-1}{a}\right\}, \quad N_1 = a, \qquad (30)$$

$$\{z_n\}_{n=1}^{N_2} = \left\{\frac{0}{b}, \frac{1}{b}, \ldots, \frac{b-1}{b}\right\}, \quad N_2 = b$$

that

$$\int_0^1 R_a(x)\,R_b(x)\,\mathrm{d}x = \frac{1}{4} + \frac{1}{12}\,\frac{(a, b)^2}{ab}, \tag{31}$$

$$\int_0^1 R_a(x)\,\mathrm{d}x \int_0^1 R_b(x)\,\mathrm{d}x = \frac{1}{4}. \tag{32}$$

The well-known formula

$$\sum_{n=1}^{N-1} n^k = \sum_{r=0}^{k} \frac{1}{k+1}\binom{k+1}{r} B_r N^{k+1-r}, \quad k \geqq 1,$$

gives

$$\int_0^1 x^k\,\mathrm{d}R_a(x) = \sum_{r=1}^{k} \frac{1}{k+1}\binom{k+1}{r}\frac{B_r}{a^{r-1}}. \tag{33}$$

Substituting (33) into (27) and after interchanging the order of the summations we see that

$$\int_0^1\int_0^1 (1-(x-y)^2)^k\,\mathrm{d}R_a(x)\,\mathrm{d}R_b(y)$$

$$= \sum_{r,s=1}^{2k} \frac{B_r}{a^{r-1}}\cdot\frac{B_s}{b^{s-1}}\cdot \sum_{\substack{i_1+i_2+i_3+i_4=k \\ 2i_1+i_3 \geqq r \\ 2i_2+i_3 \geqq s}} \frac{k!\,(-1)^{i_1+i_2}\cdot 2^{i_3}}{i_1!\,i_2!\,i_3!\,i_4!}\cdot\frac{1}{2i_1+i_3+1}\cdot$$

$$\cdot\frac{1}{2i_2+i_3+1}\cdot\binom{2i_1+i_3+1}{r}\cdot\binom{2i_2+i_3+1}{s}$$

$$= \sum_{r,s=1}^{2k} \frac{B_r}{a^{r-1}}\cdot\frac{B_s}{b^{s-1}}\cdot S_k(r, s). \tag{34}$$

Since

$$S_k(1, 1) = 2, \quad S_k(2, 1) = 0, \quad S_1(2, 2) = 0,$$

we may write

$$\int_0^1\int_0^1 [(1-(x-y)^2)^k - (1-(x-y)^2)]\,\mathrm{d}R_a(x)\,\mathrm{d}R_b(y) = \sum_{\substack{r,s=1 \\ r+s \geqq 4}}^{2k} \frac{B_r}{a^{r-1}}\cdot\frac{B_s}{b^{s-1}}\cdot S_k(r, s). \tag{35}$$

From (31), (32), (35) and (26) we get

$$\frac{1}{12}\,\frac{(a, b)^2}{ab} = \sum_{k=2}^{\infty} \frac{1}{2(2k-1)}\cdot\frac{(2k)!}{(2^k k!)^2} \sum_{\substack{r,s=1 \\ r+s \geqq 4}}^{2k} \frac{B_r}{a^{r-1}}\cdot\frac{B_s}{b^{s-1}}\cdot S_k(r, s). \tag{36}$$

Let us transform $S_k(r, s)$ into a more suitable form. Writing

$$[f(x)]_{x=x_0}^{x=x_1} = f(x_1) - f(x_0), \quad [f(x)]_{x=x_0} = f(x_0)$$

and substituting

$$\frac{1}{r!}\left[\frac{\mathrm{d}^{r-1}}{\mathrm{d}x^{r-1}} x^k\right]_{x=0}^{x=1} = \begin{cases} \frac{1}{k+1}\binom{k+1}{r} & \text{if } k \geqq r \\ 0 & \text{if } k < r \end{cases}$$

into $S_k(r, s)$ we obtain

$$S_k(r, s)$$

$$= \sum_{i_1+i_2+i_3+i_4=k} \frac{k!(-1)^{i_1+i_2} \cdot 2^{i_3}}{i_1!\, i_2!\, i_3!\, i_4!} \cdot \frac{1}{r!}\left[\frac{\mathrm{d}^{r-1}}{\mathrm{d}x^{r-1}} x^{2i_1+i_3}\right]_{x=0}^{x=1} \cdot \frac{1}{s!}\left[\frac{\mathrm{d}^{s-1}}{\mathrm{d}x^{s-1}} y^{2i_2+i_3}\right]_{y=0}^{y=1}$$

$$= \sum_{i_1+i_2+i_3+i_4=k} \frac{k!(-1)^{i_1+i_2} \cdot 2^{i_3}}{i_1!\, i_2!\, i_3!\, i_4!} \cdot \frac{1}{r!s!}\left(\left[\frac{\partial^{r+s-1}}{\partial x^{r-1}\partial y^{s-1}} x^{2i_1+i_3} y^{2i_2+i_3}\right]_{x=1,\, y=0}^{x=1,\, y=1}\right.$$

$$\left. - \left[\frac{\partial^{r+s-2}}{\partial x^{r-1}\partial y^{s-1}} x^{2i_1+i_3} y^{2i_2+i_3}\right]_{x=0,\, y=0}^{x=0,\, y=1}\right)$$

$$= \frac{1}{r!s!}\left(\left[\frac{\partial^{r+s-2}}{\partial x^{r-1}\partial y^{s-1}} (1-(x-y)^2)^k\right]_{x=1,\, y=0}^{x=1,\, y=1} - \right.$$

$$\left. - \left[\frac{\partial^{r+s-2}}{\partial x^{r-1}\partial y^{s-1}} (1-(x-y)^2)^k\right]_{x=0,\, y=0}^{x=0,\, y=1}\right). \tag{37}$$

This can be rewritten in the following way:

Let

$$\mathscr{A}_{r-1}(x, y) = \frac{\partial^{r-1}}{\partial x^{r-1}} (1-(x-y)^2)^k = \sum_{j_1+\dots+j_k=r-1} \frac{(r-1)!}{j_1!\,\dots\,j_k!} \frac{\partial^{j_1}}{\partial x^{j_1}}$$

$$(1-(x-y)^2) \dots \frac{\partial^{j_k}}{\partial x^{j_k}} (1-(x-y)^2),$$

$$\mathscr{B}_{r-1}(y) = \frac{\mathrm{d}^{r-1}}{\mathrm{d}y^{r-1}} (1-(1-y)^2)^k = \sum_{j_1+\dots+j_k=r-1} \frac{(r-1)!}{j_1!\,\dots\,j_k!} \frac{\mathrm{d}^{j_1}}{\mathrm{d}y^{j_1}}$$

$$(1-(1-y)^2) \dots \frac{\mathrm{d}^{j_k}}{\mathrm{d}y^{j_k}} (1-(1-y)^2),$$

$$\mathscr{C}_{r-1}(y) = \frac{\mathrm{d}^{r-1}}{\mathrm{d}y^{r-1}}(1-y^2)^k = \sum_{j_1+\ldots+j_k=r-1} \frac{(r-1)!}{j_1!\ldots j_k!} \frac{\mathrm{d}^{j_1}}{\mathrm{d}y^{j_1}}$$

$$(1-y^2)\ldots\frac{\mathrm{d}^{j_k}}{\mathrm{d}y^{j_k}}(1-y^2),$$

$$a_s(x, y) = \frac{\partial^s}{\partial x^s}(1-(x-y)^2) = \begin{cases} 1-(x-y)^2 & \text{if } s=0 \\ -2(x-y) & \text{if } s=1 \\ -2 & \text{if } s=2 \\ 0 & \text{if } s \geqq 3 \end{cases}$$

$$b_s(y) = \frac{\mathrm{d}^s}{\mathrm{d}y^s}(1-(1-y)^2) = \begin{cases} 1-(1-y)^2 & \text{if } s=0 \\ 2(1-y) & \text{if } s=1 \\ -2 & \text{if } s=2 \\ 0 & \text{if } s \geqq 3 \end{cases}$$

$$c_s(y) = \frac{\mathrm{d}^s}{\mathrm{d}y^s}(1-y^2) = \begin{cases} 1-y^2 & \text{if } s=0 \\ -2y & \text{if } s=1 \\ -2 & \text{if } s=2 \\ 0 & \text{if } s \geqq 3 \end{cases}$$

We see that $a_s(1, y)$ and $b_s(y)$ are distinct only if $s = 1$ and moreover $a_1(1, y) = -b_1(y)$. Since now $j_1 + \ldots + j_k = r - 1, j_n \leqq 2, n = 1, \ldots, k$ the parity of the number of $j_n = 1$ is equal to the parity of $r - 1$. Thus

$$\mathscr{A}_{r-1}(1, y) = (-1)^{r-1}\mathscr{B}_{r-1}(y).$$

Similarly

$$\mathscr{A}_{r-1}(0, y) = (-1)^{r-1}\mathscr{C}_{r-1}(y), \quad \mathscr{B}_{r-1}(0) = (-1)^{r-1}\mathscr{C}_{r-1}(1).$$

Since $b_s(1) = c_s(0)$ for $s = 0, 1, 2$ and $s \geqq 3$, we also have

$$\mathscr{B}_{r-1}(1) = \mathscr{C}_{r-1}(0).$$

Applying this in (37) we obtain

$$S_k(r, s) = \frac{(-1)^{r-1}}{r!s!}\left(\left[\frac{\mathrm{d}^{r+s-2}}{\mathrm{d}y^{r+s-2}}(1-(1-y)^2)^k\right]_{y=0}^{y=1} - \left[\frac{\mathrm{d}^{r+s-2}}{\mathrm{d}y^{r+s-2}}(1-y^2)^k\right]_{y=0}^{y=1}\right)$$

$$= \frac{(-1)^{r-1}}{r!s!}\left(2\left[\frac{\mathrm{d}^{r+s-2}}{\mathrm{d}y^{r+s-2}}(1-y^2)^k\right]_{y=0} - \right.$$

$$\left. - \left[\frac{\mathrm{d}^{r+s-2}}{\mathrm{d}y^{r+s-2}}(1-y^2)^k\right]_{y=1} \cdot (1+(-1)^{r+s-2})\right). \tag{38}$$

Cauchy's integral formula implies

$$\left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}(1-z^2)^k\right]_{z=0} = n!(-1)^{n/2}\binom{k}{n/2},$$
$$\left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}(1-z^2)^k\right]_{z=1} = n!(-1)^k.2^{2k-n}\binom{n}{n-k}, \tag{39}$$

where $\binom{k}{n/2} = 0$ for odd n and $\binom{n}{n-k} = 0$ for $k > n$.

Combining (38) and (39) we obtain that $S_k(r, s) = 0$ for odd $r + s$. Since $r + s \geqq 4$ in the sum (36) and $B_s = 0$ for each odd $s > 1$, we can replace $r \to 2r$, and $s \to 2s$. Using (39) for $r + s > k$ we see that $S_k(2r, 2s) = 0$. This gives in expression (36) that Theorem 1 holds for $X = 1$. The rest of the proof follows from the first of the next two elementary relations:

For the sake of simplicity put

$$\langle a, b\rangle = \frac{(a, b)^2}{ab}.$$

Then

$$\langle Xa, Xb\rangle = \langle a, b\rangle, \quad \langle a^X, b^X\rangle = \langle a, b\rangle^X \tag{40}$$

for all positive integers a, b, X.

Finally, observe that applying the second relation of (40), Theorem 1 can be rewritten to the form

$$\frac{1}{12}\cdot\left(\frac{(a, b)^2}{ab}\right)^Y$$
$$= \sum_{k=2}^{\infty} \frac{1}{2(2k-1)}\cdot\frac{(2k)!}{(2^k k!)^2}\cdot \sum_{\substack{r,s=1\\ 2\leqq r+s\leqq k}}^{k} \frac{1}{X^{2(r+s)-2}}\binom{2(r+s)}{2r}\cdot\frac{B_{2r}}{a^{Y(2r-1)}}\cdot\frac{B_{2s}}{b^{Y(2s-1)}}\cdot$$
$$\cdot\frac{-2}{2(r+s)(2(r+s)-1)}\cdot\left[(-1)^{r+s-1}\cdot\binom{k}{r+s-1} - \right.$$
$$\left. -(-1)^k.2^{2k-2(r+s)+2}\binom{k}{2k-2(r+s)+2}\right]$$

for every positive integer a, b, X, Y.

We finish this section by proving

Theorem 2. *In Theorem 1 the remainder* $\sum_{k=K+1}^{\infty}$ *of the infinite series does not exceed*

$$9\cdot\frac{X}{\sqrt{K}}\min\{a, b\}, \tag{41}$$

for every positive integer a, b, X, K.

Proof. If f: $[0, 1]^2 \to R$ is a continuous function, then as in the preceding section we obtain

$$\int_0^1 \int_0^1 f(x, y)\,\mathrm{d}R_{N_1}(x)\,\mathrm{d}R_{N_2}(y) = \int_0^1 \mathrm{d}R_{N_2}(y) \int_0^1 f(x, y)\,\mathrm{d}R_{N_1}(x)$$

$$= \sum_{n=1}^{N_2} \int_0^1 f(x, z_n)\,\mathrm{d}R_{N_1}(x) - N_2 \int_0^1 \mathrm{d}y \int_0^1 f(x, y)\,\mathrm{d}R_{N_1}(x)$$

$$= \sum_{n=1}^{N_2} - \int_0^1 R_{N_1}(x)\,\mathrm{d}f(x, z_n) + N_2 \int_0^1 \mathrm{d}y \int_0^1 R_{N_1}(x)\,\mathrm{d}_x f(x, y). \qquad (42)$$

Consequently, if f has a bounded variation with respect to x and y, then

$$\left| \int_0^1 \int_0^1 f(x, y)\,\mathrm{d}R_{N_1}(x)\,\mathrm{d}R_{N_2}(y) \right|$$

$$\leqq 2N_1 N_2 \min\left\{ D_{N_1}^* \sup_y \int_0^1 |\mathrm{d}_x f(x, y)|,\ D_{N_2}^* \sup_x \int_0^1 |\mathrm{d}_y f(x, y)| \right\} \qquad (43)$$

for every couple of sequences $\{y_n\}_{n=1}^{N_1}$ and $\{z_n\}_{n=1}^{N_2}$ with the discrepancy $D_{N_1}^*$ and $D_{N_2}^*$, respectively. With

$$f(x, y) = (1 - (x - y)^2)^k$$

and using

$$\int_0^1 \left| \frac{\partial}{\partial x} (1 - (x - y)^2)^k \right| \mathrm{d}x = 1 - (1 - y^2)^k - [(1 - (1 - y)^2)^k - 1] \leqq 2$$

the relation (43) gets the form

$$\left| \int_0^1 \int_0^1 (1 - (x - y)^2)^k\,\mathrm{d}R_{N_1}(x)\,\mathrm{d}R_{N_2}(y) \right| \leqq 4N_1 N_2 \cdot \min\{D_{N_1}^*, D_{N_2}^*\}. \qquad (44)$$

Moreover, applying (3) to the sequences of the form (30) we get

$$D_a^* = \frac{1}{a}, \quad D_b^* = \frac{1}{b}.$$

Hence

$$\left| \int_0^1 \int_0^1 (1 - (x - y)^2)^k\,\mathrm{d}R_a(x)\,\mathrm{d}R_b(y) \right| \leqq 4 \cdot \min\{a, b\}.$$

Using (24) we have

$$\sum_{k=K+1}^{\infty} \frac{1}{2(2k-1)} \cdot \frac{(2k)!}{(2^k k!)^2} < 0.53 \int_K^{\infty} \frac{\mathrm{d}x}{x\sqrt{x}} = 1.06 \frac{1}{\sqrt{K}}.$$

Thus, together

$$\left|\sum_{k=K+1}^{\infty}\frac{1}{2(2k-1)}\cdot\frac{(2k)!}{(2^k k!)^2}\int_0^1\int_0^1[(1-(x-y)^2)^k-(1-(x-y)^2)]\,\mathrm{d}R_a(x)\,\mathrm{d}R_b(y)\right|$$

$$\leqq 9\cdot\frac{1}{\sqrt{K}}\min\{a, b\}.$$

Finally, substitution of $a \to Xa$, $b \to Xb$ implies (41).

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НЕКОТОРЫЕ ПРИМЕНЕНИЯ ИНТЕГРАЛА ФРАНЕЛЯ—КЛУЫВЕРА, II

Oto Strauch

Резюме

В работе разложен наибольший общий делитель в ряд. В доказательстве применена теория равномерного распределения последовательностей.