N. Parhi; S. Panigrahi

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LIAPUNOV-TYPE INEQUALITY FOR HIGHER ORDER DIFFERENTIAL EQUATIONS

N. Parhi — S. Panigrahi

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ABSTRACT. In this paper, Liapunov-type inequalities are obtained for higher order nonlinear, nonhomogeneous differential equations. These inequalities are used to obtain criteria for disconjugacy of linear homogeneous equations on an interval and to show that oscillatory solutions of the equation converge to zero as $t \to \infty$. It is also shown, using these inequalities, that $(t_{m+k} - t_m) \to \infty$ as $m \to \infty$, where $1 \le k \le n-1$ and $\{t_m\}$ is an increasing sequence of zeros of an oscillatory solution of $\mathrm{D}^n y + p(t)y = 0$, $t \ge 0$, provided that $p \in L^\sigma([0,\infty),\mathbb{R})$, $1 \le \sigma < \infty$.

1. Introduction

It is known that (see [6]) if $y(t)$ is a solution of

$$y'' + p(t)y = 0 \tag{1}$$

with $y(a) = 0 = y(b)$ $(a < b)$ and $y(t) \ne 0$ for $t \in (a,b)$, then

$$\int_a^b |p(t)| \,\mathrm{d}t > 4/(b-a)\,. \tag{2}$$

In [5], Hartman obtained an inequality which is more general than (2). The inequality (2) is generalized to second order nonlinear differential equations by Eliason [2], to delay-differential equations of second order by Eliason [3], [4] and Dahiya and Singh [1] and to higher order differential equations by Pachpatte [7]. Indeed, Pachpatte derived Liapunov-type inequalities for

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the equations of the form

$$\mathrm{D}^n\big[r(t)\,\mathrm{D}^{n-1}\big(p(t)g(y'(t))\big)\big] + y(t)f\big(t,y(t)\big) = Q(t)\,, \tag{3}$$

$$\mathrm{D}^n\big[r(t)\,\mathrm{D}^{n-1}\big(p(t)h(y(t))y'(t)\big)\big] + y(t)f\big(t,y(t)\big) = Q(t)\,, \tag{3'}$$

$$\mathrm{D}^n\big[r(t)\,\mathrm{D}^{n-1}\big(p(t)h(y(t))g(y'(t))\big)\big] + y(t)f\big(t,y(t)\big) = Q(t) \tag{3''}$$

under appropriate conditions, where $n \geq 2$ is an integer and $\mathrm{D}^n = \mathrm{d}^n/\mathrm{d}t^n$. It is clear that the results in [7] are not applicable to odd order equations. In a recent work [8], the authors have obtained Liapunov-type inequality for third order equations of the form

$$y''' + p(t)y = 0\,. \tag{4}$$

This inequality is used to study many interesting properties of the zeros of an oscillatory solution of (4) (see [8; Theorems 5, 6]). In [9], Liapunov-type inequality is obtained for delay-differential equations of third order of the form

$$y'''(t) + p(t)|y(t)|^{\mu}\operatorname{sgn} y(t) + m(t)|y(t-\tau)|^{v}\operatorname{sgn} y(t-\tau) = 0\,,$$

where $p, m \in C\big([0,\infty),\mathbb{R}\big)$, $\mu > 0$, $v > 0$, and $\tau \geq 0$.

The object of this paper is to derive a Liapunov-type inequality for nth order differential equations of the form

$$\left(\frac{1}{r_{n-1}(t)}\cdots\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}y'\right)'\right)'\cdots\right)' + yf(t,y) = Q(t) \tag{5}$$

under appropriate assumptions on $r_i(t)$, $1 \leq i \leq n-1$, f and Q. Here $n \geq 2$ may be an odd or even integer. In [7; Theorem 1] it is assumed that $\alpha_2 > \alpha_3 > \cdots > \alpha_{n-1} > \alpha_n > \alpha_{n+1} > \cdots > \alpha_{2n-1}$, where these are the zeros of

$\mathrm{D}\big[p(t)g\big(y'(t)\big)\big]$, $\mathrm{D}^2\big[p(t)g\big(y'(t)\big)\big]$, …
…, $\mathrm{D}^{n-2}\big[p(t)g\big(y'(t)\big)\big]$, $r(t)\,\mathrm{D}^{n-1}\big[p(t)g\big(y'(t)\big)\big]$, $\mathrm{D}\big[r(t)\,\mathrm{D}^{n-1}\big(p(t)g(y'(t))\big)\big]$, …
…, $\mathrm{D}^{n-1}\big[r(t)\,\mathrm{D}^{n-1}\big(p(t)g(y'(t))\big)\big]$

respectively and $y(t)$ is a nontrivial solution of (3). In this work we remove this restriction on the zeros of higher order derivatives. We may observe that in [7; p. 530, Example], $y'''(3\pi/4) \neq 0$ because $y'''(t) = 2\mathrm{e}^{-t}(\cos t - \sin t)$. On the other hand, $y'''(\pi/4) = 0$, but $\pi/4 \notin (\pi/2, 3\pi/2)$ and $y'''(5\pi/4) = 0$, but $5\pi/4 \not< \pi$. Although this example does not illustrate [7; Theorem 1], it has motivated us to remove the restriction on the zeros of higher order derivatives of the solution of (3). Further, we show that every oscillatory solution of (5) converges to zero as $t \to \infty$ with the help of Liapunov-type inequality. We also generalize a theorem of Patula (see [10; Theorem 2]) to higher order equations.

2. Main results

Equation (5) may be written as

$$\mathrm{D}^n y + y f(t,y) = Q(t)\,, \tag{6}$$

where $n \geq 2$ is an integer,

$$\mathrm{D} y = \frac{1}{r_1(t)} y'\,, \qquad \mathrm{D}^i y = \frac{1}{r_i(t)} (\mathrm{D}^{i-1} y)'\,,$$

$2 \leq i \leq n$ and $r_n(t) \equiv 1$. Suppose that

(C_1) $r_i \colon I \to \mathbb{R}$ is continuous and $r_i(t) > 0$, $1 \leq i \leq n-1$, and $Q \colon I \to \mathbb{R}$ is continuous, where I is a real interval.

(C_2) $f \colon I \times \mathbb{R} \to \mathbb{R}$ is continuous such that $|f(t,y)| \leq W(t,|y|)$, where $W \colon I \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $W(t,u) \leq W(t,v)$ for $0 \leq u < v$ and $\mathbb{R}^+ = [0,\infty)$.

Following Pachpatte [7], we define

$$\begin{aligned} &E\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); z(s_{n-1})\big) \\ &= r_2(t) \int_{\alpha_1}^{t} r_3(s_2) \int_{\alpha_2}^{s_2} r_4(s_3) \dots \int_{\alpha_{n-3}}^{s_{n-3}} r_{n-1}(s_{n-2}) \int_{\alpha_{n-2}}^{s_{n-2}} z(s_{n-1})\, \mathrm{d}s_{n-1}\, \mathrm{d}s_{n-2} \dots \mathrm{d}s_2\,, \end{aligned}$$

where $z(t)$ is a real valued continuous function defined on $[a,b] \subset I$ $(a < b)$ and $\alpha_1, \alpha_2, \dots, \alpha_{n-2}$ are suitable points in $[a,b]$, and

$$\begin{aligned} &\overline{E}\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); z(s_{n-1})\big) \\ &= r_2(t) \left| \int_{\alpha_1}^{t} r_3(s_2) \left| \int_{\alpha_2}^{s_2} r_4(s_3) \dots \right.\right. \\ &\qquad \dots \left| \int_{\alpha_{n-3}}^{s_{n-3}} r_{n-1}(s_{n-2}) \left| \int_{\alpha_{n-2}}^{s_{n-2}} z(s_{n-1})\, \mathrm{d}s_{n-1} \right| \mathrm{d}s_{n-2} \right| \dots \left| \mathrm{d}s_2 \right|. \end{aligned}$$

THEOREM 1. *Suppose that* (C_1) *and* (C_2) *hold. Let* $\alpha_1, \dots, \alpha_{n-2} \in [a,b]$, *where* $\alpha_1, \dots, \alpha_{n-2}$ *are zeros of* $\mathrm{D}^2 y(t)$, $\mathrm{D}^3 y(t)$, $\dots$, $\mathrm{D}^{n-2} y(t)$, $\mathrm{D}^{n-1} y(t)$ *respectively,* $[a,b] \subset I$ $(a < b)$ *and* $y(t)$ *is a nontrivial solution of* (6) *with* $y(a) = 0 = y(b)$. *If* c *is a point in* (a,b) *where* $|y(t)|$ *attains maximum and*

$M = \max\{|y(t)| : t \in [a,b]\} = |y(c)|$, *then*

$$4 \le \left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)\left(\int_a^b \Big[\overline{E}\big(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)\big) + \frac{1}{M}\overline{E}\big(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|\big)\Big]\,\mathrm{d}s_1\right) \tag{7}$$

for $n \ge 3$ *and*

$$4 \le \left(\int_a^b r_1(t)\,\mathrm{d}t\right)\left[\int_a^b W(t,M)\,\mathrm{d}t + \frac{1}{M}\int_a^b |Q(t)|\,\mathrm{d}t\right] \tag{8}$$

for $n = 2$.

P r o o f . Let $n \ge 3$. Integrating (6) from α_{n-2} to $t \in [a,b]$, we obtain

$$\mathrm{D}^{n-1} y(t) + \int_{\alpha_{n-2}}^{t} y(s_{n-1}) f\big(s_{n-1}, y(s_{n-1})\big)\,\mathrm{d}s_{n-1} = \int_{\alpha_{n-2}}^{t} Q(s_{n-1})\,\mathrm{d}s_{n-1},$$

that is,

$$\big(\mathrm{D}^{n-2} y(t)\big)' + r_{n-1}(t)\int_{\alpha_{n-2}}^{t} y(s_{n-1}) f\big(s_{n-1}, y(s_{n-1})\big)\,\mathrm{d}s_{n-1} = r_{n-1}(t)\int_{\alpha_{n-2}}^{t} Q(s_{n-1})\,\mathrm{d}s_{n-1}.$$

Further integration from α_{n-3} to $t \in [a,b]$ yields

$$\mathrm{D}^{n-2} y(t) + \int_{\alpha_{n-3}}^{t} r_{n-1}(s_{n-2})\left(\int_{\alpha_{n-2}}^{s_{n-2}} y(s_{n-1}) f\big(s_{n-1}, y(s_{n-1})\big)\,\mathrm{d}s_{n-1}\right)\mathrm{d}s_{n-2} = \int_{\alpha_{n-3}}^{t} r_{n-1}(s_{n-2})\left(\int_{\alpha_{n-2}}^{s_{n-2}} Q(s_{n-1})\,\mathrm{d}s_{n-1}\right)\mathrm{d}s_{n-2}.$$

Proceeding as above we obtain

$$\begin{aligned}
&\mathrm{D}^2 y(t) + \int_{\alpha_1}^{t} r_3(s_2) \int_{\alpha_2}^{s_2} r_4(s_3) \dots \\
&\dots \int_{\alpha_{n-3}}^{s_{n-3}} r_{n-1}(s_{n-2}) \left(\int_{\alpha_{n-2}}^{s_{n-2}} y(s_{n-1}) f(s_{n-1}, y(s_{n-1}))\, \mathrm{d}s_{n-1} \right) \mathrm{d}s_{n-2} \dots \mathrm{d}s_2 \\
&= \int_{\alpha_1}^{t} r_3(s_2) \int_{\alpha_2}^{s_2} r_4(s_3) \dots \int_{\alpha_{n-3}}^{s_{n-3}} r_{n-1}(s_{n-2}) \int_{\alpha_{n-2}}^{s_{n-2}} Q(s_{n-1})\, \mathrm{d}s_{n-1}\, \mathrm{d}s_{n-2} \dots \mathrm{d}s_2 ,
\end{aligned}$$

that is,

$$\begin{aligned}
(\mathrm{D}\, y(t))' + E\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2});\, y(s_{n-1}) f(s_{n-1}, y(s_{n-1}))\big) \\
= E\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2});\, Q(s_{n-1})\big).
\end{aligned}$$

Hence

$$\begin{aligned}
\left|(\mathrm{D}\, y(t))'\right| \le M\overline{E}\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2});\, W(s_{n-1}, M)\big) \\
+\overline{E}\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2});\, |Q(s_{n-1})|\big) .
\end{aligned} \tag{9}$$

Since

$$M = |y(c)| = \left| \int_a^c y'(s_1)\, \mathrm{d}s_1 \right| \le \int_a^c |y'(s_1)|\, \mathrm{d}s_1$$

and

$$M = |y(c)| = \left| \int_c^b y'(s_1)\, \mathrm{d}s_1 \right| \le \int_c^b |y'(s_1)|\, \mathrm{d}s_1 ,$$

then

$$2M \le \int_a^b |y'(s_1)|\, \mathrm{d}s_1 .$$

First, using Cauchy-Schwarz inequality and then integrating by parts, we obtain

$$
\begin{aligned}
4M^2 &\le \left[\int_a^b |y'(s_1)|\,\mathrm{d}s_1\right]^2 \\
&= \left[\int_a^b \big(r_1(s_1)\big)^{\frac12}\frac{1}{\big(r_1(s_1)\big)^{\frac12}}|y'(s_1)|\,\mathrm{d}s_1\right]^2 \\
&\le \left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)\left(\int_a^b \frac{1}{r_1(s_1)}\big(y'(s_1)\big)^2\,\mathrm{d}s_1\right) \\
&= \left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)\left(\left[\frac{y'(s_1)y(s_1)}{r_1(s_1)}\right]_a^b - \int_a^b (\mathrm{D}\,y)'(s_1)y(s_1)\,\mathrm{d}s_1\right) \\
&= -\left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)\int_a^b (\mathrm{D}\,y)'(s_1)y(s_1)\,\mathrm{d}s_1 \\
&\le \left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)\int_a^b |(\mathrm{D}\,y)'(s_1)||y(s_1)|\,\mathrm{d}s_1 .
\end{aligned}
\tag{10}
$$

Use of (9) yields

$$
\begin{aligned}
4M^2 \le \left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)\cdot
&\cdot\left[M^2\int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M))\,\mathrm{d}s_1\right. \\
&\left. + M\int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|)\,\mathrm{d}s_1\right],
\end{aligned}
$$

that is,

$$
\begin{aligned}
4 \le \left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)&\left[\int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M))\,\mathrm{d}s_1\right. \\
&\left. + \frac{1}{M}\int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|)\,\mathrm{d}s_1\right].
\end{aligned}
$$

If $n = 2$, then (6) has the form

$$(\mathrm{D}\,y)'(t) + y(t)f\big(t, y(t)\big) = Q(t)\,.$$

Hence (10) yields

$$4M^2 \le \left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)\left[\int_a^b |y(t)|^2\big|f\big(t, y(t)\big)\big|\,\mathrm{d}t + \int_a^b |y(t)||Q(t)|\,\mathrm{d}t\right],$$

that is,

$$4 \le \left(\int_a^b r_1(s_1)\,\mathrm{d}s_1\right)\left[\int_a^b W(t, M)\,\mathrm{d}t + \frac{1}{M}\int_a^b |Q(t)|\,\mathrm{d}t\right].$$

Thus the proof of the theorem is complete. □

Remark.

(i) If $n \ge 3$ and $a = a_1, a_2, \dots, a_{n-1}, a_n = b$ are consecutive zeros of a solution $y(t)$ of (6), then $\mathrm{D}^2 y(t), \mathrm{D}^3 y(t), \dots, \mathrm{D}^{n-2} y(t)$ and $\mathrm{D}^{n-1} y(t)$ have zeros in $[a, b]$.

(ii) If $y(t) = (t-1)^{n-1}(t-2)^{m-1}$, $m \ge n \ge 2$, then $y(t)$, $y'(t)$, $y''(t), \dots$ $\dots, y^{(n-2)}(t), y^{(n-1)}(t)$ have zeros in $[1, 2]$ and $y(t) \ne 0$ for $t \in (1, 2)$.

Remark. For equation (4), inequality (7) takes the form

$$4 \le (b-a)\int_a^b \left|\int_{\alpha_1}^{s_1} |p(s_2)|\,\mathrm{d}s_2\right|\mathrm{d}s_1 \le (b-a)^2\int_a^b |p(t)|\,\mathrm{d}t\,,$$

which is same as the inequality in [8; Theorem 1]. For equation (1), inequality (8) takes the form (2).

Example. Consider

$$y^{(4)} + 4y = 0\,, \qquad t \ge 0\,.$$

Clearly, $y(t) = \mathrm{e}^{-t}\cos t$ is a solution of the equation with $y\big(\frac{\pi}{2}\big) = 0 = y\big(\frac{3\pi}{2}\big)$, $y''(\pi) = 0$, $y'''\big(\frac{5\pi}{4}\big) = 0$. From Theorem 1 it follows that

$$4 \le \pi\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \overline{E}\big(s_1, r_2(s_1), r_3(s_2); 4\big)\,\mathrm{d}s_1\,,$$

where

$$\overline{E}(s_1, r_2(s_1), r_3(s_2); 4)$$

$$= 4\left|\int_{\pi}^{s_1} \left|\int_{\frac{5\pi}{4}}^{s_2} \mathrm{d}s_3\right| \mathrm{d}s_2\right|$$

$$= 4\left|\int_{\pi}^{s_1} \left|\left(s_2 - \frac{5\pi}{4}\right)\right| \mathrm{d}s_2\right|$$

$$= \begin{cases} 4\left[\frac{5\pi}{4}s_1 - \frac{1}{2}s_1^2 - \frac{3\pi^2}{4}\right], & \text{if } s_2 < \frac{5\pi}{4},\ \pi < s_1 \text{ or } s_2 > \frac{5\pi}{4},\ s_1 < \pi, \\ 4\left[\frac{3\pi^2}{4} + \frac{1}{2}s_1^2 - \frac{5\pi}{4}s_1\right], & \text{if } s_2 < \frac{5\pi}{4},\ s_1 < \pi \text{ or } s_2 > \frac{5\pi}{4},\ s_1 > \pi. \end{cases}$$

Hence

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \overline{E}(s_1, r_2(s_1), r_3(s_2); 4)\ \mathrm{d}s_1$$

$$= \begin{cases} -\frac{\pi^3}{6}, & \text{if } s_2 < \frac{5\pi}{4},\ \pi < s_1 \text{ or } s_2 > \frac{5\pi}{4},\ \pi > s_1, \\ \frac{\pi^3}{6}, & \text{if } s_2 < \frac{5\pi}{4},\ s_1 < \pi \text{ or } s_2 > \frac{5\pi}{4},\ s_1 > \pi. \end{cases}$$

As $\overline{E} > 0$, then $s_2 < \frac{5\pi}{4}$, $s_1 < \pi$ or $s_2 > \frac{5\pi}{4}$, $s_1 > \pi$ and

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \overline{E}(s_1, r_2(s_1), r_3(s_2); 4)\ \mathrm{d}s_1 = \frac{\pi^3}{6}.$$

Thus, by Theorem 1, $4 < \frac{\pi^4}{6}$ or $24 < \pi^4$, which is obviously true.

Remark. Under appropriate assumptions on g and h, Liapunov-type inequalities can be derived for (3), (3') and (3'').

Remark. If

$$4 > \left(\int_a^b r_1(s_1)\ \mathrm{d}s_1\right)\left(\int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |p(s_{n-1})|)\ \mathrm{d}s_1\right),$$

then

$$\mathrm{D}^n y + p(t) = 0 \tag{11}$$

is disconjugate on $[a,b]$, where p is a real-valued continuous function on $[a,b]$.

Indeed, if (11) is not disconjugate on $[a,b]$, then it admits a nontrivial solution $y(t)$ which has n zeros in $[a,b]$. Let these zeros be given by $a \le a_1 < a_2 < \dots$

$\dots < a_{n-1} < a_n \le b$. Then $D^2 y(t)$, $D^3 y(t)$, ..., $D^{n-1} y(t)$ have zeros in $[a_1, a_n]$. From Theorem 1 it follows that

$$4 \le \left(\int_{a_1}^{a_n} r_1(s_1)\, ds_1\right)\left(\int_{a_1}^{a_n} \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |p(s_{n-1})|)\, ds_1\right)$$

$$\le \left(\int_a^b r_1(s_1)\, ds_1\right)\left(\int_a^b \overline{E}(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |p(s_{n-1})|)\, ds_1\right),$$

a contradiction. Thus (11) is disconjugate on $[a,b]$.

THEOREM 2. *Let* (C_1) *and* (C_2) *hold. Let* $\alpha_1, \alpha_2, \dots, \alpha_{n-3}, \alpha_{n-2}$ *be zeros of* $D^2 y(t)$, $D^3 y(t)$, ..., $D^{n-2} y(t)$, $D^{n-1} y(t)$ *in* $[a,b] \subset I$ $(a < b)$ *respectively, where* $y(t)$ *is a nontrivial solution of*

$$D^n y + y f(t,y) = 0$$

with $y(a) = 0 = y(b)$. *If* c *is a point in* (a,b) *where* $|y(t)|$ *attains maximum, then*

$$\left(\int_a^c r_1(t)\, dt\right)^{-1} < \infty \qquad \text{and} \qquad \left(\int_c^b r_1(t)\, dt\right)^{-1} < \infty .$$

P r o o f. Let $M = \max\{|y(t)| : t \in [a,b]\} = |y(c)|$. Then $y'(c) = 0$. Since

$$y(c) = \int_a^c y'(t)\, dt,$$

using Cauchy-Schwarz inequality and integrating by parts we obtain

$$M^2 = \left(\int_a^c y'(t)\, dt\right)^2 = \left[\int_a^c r_1^{\frac{1}{2}}(t) r_1^{-\frac{1}{2}}(t) y'(t)\, dt\right]^2$$

$$= \left(\int_a^c r_1(t)\, dt\right)\left[\int_a^c r_1^{-1}(t)\big(y'(t)\big)^2\, dt\right]$$

$$= \left(\int_a^c r_1(t)\, dt\right)\left(\Big[r_1^{-1}(t) y'(t) y(t)\Big]_a^c - \int_a^c (D\, y)'(t) y(t)\, dt\right)$$

$$\le \left(\int_a^c r_1(t)\, dt\right)\left(\int_a^c |(D\, y)'(t)||y(t)|\, dt\right)$$

$$\le M\left(\int_a^c r_1(t)\, dt\right)\left(\int_a^b |(D\, y)'(t)|\, dt\right).$$

Proceeding as in Theorem 1 we obtain

$$|(\mathrm{D}\,y)'(t)| \le M\overline{E}\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)\big)\,.$$

Hence

$$\left(\int_a^c r_1(t)\,\mathrm{d}t\right)^{-1} \le \int_a^b \overline{E}\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)\big)\,\mathrm{d}t < \infty\,. \tag{12}$$

Thus c cannot be very close to a because

$$\lim_{c\to a+}\left(\int_a^c r_1(t)\,\mathrm{d}t\right)^{-1} = \infty\,.$$

Next we show that c cannot be very close to b. Since

$$|y(c)| = \left|\int_c^b y'(t)\,\mathrm{d}t\right|,$$

then proceeding as above we obtain

$$\begin{aligned} M^2 &= \left[\int_c^b y'(t)\,\mathrm{d}t\right]^2 \\ &\le \left(\int_c^b r_1(t)\,\mathrm{d}t\right)\left(\int_c^b |(\mathrm{D}\,y)'(t)||y(t)|\,\mathrm{d}t\right) \\ &\le M\left(\int_c^b r_1(t)\,\mathrm{d}t\right)\left(\int_a^b |(\mathrm{D}\,y)'(t)|\,\mathrm{d}t\right) \\ &\le M^2\left(\int_c^b r_1(t)\,\mathrm{d}t\right)\left(\int_a^b \overline{E}\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)\big)\,\mathrm{d}t\right). \end{aligned}$$

Hence

$$\left(\int_c^b r_1(t)\,\mathrm{d}t\right)^{-1} \le \int_a^b \overline{E}\big(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); W(s_{n-1}, M)\big)\,\mathrm{d}t < \infty\,.$$

Thus c cannot be very close to b. This completes the proof of the theorem. □

Remark. Theorem 2 need not hold if $\alpha_i \notin [a, b]$ for some $i \in \{1, \dots, n-2\}$.

DEFINITION. A solution $y(t)$ of (6) is said to be *oscillatory* if there exists a sequence $\langle t_m \rangle \subset [0,\infty)$ such that $y(t_m) = 0$, $m \geq 1$, and $t_m \to \infty$ as $m \to \infty$.

THEOREM 3. *Let $p \in L^\sigma([0,\infty),\mathbb{R})$, where $1 \leq \sigma < \infty$. Let $r_i(t) \leq K$ for $t \geq 0$ and $1 \leq i \leq n-1$, where $K > 0$ is a constant. If $\langle t_m \rangle$ is an increasing sequence of zeros of an oscillatory solution $y(t)$ of*

$$\mathrm{D}^n y + p(t)y = 0, \qquad t \geq 0,$$

such that $\alpha_1,\dots,\alpha_{n-2} \in (t_m, t_{m+k})$, $1 \leq k \leq n-1$, for every large m, then $(t_{m+k} - t_m) \to \infty$, as $m \to \infty$, where $\alpha_1,\dots,\alpha_{n-2}$ are the zeros of $\mathrm{D}^2 y(t)$, $\mathrm{D}^3 y(t),\dots$, $\mathrm{D}^{n-2} y(t)$, $\mathrm{D}^{n-1} y(t)$, respectively.

Remark. If $n \geq 3$ and $k = n-1$, then $t_m, t_{m+1},\dots,t_{m+n-1}$ are n-consecutive zeros of $y(t)$ for every m, and hence $\alpha_1,\dots,\alpha_{n-2} \in (t_m, t_{m+n-1})$. If $n = 2$, such a condition is not required (see [10; Theorem 2]).

Proof of the theorem. If possible, let there exist a subsequence $\langle t_{m_i} \rangle$ of $\langle t_m \rangle$ such that $(t_{m_i+k} - t_{m_i}) \leq M$ for every i, where $M > 0$ is a constant. Let $\max\{|y(t)| : t \in [t_{m_i}, t_{m_i+k}]\} = |y(s_i)|$, where $s_i \in (t_{m_i}, t_{m_i+k})$. Since $p \in L^\sigma([0,\infty),\mathbb{R})$, then

$$\int_0^\infty |p(t)|^\sigma \, \mathrm{d}t < \infty .$$

Hence

$$\int_t^\infty |p(t)|^\sigma \, \mathrm{d}s \to 0 \qquad \text{as} \quad t \to \infty .$$

Thus, for $1 < \sigma < \infty$, we may have

$$\int_{t_{m_i}}^\infty |p(t)|^\sigma \, \mathrm{d}t < \left[K^{n-1} M^{n-1+\frac{1}{\mu}}\right]^{-\sigma}$$

for large i, where $\frac{1}{\mu} + \frac{1}{\sigma} = 1$. From (12) we obtain

$$\left(\int_{t_{m_i}}^{s_i} r_1(t) \, \mathrm{d}t\right)^{-1} \leq K^{n-2}\left(t_{m_i+k} - t_{m_i}\right)^{n-2} \int_{t_{m_i}}^{t_{m_i+k}} |p(t)| \, \mathrm{d}t ,$$

that is,

$$1 \leq K^{n-1}\left(t_{m_i+k} - t_{m_i}\right)^{n-1} \int_{t_{m_i}}^{t_{m_i+k}} |p(t)| \, \mathrm{d}t . \tag{13}$$

The use of Hölder's inequality yields

$$1 \le K^{n-1}\big(t_{m_i+k} - t_{m_i}\big)^{n-1}\big(t_{m_i+k} - t_{m_i}\big)^{\frac{1}{\mu}} \left[\int\limits_{t_{m_i}}^{t_{m_i+k}} |p(t)|^{\sigma}\, \mathrm{d}t\right]^{\frac{1}{\sigma}}$$

$$\le K^{n-1}\big(t_{m_i+k} - t_{m_i}\big)^{n-1+\frac{1}{\mu}} \left[\int\limits_{t_{m_i}}^{\infty} |p(t)|^{\sigma}\, \mathrm{d}t\right]^{\frac{1}{\sigma}}$$

$$< K^{n-1}M^{n-1+\frac{1}{\mu}}\left[K^{n-1}M^{n-1+\frac{1}{\mu}}\right]^{-1} = 1\,,$$

a contradiction. If $\sigma = 1$, then we choose i large enough such that

$$\int\limits_{t_{m_i}}^{\infty} |p(t)|\, \mathrm{d}t < (KM)^{-(n-1)}\,.$$

Hence from (13) we obtain

$$1 \le (KM)^{n-1} \int\limits_{t_{m_i}}^{\infty} |p(t)|\, \mathrm{d}t < 1\,,$$

a contradiction. Hence the theorem is proved. □

EXAMPLE. Consider

$$t^4 y^{(iv)} - 24y = 0\,, \qquad t \ge 1\,. \tag{14}$$

Clearly, the conditions of Theorem 3 are satisfied. A basis of the solution space of (14) is

$$\left\{\frac{1}{t},\, t^4,\, t^{\frac{3}{2}}\cos\left(\frac{\sqrt{15}}{2}\log t\right),\, t^{\frac{3}{2}}\sin\left(\frac{\sqrt{15}}{2}\log t\right)\right\}.$$

The zeros of

$$u(t) = t^{\frac{3}{2}}\cos\left(\frac{\sqrt{15}}{2}\log t\right)$$

are given by

$$t_n = \exp\left(\frac{1}{\sqrt{15}}(2n-1)\pi\right), \qquad n = 1, 2, 3, \dots\,.$$

Hence

$$t_{n+3} - t_n = \mathrm{e}^{\frac{2n\pi}{\sqrt{15}}}\left[\mathrm{e}^{\frac{5\pi}{\sqrt{15}}} - \mathrm{e}^{-\frac{\pi}{\sqrt{15}}}\right] \to \infty$$

as $n \to \infty$. We may note that the zeros of $u''(t)$ are expressed as

$$t_n^* = \exp\left[\frac{2}{\sqrt{15}}\left\{\tan^{-1}\left(-\frac{3}{\sqrt{15}}\right) - n\pi\right\}\right], \qquad n = 1, 2, \dots,$$

and hence $t_n^* \notin [t_n, t_{n+1}]$ for any large n. However, $u''(t)$ has a zero in (t_n, t_{n+2}) for every n. As the zeros of $u'''(t)$ are given by

$$t_n^{**} = \exp\left[\frac{2}{\sqrt{15}}\left\{\tan^{-1}\sqrt{\frac{3}{5}} - n\pi\right\}\right], \qquad n = 1, 2, \dots,$$

then $t_n^{**} \notin [t_n, t_{n+2}]$ for any large n. But $u'''(t)$ has a zero in (t_n, t_{n+3}). If

$$v(t) = \frac{1}{t} + t^{\frac{3}{2}} \cos\left(\frac{\sqrt{15}}{2}\log t\right) = t^{\frac{3}{2}}\left[t^{-\frac{5}{2}} + \cos\left(\frac{\sqrt{15}}{2}\log t\right)\right],$$

then it is an oscillatory solution of (14). The zeros of $v(t)$ may be written as

$$t_n^{\frac{-5}{2}} + \cos\left(2n\pi + \frac{\sqrt{15}}{2}\log t_n\right) = 0, \qquad n = 1, 2, \dots. \tag{15}$$

From Theorem 3 it follows that $t_{n+3} - t_n \to \infty$ as $n \to \infty$. However, it is not easy to show that $t_{n+3} - t_n \to \infty$ as $n \to \infty$ using (15).

As an application of Theorem 1, we prove the following theorem.

THEOREM 4. *Suppose that* (C_1) *and* (C_2) *hold with* $I = [0, \infty)$. *Let there exist a continuous function* $H\colon I \to \mathbb{R}^+$ *such that* $W(t, L) \le H(t)$ *for every constant* $L > 0$. *Let*

$$\int_0^\infty r_1(t)\, dt < \infty.$$

If

$$\int_0^\infty \overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|)\, dt < \infty$$

and

$$\int_0^\infty \overline{E}(t, r_2(t), r_3(s_2), \dots, r_{n-1}(s_{n-2}); H(s_{n-1}))\, dt < \infty,$$

for $n \ge 3$ *and*

$$\int_0^\infty H(t)\, dt < \infty \quad \textit{and} \quad \int_0^\infty |Q(t)|\, dt < \infty \qquad \textit{for} \quad n = 2,$$

then every oscillatory solution of (6) *converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be an oscillatory solution of (6) on $[T_y, \infty)$, $T_y \geq 0$. To complete the proof of the theorem, it is enough to show that $\limsup_{t\to\infty} |y(t)| = 0$. If possible, let $\limsup_{t\to\infty} |y(t)| = \lambda > 0$. Choose $0 < d < \lambda/2$. From the given assumptions it follows that it is possible to choose a large $T_0 > 0$ such that, for $t \geq T_0$,

$$\int_t^\infty r_1(s_1)\,\mathrm{d}s_1 < 1,$$

$$\int_t^\infty \overline{E}\big(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|\big)\,\mathrm{d}s_1 < d$$

and

$$\int_t^\infty \overline{E}\big(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); H(s_{n-1})\big)\,\mathrm{d}s_1 < 1$$

for $n \geq 3$ and $\int_t^\infty H(s)\,\mathrm{d}s < 1$ and $\int_t^\infty |Q(s)|\,\mathrm{d}s < d$ for $n = 2$. Since $y(t)$ is oscillatory, we can find a $t_1 > T_0$ such that $y(t_1) = 0$. Let $T_0^* > t_1$ be such that $\alpha_1, \alpha_2, \dots, \alpha_{n-3}, \alpha_{n-2} \in [t_1, T_0^*]$, where $\alpha_1, \alpha_2, \dots, \alpha_{n-3}, \alpha_{n-2}$ are the zeros of $\mathrm{D}^2 y(t)$, $\mathrm{D}^3 y(t)$, …, $\mathrm{D}^{n-2} y(t)$, $\mathrm{D}^{n-1} y(t)$ respectively. Further, $\limsup_{t\to\infty} |y(t)| > 2d$ implies that we can find a $T_0^{**} > t_1$ such that $\sup\{|y(t)| : t \in [t_1, T_0^{**}]\} > d$. Let $T_1 = \max\{T_0^*, T_0^{**}\}$. Let $t_2 > T_1$ such that $y(t_2) = 0$. If $M = \max\{|y(t)| : t \in [t_1, t_2]\}$, then $M > d$. From Theorem 1 we obtain (7) for $n \geq 3$ and (8) for $n = 2$, with $a = t_1$ and $b = t_2$. Hence, for $n \geq 3$,

$$\begin{aligned}4 \leq \bigg(\int_{t_1}^\infty r_1(s_1)\,\mathrm{d}s_1\bigg)\bigg(\int_{t_1}^\infty \Big[\overline{E}\big(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); H(s_{n-1})\big) \\ + \frac{1}{M}\overline{E}\big(s_1, r_2(s_1), r_3(s_2), \dots, r_{n-1}(s_{n-2}); |Q(s_{n-1})|\big)\Big]\,\mathrm{d}s_1\bigg) \\ < \Big(1 + \frac{d}{M}\Big) < 2,\end{aligned}$$

a contradiction. For $n = 2$,

$$4 \le \left(\int_{t_1}^{\infty} r_1(s_1)\, \mathrm{d}s_1 \right) \left[\int_{t_1}^{\infty} H(t)\, \mathrm{d}t + \frac{1}{M} \int_{t_1}^{\infty} |Q(t)|\, \mathrm{d}t \right]$$
$$< \left[1 + \frac{d}{M} \right] < 2,$$

a contradiction. Hence $\limsup_{t\to\infty} |y(t)| = 0$. Thus the theorem is proved. □

Following example illustrates Theorem 4.

EXAMPLE. Consider

$$\left(\mathrm{e}^t (\mathrm{e}^t y')' \right)' + 4y = 10\,\mathrm{e}^{-t} \cos t + 4\,\mathrm{e}^{-3t} \sin t, \qquad t \ge 0.$$

Thus $r_1(t) = \mathrm{e}^{-t}$, $r_2(t) = \mathrm{e}^{-t}$, $f(t,y) = 4$, and hence $H(t) = 4$. Clearly, $y(t) = \mathrm{e}^{-3t} \sin t$ is a solution of the equation with $y(0) = 0$ and $\left(\mathrm{e}^t y'(t)\right)' = 0$ for $t = \pi/4$. Hence $\alpha_1 = \pi/4$. Since

$$\overline{E}(s_1, r_2(s_1); H(s_2)) = 4\,\mathrm{e}^{-s_1} \left(s_1 - \frac{\pi}{4} \right) \qquad \text{for} \quad s_1 > \pi/4,$$

and

$$\overline{E}(s_1, r_2(s_1); |Q(s_2)|) \le -10\,\mathrm{e}^{-2s_1} - \frac{4}{3}\,\mathrm{e}^{-4s_1} + 10\,\mathrm{e}^{-(s_1 + \frac{\pi}{4})} + \frac{4}{3}\,\mathrm{e}^{-(s_1 + \frac{\pi}{4})}$$

for $s_1 > \pi/4$, then

$$\int_0^{\infty} \overline{E}(s_1, r_2(s_1); H(s_2))\, \mathrm{d}s_1 = 4$$

and

$$\int_0^{\infty} \overline{E}(s_1, r_2(s_1); |Q(s_2)|)\, \mathrm{d}s_1 \le \frac{16}{3} - 10\,\mathrm{e}^{-\frac{\pi}{4}} - \frac{4}{3}\,\mathrm{e}^{-\frac{3\pi}{4}}.$$

From Theorem 4 it follows that every oscillatory solution of the equation tends to zero as $t \to \infty$. Indeed, $y(t) = \mathrm{e}^{-3t} \sin t \to 0$ as $t \to \infty$.

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Department of Mathematics
Berhampur University
Berhampur-760007
INDIA