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Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

ON THE FUNCTION a_p, $p^{a_p(n)} \parallel n$ $(n > 1)$

TIBOR ŠALÁT[1]

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ABSTRACT. Some elementary properties of the arithmetical function $a_p(n)$ ($= \mathrm{ord}_p\, n$) are studied in this paper.

Introduction

Let p be a prime number. Then the function a_p is defined in the following way: $a_p(1) = 0$ and if $n > 1$, then $p^{a_p(n)} \parallel n$, i.e. $p^{a_p(n)} \mid n$, but $p^{a_p(n)+1} \nmid n$. In this paper we shall study some fundamental properties of the arithmetic function a_p.

1. Elementary properties of a_p and the average order of a_p

The function a_p is obviously completely additive, i.e.

$$a_p(n_1 \cdot n_2) = a_p(n_1) + a_p(n_2)$$

for arbitrary $n_1, n_2 \in \mathbb{N}$.

First of all we shall prove two simple results on a_p.

PROPOSITION 1.1. *Let p be a fixed prime number. Then the series*

$$\sum_{n=1}^{\infty} \frac{a_p(n)}{n^t}$$

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converges for $t > 1$ *and diverges for* $t \leqq 1$.

P r o o f. Let $t > 1$. Since $p^{a_p(n)} \mid n$ $(n > 1)$, we get

$$a_p(n) \leqq \frac{\log n}{\log p} \qquad (n = 1, 2, \dots).$$

Hence

$$\sum_{n=1}^{\infty} \frac{a_p(n)}{n^t} \leqq \frac{1}{\log p} \sum_{n=1}^{\infty} \frac{\log n}{n^t} < +\infty .$$

Let $t \leqq 1$. Then

$$\sum_{n=1}^{\infty} \frac{a_p(n)}{n^t} \geqq \sum_{n:\, a_p(n) \geqq 1} \frac{a_p(n)}{n^t} .$$

If $a_p(n) \geqq 1$, then $n = kp$, $k \geqq 1$. The series on the right-hand side contains each term

$$\frac{a_p(kp)}{(kp)^t} \qquad (k \geqq 1).$$

Therefore

$$\sum_{n=1}^{\infty} \frac{a_p(n)}{n^t} \geqq \sum_{k=1}^{\infty} \frac{a_p(kp)}{(kp)^t} \geqq \frac{1}{p^t} \sum_{k=1}^{\infty} \frac{1}{k^t} = +\infty .$$

□

In the following result we shall describe the behaviour of the differences $a_p(n+1) - a_p(n)$ $(n = 1, 2, \dots)$.

PROPOSITION 1.2. *The set*

$$\big(a_p(n+1) - a_p(n)\big)'_n$$

of all limit points of the sequence $\big(a_p(n+1) - a_p(n)\big)_{n=1}^{\infty}$ *contains* $+\infty$ *and all integers if* p *is an odd prime number and it contains* $+\infty$ *and all non-zero integers if* $p = 2$.

P r o o f. First of all observe that, if $n_k = p^k - 1$ $(k = 1, 2, \dots)$, then

$$\lim_{k \to \infty} \big(a_p(n_k + 1) - a_p(n_k)\big) = \lim_{k \to \infty} k = +\infty .$$

Further, let k be a fixed positive integer. We put $n_s = sp^k - 1$, where s runs over all positive integers which are not divisible by p. Then we get

$$a_p(n_s + 1) - a_p(n_s) = k$$

for each s. The assertion for $-k < 0$ can be proved by choosing $n_s = sp^k$.

If $p > 2$, then we put $n_s = sp + 1$, where $p \nmid s$. Then $a_p(n_s + 1) - a_p(n_s) = 0 - 0 = 0$.

Finally, it can be easily checked that $a_2(n+1) - a_2(n) \neq 0$ for every $n \in \mathbb{N}$. □

Put

$$S(a_p, n) = \frac{a_p(1) + a_p(2) + \cdots + a_p(n)}{n} \qquad (n = 1, 2, \ldots).$$

THEOREM 1.3. *We have*

$$\lim_{n\to\infty} S(a_p, n) = \frac{1}{p-1}.$$

P r o o f. On account of the complete additivity of a_p we get

$$S(a_p, n) = \frac{1}{n} \sum_{k=1}^{n} a_p(k) = \frac{1}{n} a_p(n!).$$

But for $a_p(n!)$ we have

$$a_p(n!) = \sum_{k=1}^{b_n} \left[\frac{n}{p^k} \right],$$

where $b_n = \left[\frac{\log n}{\log p} \right]$ (cf. [3; p. 342, Theorem 416]).

Using this fact a simple estimation yields

$$\frac{1 - \left(\frac{1}{p}\right)^{b_n}}{1 - \frac{1}{p}} - \frac{b_n}{n} < S(a_p, n) \leqq \frac{1}{p} \frac{1 - \left(\frac{1}{p}\right)^{b_n}}{1 - \frac{1}{p}}.$$

From this the assertion follows at once. □

2. Level sets of the function a_p

For $k \geqq 0$ we put

$$T_k = \{n : \ a_p(n) = k\} = a_p^{-1}(\{k\}) .$$

THEOREM 2.1. *We have*

$$d(T_k) = \lim_{x\to\infty} \frac{T_k(x)}{x} = \frac{p-1}{p^{k+1}} \qquad (k = 0, 1, 2, \dots)$$

($d(T_k)$ denotes the asymptotic density of T_k).

P r o o f . Let $T_k(x)$ ($x > 0$) denote the number of elements of T_k which are not greater than x. A positive integer n belongs to T_k if and only if it has the form bp^k, where $p \nmid b$. From this we get

$$T_k(x) = \left[\frac{x}{p^k}\right] - \left[\frac{\left[\frac{x}{p^k}\right]}{p}\right] .$$

A simple estimation gives

$$x\frac{p-1}{p^{k+1}} - 2 \leqq T_k(x) \leqq x\frac{p-1}{p^{k+1}} + 2 .$$

The theorem follows. □

R e m a r k 2.1. In [2] (see also [4]), the concept of statistical convergence is introduced. A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is said to be statistically convergent to $x \in \mathbb{R}$ (shortly: $\lim\operatorname{stat} x_n = x$) provided that for each $\varepsilon > 0$ we have $d\big(A(\varepsilon)\big) = 0$, where $A(\varepsilon) = \{n : \ |x_n - x| \geqq \varepsilon\}$, d being the asymptotic density. Theorem 2.1 says that

$$d(T_k) = \frac{p-1}{p^{k+1}} > 0 \qquad (k = 0, 1, \dots) .$$

From this it easily follows that $\big(a_p(n)\big)_{n=1}^{\infty}$ is not a statistically convergent sequence.

3. Sets $\{n : a_p(n) \mid n\}$

In the paper [1], the sets of the form $M_f = \{n : f(n) \mid n\}$ are investigated, where f is an arithmetical function with integer values. In [1], the density of M_f is determined for various functions f (e.g. for $\omega(n)$ – the number of distinct primes that divide n, $s(n)$ – the digital sum of n a.s.o.). In connection with these results we prove the following theorem.

THEOREM 3.1. *For each prime number p we have*

$$d(M_{a_p}) = (p-1) \sum_{(k,p)=1} \frac{1}{kp^{k+1}} + (p-1) \sum_{(k,p)>1} \frac{1}{kp^{k-s_k+1}},$$

where $p^{s_k} \| k$.

Proof. Obviously we have

$$M_{a_p} = \bigcup_{k=1}^{\infty} B_k, \tag{1}$$

where

$$B_k = \{n : a_p(n) = k \wedge k \mid n\} \qquad (k = 1, 2, \dots).$$

Let $x > 0$. We shall try to calculate the number $B_k(x)$ of all $n \in B_k$ not exceeding x.

For k we have two possibilities: 1. $p \nmid k$, 2. $p \mid k$.

1. Let $p \nmid k$. A positive integer n belongs to B_k if and only if it has the form $n = kp^k n_1$, where $p \nmid n_1$. From this we get

$$B_k(x) = \left[\frac{x}{kp^k}\right] - \left[\frac{\left[\frac{x}{kp^k}\right]}{p}\right] = c_k(x).$$

2. Let $p \mid k$. Then there is an s_k, $1 \leqq s_k \leqq \left[\frac{\log k}{\log p}\right]$, such that $p^{s_k} \| k$. A positive integer belongs to B_k if and only if it has the form $n = kp^{k-s_k} n_1$, where $p \nmid n_1$.

From this we get

$$B_k(x) = \left[\frac{x}{kp^{k-s_k}}\right] - \left[\frac{\left[\frac{x}{kp^{k-s_k}}\right]}{p}\right] = d_k(x).$$

Since the sets on the right-hand side of (1) are pairwise disjoint, we get

$$M_{a_p}(x) = \sum_{(k,p)=1} c_k(x) + \sum_{(k,p)>1} d_k(x) = S_1(x) + S_2(x) . \tag{2}$$

The summands corresponding to k's greater than $m_x = \left[2\frac{\log x}{\log p}\right]$ are zero. This is evident for $S_1(x)$ and for $S_2(x)$ it can be seen as follows. If $\frac{x}{p^{k-s_k}} < 1$, then $d_k(x) = 0$. Since $s_k \leqq \left[\frac{\log k}{\log p}\right] \leqq \frac{k}{2}$, we have $\frac{x}{p^{k-s_k}} \leqq \frac{x}{p^{\frac{k}{2}}}$. Hence, if $\frac{x}{p^{\frac{k}{2}}} < 1$, i.e. if $k > 2\frac{\log x}{\log p}$, then $d_k(x) = 0$. So we can suppose that $k \leqq m_x$.

So we get

$$S_1(x) = \sum_{k \leqq m_x,\, (k,p)=1} c_k(x) , \tag{3}$$

$$S_2(x) = \sum_{k \leqq m_x,\, (k,p)>1} d_k(x) . \tag{4}$$

Simple estimations give

$$x\frac{p-1}{kp^{k+1}} - 2 < c_k(x) < x\frac{p-1}{kp^{k+1}} + 2 ,$$

$$x\frac{p-1}{kp^{k-s_k+1}} - 2 < d_k(x) < x\frac{p-1}{kp^{k-s_k+1}} + 2 .$$

So we get

$$c_k(x) = x\frac{p-1}{kp^{k+1}} + O(1) , \qquad d_k(x) = x\frac{p-1}{kp^{k-s_k+1}} + O(1) . \tag{5}$$

From (3), (4), (5) we obtain

$$S_1(x) = x(p-1) \sum_{k \leqq m_x,\, (k,p)=1} \frac{1}{kp^{k+1}} + O(m_x) ,$$

$$S_2(x) = x(p-1) \sum_{k \leqq m_x,\, (k,p)>1} \frac{1}{kp^{k-s_k+1}} + O(m_x) .$$

Hence, according to the definition of m_x,

$$x^{-1} M_{a_p}(x) = (p-1) \sum_{k \leqq m_x,\, (k,p)=1} \frac{1}{kp^{k+1}} + (p-1) \sum_{k \leqq m_x,\, (k,p)>1} \frac{1}{kp^{k-s_k+1}} + o(1).$$

By $x \to \infty$, we get from this

$$d(M_{a_p}) = \lim_{x\to\infty} \frac{M_{a_p(x)}}{x} = (p-1) \sum_{(k,p)=1} \frac{1}{kp^{k+1}} + (p-1) \sum_{(k,p)>1} \frac{1}{kp^{k-s_k+1}},$$

where $p^{s_k} \parallel k$. □

The following result on the behaviour of the sequence $\big(d(M_{a_p})\big)_p$ (p runs over all primes) is a simple consequence of Theorem 3.1.

THEOREM 3.2. *We have* $\lim_{p\to\infty} d(M_{a_p}) = 0$.

P r o o f. Simple estimations yield

$$d(M_{a_p}) \leqq (p-1)\left(\frac{1}{p^2} + \sum_{k\geqq 2,\, (k,p)=1} \frac{1}{kp^{k+1}}\right) + (p-1)\left(\frac{1}{pp^p} + \sum_{k>p,\, (k,p)>1} \frac{1}{kp^{k-s_k+1}}\right) = S_1 + S_2.$$

Further,

$$S_1 < \frac{p-1}{p^2} + (p-1)\int_2^\infty \frac{\mathrm{d}t}{p^{t+1}} = \frac{p-1}{p^2} + \frac{p-1}{p^2 \log p} \to 0 \qquad \text{by} \quad p \to \infty,$$

$$S_2 = \frac{p-1}{p^{p+1}} + (p-1) \sum_{k>p,\, (k,p)>1} \frac{1}{kp^{k-s_k+1}}.$$

But $k - s_k \geqq k - \left[\frac{\log k}{\log p}\right] > \frac{k}{2}$ for $k > p$. Thus

$$S_2 < \frac{p-1}{p^{p+1}} + (p-1)\int_p^\infty \frac{\mathrm{d}t}{p^{\frac{t}{2}}} = \frac{p-1}{p^{p+1}} + 2\frac{p-1}{p^{\frac{p}{2}}\log p} \to 0 \qquad \text{by} \quad p \to \infty.$$

□

4. Density and statistical convergence of the sequence $\left(\log p \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$

In [5] O. Strauch has proved the following result:

THEOREM 4.1. *The sequence* $\left(\log p \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ *is dense in the interval* $(0,1)$.

Proof. We shall outline the proof of O. Strauch.

Let n runs over all numbers of the form $p^\alpha q^\beta$, where q is a fixed prime number different from p and α, β are positive integers.

Let $x \in (0,1)$. Then $x = \frac{1}{1+y}$, where $y > 0$. Let $\varepsilon > 0$. The density of rational numbers in $\mathbb{R}$ implies the existence of positive integers α, β such that

$$\left| y - \frac{\beta \log q}{\alpha \log p} \right| < \varepsilon . \tag{5'}$$

If $n = p^\alpha q^\beta$, then we have

$$\log \frac{a_p(n)}{\log n} = \log p \frac{\alpha}{\alpha \log p + \beta \log q} = \left(1 + \frac{\beta \log q}{\alpha \log p}\right)^{-1} . \tag{5''}$$

From (5'), (5") we get

$$\left| x - \log p \frac{a_p(n)}{\log n} \right| = \left| \frac{1}{1+y} - \frac{1}{1 + \frac{\beta \log q}{\alpha \log p}} \right| < \left| y - \frac{\beta \log q}{\alpha \log p} \right| < \varepsilon .$$

The theorem follows. □

THEOREM 4.2. *We have*

$$\lim \operatorname{stat} \log p \frac{a_p(n)}{\log n} = 0 .$$

Proof. Let $\varepsilon > 0$, put $A(\varepsilon) = \left\{ n > 1 : \log p \frac{a_p(n)}{\log n} \geqq \varepsilon \right\}$.

Let $\eta > 0$. Choose an integer $K > 0$ such that

$$p^{-K} < \eta . \tag{6}$$

Then there exists an n_0 such that for $n > n_0$ we have

$$n^{\varepsilon} > p^{K} . \tag{7}$$

Let $n \in A(\varepsilon)$, $n > n_0$. Then, according to (6), (7), we have $\varepsilon \log n > K \log p$ and $a_p(n) \geqq \frac{\varepsilon \log n}{\log p} > K$. Therefore

$$A(\varepsilon) \subseteqq \{2, 3, \dots, n_0\} \cup \{n > n_0 : \ p^K \mid n\} . \tag{8}$$

It follows from (8) and (6) that

$$\limsup_{n \to \infty} \frac{A(\varepsilon)(n)}{n} \leqq \frac{1}{p^K} < \eta .$$

Since $\eta > 0$ is an arbitrary positive number, we get $d\big(A(\varepsilon)\big) = 0$. □

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