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UNIQUELY EDGE COLOURABLE GRAPHS

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Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

The aim of this paper is to extend known results concerning the (finite simple) uniquely edge colourable graphs into further classes of graphs. All finite uniquely k-edge colourable graphs (including multigraphs) with $4 \leq k < \aleph_0$ are constructed and enumerated. Similarly, all uniquely k-edge colourable graphs with $k \geq \aleph_0$ are described. Cases $k < 3$ and $k = 3$ are considered as well. The results were presented at the Third Czechoslovak Symposium on Graph Theory held in Prague, 1982 [3].

1. Introduction

All *graphs* considered in this paper are undirected and *loopless* (= without loops), but we admit *multigraphs* (= graphs with multiple edges) as well as infinite graphs. A graph is said to be *simple* if it contains no multiple edges. Isomorphic graphs are usually not considered as different. By the *order [size]* of a graph we mean the number of its vertices [edges, respectively].

Let k be a cardinal number. A graph G is said to be *uniquely k-edge colourable* if 1. G has chromatic index k; 2. every admissible edge colouring (adjacent edges have always different colours) of G by k colours induces the same partition of the edge set of G. (In other words, G has a unique decomposition D^G into k factors of maximum degree one and no decomposition into less than k such factors.)

The class of all uniquely k-edge colourable graphs will be denoted by J_k. A graph is said to be *uniquely edge colourable* if it is uniquely k-edge colourable for a cardinal number k.

Finite simple uniquely edge colourable graphs were treated in papers [5, 6, 8, 13, 14, 16, 17] and in books [1, 7]. Our aim is to describe all uniquely edge colourable graphs but this aim at the present time is reached only partly. Evidently, we may restrict ourselves to graphs without isolated vertices.

2. The case $k<3$

For $k<3$ the situation is very simple:

Proposition 1. *For $k<3$ all uniquely k-edge colourable graphs without isolated vertices are:*

1. *the empty graph if $k=0$,*
2. *1-regular graphs if $k=1$,*
3. *even cycles and all paths (finite, one-sided infinite, two-sided infinite) of length $\geqslant 2$ if $k=2$.*

Only one of these graphs, the cycle of length 2, is not simple.

Proof. This follows directly from the definition (see also the papers cited above).

3. Examples and auxiliary general results

Examples of uniquely k-edge colourable graphs for $k\geqslant 3$ can be easily constructed as graphs G satisfying the following conditions C(i, k), where i and k are cardinal numbers, $1\leqslant i\leqslant 5$ and $3\leqslant k$. Moreover, if $3\leqslant i\leqslant 5$, we suppose that k is finite and the vertices of G are denoted by v_1, v_2, v_3, ...

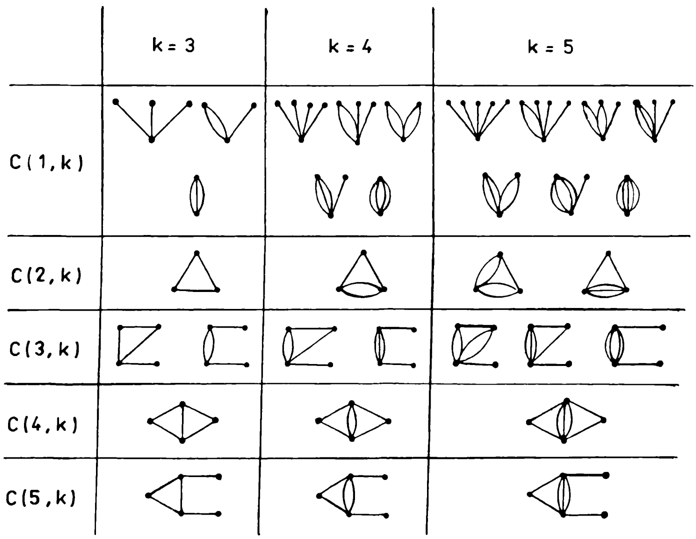

Fig. 1

Condition C(1, k). G has size k and contains at least one vertex incident with all edges of G.

Condition C(2, k). G has order 3, size k and contains at least one triangle.

Condition C(3, k). G has order 4 and size $k+1$; it contains edges v_1v_2 and v_3v_4; each of the remaining $k-1$ edges joins v_3 to v_1 or v_2.

Condition C(4, k). G has order 4 and size $k+2$; it contains edges v_1v_3, v_1v_4, v_2v_3, v_2v_4 and $k-2$ edges joining v_1 and v_2.

Condition C(5, k). G has order 5 and size $k+2$; it contains edges v_1v_3, v_1v_4, v_2v_3, v_2v_5 and $k-2$ edges joining v_1 and v_2.

It is easy to show that for every admissible i and k there exists at least one graph G satisfying Condition C(i, k) and that all these graphs are uniquely k-edge colourable. For $k \in \{3, 4, 5\}$ all graphs G satisfying Conditions C(i, k), $i \in \{1, 2, 3, 4, 5\}$ are drawn on Fig. 1. Evidently, no graph can fulfil two different Conditions C(i, k).

Denote by $p(k)$ the number of partitions of a positive integer k (i.e., the number of unordered decompositions of k into a sum of positive integers). The values of $p(k)$ for $k = 1, 2, \ldots, 100$ are given in [9]. They can be easily calculated using the following recurrence formula (whose validity follows also from [9]):

$$p(k)=\sum_{s=1}^{\infty}(-1)^{s-1}(p(k-\tfrac{1}{2}s(3s-1))+p(k-\tfrac{1}{2}s(3s+1)))=$$
$$=p(k-1)+p(k-2)-p(k-5)-p(k-7)+$$
$$+p(k-12)+p(k-15)-p(k-22)-p(k-26)+\ldots,$$

where we put $p(0)=1$ and $p(x)=0$ for $x<0$. Evidently, in the infinite series for $p(k)$, given a fixed k, only finitely many (exactly $[(\sqrt{(24k+1)}+1)/6]$) indices s produce a non-zero term. We need here only some small values of $p(k)$ given in Table 1.

Table 1.

k	0	1	2	3	4	5	6	7	8	9	10
$p(k)$	1	1	2	3	5	7	11	15	22	30	42

k	11	12	13	14	15	16	17	18	19	20
$p(k)$	56	77	101	135	176	231	297	385	490	627

Lemma 1. *Let k be an integer, $k \geqslant 3$. Then there exists exactly*

$$N(k)=p(k)+\left[\frac{k^2+3}{12}\right]+\left[\frac{k+5}{2}\right]$$

graphs fulfilling one of Conditions C(1, k), C(2, k), ..., C(5, k) *and all these graphs are finite, connected, planar and uniquely k-edge colourable.*

Proof. Determine the number of (nonisomorphic) graphs fulfilling Conditions C(i, k). For $i=1$ this number is obviously equal to $p(k)$, for $i=2$ this number is equal to the number $p_3(k)$ of partitions of k into 3 parts, which by [8] equals

$$p_3(k)=\left\lceil\frac{k^2+3}{2}\right\rceil.$$

If $i=3$, then the number of enumerated graphs is equal to the number of expressions $k-1=a+b$ with nonnegative integers $a\geqslant b$, i.e. to the number $p_2(k+1)$ of partitions of $k+1$ into two (positive) parts and we evidently have

$$p_2(k+1)=\left\lceil\frac{k+1}{2}\right\rceil.$$

For $i=4$ and $i=5$ Condition C(i, k) is fulfilled by exactly one graph. Thus

$$N(k)=p(k)+p_3(k)+p_2(k+1)+1+1=$$
$$=p(k)+\left\lceil\frac{k^2+3}{2}\right\rceil+\left\lceil\frac{k+5}{2}\right\rceil.$$

The second assertion can be proved by the systematic examination of the possibilities allowed by Conditions C(i, k). The proof follows.

In Table 2 we give the values of $N(k)$ for $k=3, 4, \ldots, 20$.

Table 2.

k	3	4	5	6	7	8	9	10
$N(k)$	8	10	14	19	25	33	44	57

k	11	12	13	14	15	16	17	18	19	20
$N(k)$	74	97	124	160	205	262	332	423	532	672

Note that for $k=3$ Lemma 1 does not determine all (even finite, connected and planar) uniquely k-edge colourable graphs. There exist finite connected planar uniquely 3-edge colourable graphs satisfying none of Conditions C(i, 3), e.g. K_4, the complete graph on 4 vertices. The case $k=3$ will be studied in detail in the next section. The question whether the converse to Lemma 1 is true in the case $k\geqslant 4$ will be dealt with in Section 5.

Lemma 2 (cf. [8]). *Let j and k be cardinal numbers such that $1\leqslant j\leqslant k$ and let $G\in J_k$. Then every subgraph of G generated by the edges of fixed j colours belongs to J_j.*

Proof. This is obvious.

As already mentioned, studying the class J_k we need not consider graphs containing isolated vertices. The following lemma allows us to restrict ourselves to connected graphs.

Lemma 3. *Let $k \geq 2$, $G \in J_k$. Then G is connected if and only if G has no isolated vertices.*

Proof. Let G have no isolated vertices. It is sufficient to prove that any two edges of G are in a path or in a cycle of G. But this follows from Lemma 2 (for $j=2$) and the last assertion of Proposition 1. The converse assertion is trivial.

Lemma 4 (cf. [5]). *Let $k \geq 2$, $G \in J_k$. If A and B are the sets of the edges of two different colours in G, then either $|A| = |B|$, or A and B are finite and $|A| - |B| \in \{1, -1\}$.*

Proof. This also follows from Lemma 2 ($j=2$) and the last assertion of Proposition 1.

Lemma 5. *Let $k \geq 2$, $G \in J_k$. Then we have:*

(i) *If G has multiple edges, then their colours occur in G exactly once.*

(ii) *Supposing* (i), *the colours of all other edges occur in G exactly once or twice.*

(iii) *The edges of any two different colours generate a connected subgraph of G* (cf. [8]).

Proof. (i) Let e_1 and e_2 be different edges of G with the same end vertices. By Lemma 2 ($j=2$) and Proposition 1 the edges of colours of e_1 and e_2 generate a subgraph of G that is a cycle of length two.

(ii) This follows from Lemma 4.

(iii) This follows from Lemmas 2 and 3.

Lemma 6. *Let k be a positive integer, $k \geq 2$ and let G be a finite connected graph from J_k. Then for every positive integer $i < k$ the number s_i of vertices of degree i in G is finite and we have:*

(i) $$\sum_{i=1}^{k-1} i(k-i)s_i \leq k(k-1).$$

(ii) $$1 \neq s_1 + s_2 + \ldots + s_{k-1} \leq k.$$

Proof. (i) By Lemma 2 the edges of arbitrary two different colours generate a subgraph of G belonging to J_2 which will be called a *bicoloured subgraph* of G. Every vertex having degree $i < k$ in G has degree 1 in exactly $i(k-i)$ bicoloured subgraphs of G. As the number of bicoloured subgraphs of G is $\binom{k}{2}$ and each of them has at most two vertices of degree 1, the sum of the numbers of vertices of degree 1 in bicoloured subgraphs of G is at most $k(k-1)$ and (i) follows.

(ii) The inequality $\leq$ follows from (i) and the trivial inequality $i(k-i) \geq k-1$ holding for every $i \in \{1, 2, \ldots, k-1\}$. If $s_1 + s_2 + \ldots + s_{k-1} = 1$, then in G there exists a bicoloured subgraph with only one vertex of degree 1, which is impossible in a finite graph.

Lemma 7. *Let k be a positive integer, $k \geq 2$ and let G be an infinite connected* *graph from J_k. Then G is simple and all vertices of G have degree k with a possible exception of a unique vertex of degree $i \in \{1, 2, \ldots, k-1\}$.*

Proof. The first assertion follows from Lemma 5. As G has chromatic index k, G cannot have a vertex of degree $>k$. Suppose that G has two vertices, say u and v, of degree $<k$. Let D^G be the decomposition of G into factors of (maximum) degree 1. Denote by $F(u)$ $[F(v)]$ the set of factors from D^G containing at least one edge incident with u $[v$, respectively]. Evidently,

$$D^G = D_1 \cup D_2 \cup D_3 \cup D_4,$$

where

$$\begin{aligned} D_0 &= F(u) \cap F(v), \\ D_1 &= F(u) - F(v), \\ D_2 &= F(v) - F(u), \\ D_3 &= D^G - (F(u) \cup F(v)). \end{aligned}$$

It is easy to see that at least one of the following two cases must occur:
(i) $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$.
(ii) $D_0 \neq \emptyset$ and $D_3 \neq \emptyset$.

Pick factors $F_1, F_2 \in D^G$ so that if (i) holds, then $F_1 \in D_1$, $F_2 \in D_2$; otherwise let $F_1 \in D_0$, $F_2 \in D_3$. In both cases $F_1 \cup F_2$ is a finite path so that F_1 and F_2 have a finite number of edges. By Lemma 4, all factors from D^G have the same property. As $|D^G| = k < \aleph_0$, it follows that G has a finite number of edges, which is impossible because G is infinite and (by Lemma 3) connected.

4. The case $k=3$

In the case $k=3$ a complete description of the uniquely k-edge colourable graphs is known neither for finite graphs nor for infinite ones. Therefore we show here only how our investigation can be reduced and we formulate main open problems. In fact, the content of the most of this section is known (usually in a slightly different formulation) from the quoted sources [1, 5, 6, 7, 8, 13, 14, 15, 17].

The next result shows that for $k=3$ Lemma 1 gives all connected uniquely k-edge colourable multigraphs. Moreover, we determine how many vertices of degree $<k$ a graph from J_k can have (evidently, it cannot have vertices of degree $>k$).

Proposition 2. *Let G be a connected uniquely 3-edge colourable graph with exactly s vertices of degree <3. Then we have:*

(i) *If G is a multigraph, then G is isomorphic to one of three multigraphs drawn for $k=3$ in Fig. 1.*

(ii) *If G is finite, then $s \in \{0, 2, 3\}$.*

(iii) *If G is infinite, then $s \in \{0, 1\}$.*

Proof. (i) easily follows from Lemma 5, (ii) from Lemma 6, and (iii) from Lemma 7.

The problem of constructing all graphs from J_3 is thus reduced to connected simple graphs with a "small" number of vertices of degree <3. Moreover, we can suppose that this degree is 2. In fact, deleting a vertex of degree 1 (and the edge adjacent to it) in a graph G from J_3 leads to a graph G' with the same number of vertices of degree <3, but instead of a vertex of degree 1 we get a vertex of degree 2. Evidently, $G' \in J_3$, or G is one of the first two graphs of Fig. 1. In such a way the vertices of degree 1 can be successively deleted from the graph. Thus our problem in the case $k=3$ for infinite graphs can be formulated in the following way:

Problem 1. Find all infinite connected uniquely 3-edge colourable graphs in which all vertices are of degree 3 with a possible exception of one vertex of degree 2.

According to Lemma 7 every such graph is simple. Examples of these graphs are in Fig. 2.

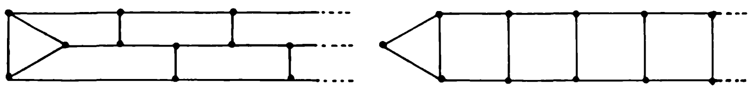

Fig. 2

The following two reductions, namely reductions to cubic and cyclically connected graphs will be applied only to finite graphs although the second one is in a certain sense applicable also to infinite graphs.

The reduction to cubic graphs is based on the following Lemma 8. The symbol $G-v$ $[G-e]$ will denote the graph formed from G by deleting vertex v [edge e, respectively]. (If we delete a vertex v from G, then we must delete also all edges incident with v.)

Lemma 8. *Let v $[e]$ be a vertex [an edge, respectively] of a finite cubic simple graph G. Then the following three assertions are equivalent:* (i) $G \in J_3$. (ii) $G-v \in J_3$. (iii) $G-e \in J_3$.

Proof. G is a cubic graph of order $2n$ and size $3n$ (where n is an integer, $n \geq 2$) and every factor of the decomposition D^G of G into factors of (maximum) degree 1 has size n. The graph $G-v$ has order $2n-1$, size $3n-3$, 3 vertices of degree 2, $2n-4$ vertices of degree 3 and every factor of D^{G-v} has size $n-1$. The graph $G-e$ has order $2n$, size $3n-1$, 2 vertices of degree 2 and $2n-2$ vertices of degree 3. Two factors of D^{G-e} have size n and the third one has size $n-1$. Using these facts the equivalences (i)$\Leftrightarrow$(ii) and (i)$\Leftrightarrow$(iii) can be easily established. In every case it is sufficient to prove the validity of two conditions from the definition of uniquely k-edge colourable graphs, which is easy. The lemma follows.

It is evident that each graph having all vertices of degree 3 with an exception of exactly two [three] vertices of degree 2 can be represented in the form $G-e$ $[G-v]$, where G is a cubic graph with an edge e [a vertex v, respectively]. Thus the problem of finding all finite uniquely 3-edge colourable graphs is reduced to cubic graphs.

A connected graph is said to be *cyclically* 4-*connected* if the removal of less than 4 edges from it cannot yield a graph with two components each containing a cycle.

Suppose that G is a finite simple uniquely 3-edge colourable graph that is not cyclically 4-connected. Let E be a set of less than 4 edges whose removal yields a graph with two components G_1 and G_2, each containing a cycle. As $G \in J_3$, we cannot have $|E|=1$ or $|E|=2$ (evidently — cf. [10] — two edges of E must belong to the same factor of D^G). In both cases mutual exchanging of two colours in G_2 leads to a new edge colouring of G by 3 colours. Therefore $|E|=3$ and the three edges of E have 6 end vertices (otherwise there would be a set E' of less than 3 edges whose removal would yield a graph with two components each containing a cycle, a contradiction to previous considerations). Evidently there exist cubic graphs H_1 and H_2 with vertices v_1 and v_2, respectively, such that $H_i - v_i$ is isomorphic to G_i $(i=1,2)$. Obviously, the three edges of E belong to mutually different factors of D^G (cf. [10] or the proof of our Lemma 8). Therefore H_1 and H_2 are finite simple cubic graphs from J_3 of an order smaller than that of G. Thus the graphs that are not cyclically 4-connected need not be considered and we can formulate the next problem.

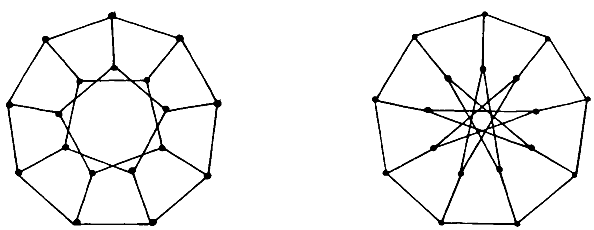

Fig. 3

Problem 2 (cf. [5, 6, 8, 13, 17]). Describe all finite, simple, uniquely 3-edge colourable, and cyclically 4-connected cubic graphs.

Only two examples of such graphs are known: The first one is planar, namely K_4, the complete graph on 4 vertices. The second one is nonplanar, namely the generalized Petersen graph on 18 vertices denoted usually by $P(9, 2)$ or $P(9, 4)$ whose two diagrams are given in Fig. 3.

This graph has been found independently by three authors [5, 6, 12, 15]. Ninčák [12] constructed it in connection with the following related problem.

Problem 2′ (cf. [2, 8, 12, 13, 14, 15]). Describe all finite simple cyclically 4-connected cubic graphs with exactly 3 hamiltonian cycles.

Thomason [14] proved that the classes of graphs determined by Problems 2 and 2′ are different (in fact, the first class is a proper subclass of the second). However it is possible that in the case of planar graphs these two classes coincide and they consist of a unique graph K_4. This is equivalent to the conjecture that every finite simple cubic planar graph that has exactly 3 hamiltonian cycles (especially, every finite simple cubic planar uniquely 3-edge colourable graph) contains a triangle so that it can be obtained from K_4 by successively replacing a vertex by a triangle (cf. [2, 8, 13, 17], or Cantoni's conjecture in [15]).

5. The case $4 \leqslant k < \aleph_0$

R. J. Wilson [16] formulated a conjecture that for every integer $k \geqslant 4$ every finite connected simple and uniquely k-edge colourable graph is isomorphic to the complete bipartite graph $K(1, k)$ (i.e., the *star* of size k). This conjecture has been proved by A. G. Thomason [13]:

Lemma 9 [13]. *Let k be an integer, $k \geqslant 4$. Then every finite connected simple uniquely k-edge colourable graph is isomorphic to $K(1, k)$.*

Our aim is to extend this result to graphs with multiple edges (multigraphs).

Theorem 1. *A connected multigraph is uniquely 4-edge colourable if and only if it is isomorphic to some of 9 multigraphs given for $k = 4$ in Fig. 1.*

Proof. According to Lemma 1 all 9 multigraphs given in Fig. 1 for $k = 4$ are connected and uniquely 4-edge colourable. To prove the converse, let G be a connected uniquely 4-edge colourable multigraph. Denote by m the multiplicity of G, i.e., the maximum multiplicity of an edge of G. Evidently $m \in \{2, 3, 4\}$. Distinguish 3 cases:

1. $m = 4$. Then, according to Lemma 5, G is the 5th graph of case $k = 4$ in Fig. 1.
2. $m = 3$. Then, according to Lemma 5, G is the 4th or the 8th graph of the case $k = 4$ in Fig. 1.
3. $m = 2$. Pick a 2-cycle in G. According to Lemma 5 the colours of its edges occur in G only once and the remaining two colours once or twice; thus the number of the corresponding edges is 2, 3 or 4. By Lemma 2 and Proposition 1, these edges generate a path of length 2, 3, or 4, or a cycle of length 2, or 4. Using Lemmas 3 and 5 it is easy to show that G is isomorphic to one of the remaining 6 multigraphs.

Corollary. *A finite connected graph is uniquely 4-edge colourable if and only if it is isomorphic to one of 10 graphs given in Fig. 1 for $k=4$.*

Proof. This follows from Lemma 9 and Theorem 1.

Remark. This Corollary can be proved also without using Lemma 9, but then the proof is longer.

Theorem 2. *Let k be an integer, $k \geqslant 4$. A finite connected graph G is uniquely k-edge colourable if and only if G fulfils some of Conditions* C(1, k), C(2, k), ..., C(5, k).

Proof. If G fulfils a condition C(i, k), then, according to Lemma 1, G is uniquely k-edge colourable. The converse assertion will be proved by induction. For $k=4$ this follows from Corollary to Theorem 1. Let n be an integer, $n \geqslant 4$ and suppose that each finite graph from J_n fulfils one of Conditions C(i, n), $i \in \{1, 2, 3, 4, 5\}$. Let G be a finite connected graph from J_{n+1} and suppose that its edge colouring by colours 1, 2, ..., $n+1$ is given. Let $G(j)$, $j=1, 2, \ldots, n+1$ be the subgraph of G generated by the edges of colours $\neq j$. If we denote the size of G by q and the size of $G(j)$ by q_j, we have:

$$q = \frac{q_1 + q_2 + \ldots + q_{n+1}}{n}, \quad n \leqslant q_j \leqslant n+2,$$

$$\frac{(n+1)n}{n} \leqslant q \leqslant \frac{(n+1)(n+2)}{n}.$$

It follows that

$$n+1 \leqslant q \leqslant n+3.$$

As G has at most $n+3$ edges coloured by $n+1$ colours, there exists a colour used only once. If we delete the corresponding edge (and, may be, also an isolated vertex), we get a finite connected graph G'. By Lemma 2 we have $G' \in J_n$. According to the induction hypothesis, $G = G'$ fulfils a condition C(i, n), $i \in \{1, 2, 3, 4, 5\}$. Add an edge to G' in such a way that we get a graph G'' that belongs to J_{n+1}. It is easy to check that $G = G''$ fulfils either Condition C(i, $n+1$), or (if $i=1$), may be, Condition C($i+1$, $n+1$). The theorem follows.

Corollary 1. *For every integer $k \geqslant 4$ there exist exactly $N(k)$ nonisomorphic finite connected uniquely k-edge colourable graphs, where $N(k)$ is defined in Lemma 1.*

Proof. This follows from Lemma 1 and Theorem 2.

Remark. For $k=3$, Theorem 2 and its Corollary 1 are not valid (see Section 4 of this paper).

Problem 3. Describe all infinite connected uniquely k-edge colourable graphs for $4 \leqslant k < \aleph_0$.

Remarks. 1. If graphs of Problem 3 exist, they must satisfy necessary conditions given in Lemma 7.

2. No example is known and we do not know if it exists.

The following result is a contribution to the question which graphs have the chromatic index equal to the (maximum) degree [7, 17].

Corollary 2 (for finite simple graphs see Theorems 3 of [5] or [8]). *Let k be a positive integer. Let G be a uniquely k-edge colourable graph. Then G has (maximum) degree less than k if and only if $k \geq 3$ and G fulfils Condition* C(2, k).

Proof. For $k \leq 3$ the assertion is obvious, for $k \geq 4$ and finite graphs it follows from Theorem 2, for infinite graphs from Lemma 7.

Note that the only simple graph satisfying C(2, k) for $k = 3$ is K_3, the complete graph of order 3; for $k \geq 4$ no such simple graphs exist.

6. The case $k \geq \aleph_0$

The case of an infinite k ($k \geq \aleph_0$) is relatively easy, even for vertex colourings (see Proposition 3 below).

Given a cardinal number k, a graph G is said to be *uniquely k-vertex colourable*, if G has chromatic number k and every admissible vertex colouring of G by k colours induces the same partition of the vertex set of G. Evidently, a graph is uniquely k-edge colourable iff its line graph is uniquely k-vertex colourable.

Proposition 3. *Let k be an infinite cardinal number. Then every connected uniquely k-vertex [k-edge] colourable graph has exactly k vertices [edges] and these vertices [edges] are mutually adjacent. Only one of these graphs, the complete graph of order k [the star of size k, respectively], is simple.*

Proof. Let G be a connected uniquely k-vertex[k-edge]colourable graph. As G has chromatic number [chromatic index] k, G has at least k vertices [edges]. We assert that in every vertex [edge] colouring of G by k colours the vertices [edges] have mutually different colours. Otherwise, taking two vertices [edges] of the same colour and changing the colour of one of them into a new colour a new vertex [edge] colouring of G by k colours could be obtained inducing a different partition of the vertex [edge] set, a contradiction. Therefore G has exactly k vertices [edges] coloured in every admissible colouring by different colours. If there exist in G two non-adjacent vertices [edges], the identification of their colours leads to a different partition of the vertex [edge] set, a contradiction again. The rest of the proof is obvious.

Corollary. *Let k be an infinite cardinal number. Then a connected graph G is uniquely k-edge colourable if and only if G satisfies Condition* C(1, k) *or Condiction* C(2, k).

Proof. This is a consequence of Proposition 3.

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ОДНОЗНАЧНО РЕБЕРНО ОКРАШЕННЫЕ ГРАФЫ

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Резюме

В статье построены и перечислены все конечные k-реберно окрашенные графы для произвольного натурального числа $k \geq 4$ и охарактеризованы все однозначно k-реберно окрашенные графы для любого бесконечного крадинального числа k. Случаи $k \leq 3$ также исследуются.