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Conrad [3] studied the system of all convex l-subgroups of a lattice ordered group G. Jakubík and Kolibiar [7] investigated pairs of distributive lattices L_1 and L_2 with the same underlying set such that the system of all convex sublattices of L_1 coincides with the system of all convex sublattices of L_2. They proved that L_1 and L_2 can differ only by duality of a direct factor.

The paper presented is a contribution to the investigation of an analogous question concerning partially ordered groups. In the paper there are studied pairs of isolated abelian partially ordered groups $(H, \leqq)$, $(H, \leqq')$ with the same underlying set and the same group operation such that the system of all convex subgroups of $(H, \leqq)$ coincides with the system of all convex subgroups of $(H, \leqq')$. It will be shown that instead of direct factors (as in the case examined by Jakubík and Kolibiar in [7]) we have now to deal with certain subdirect factors of $(H, \leqq)$ and $(H, \leqq')$, respectively, which are either linearly ordered or trivially ordered. For the main results concerning partially ordered groups cf. 2.2, 2.3 and 2.20. A related question for abelian lattice ordered groups is dealt with in Theorem 3.1.

In Section 4 there are investigated mixed products with factors which are either linearly ordered or trivially ordered.

In Section 5 we obtain necessary and sufficient conditions for certain particular cases.

1. Preliminaries

First we recall some notions and denotations which will be used in the paper. Throughout this paper let $\leqq \sim$ denote for a partially ordered group (notation po-group) $(G, \leqq)$ the dual of $\leqq$. The group operations in po-groups will be written additively.

$(R, \leqq)$, $(Q, \leqq)$ and $(Z, \leqq)$ will denote additive groups of all real numbers, rational numbers and integers with the natural order. The set of all positive integers will be denoted by N.

Let $(G, \leqq)$ be a po-group. A subgroup C of $(G, \leqq)$ is said to be convex if $a, c \in C$, $b \in G$ and $a \leqq b \leqq c$ imply $b \in C$.

We say that a po-group $(G, \leqq)$ is isolated if $a \in G$ and $na \geqq 0$ for some $n \in N$ imply $a \geqq 0$.

A subgroup A of an abelian group G is called pure if the equation $nx = g$, where $g \in A$ and $n \in N$, is solvable in A whenever it is solvable in the whole group.

Let Γ be a partially ordered set (notation po-set) and for each $i \in \Gamma$ let $(H_i, \leqq_i)$ be a nontrivial po-group. Let $V = V[\Gamma, H_i]$ be the following subset of the large direct sum of the H_i. An element $v = (\dots, v_i, \dots)$ belongs to V if and only if $S_v = \{i \in \Gamma;\ v_i \neq 0\}$ contains no infinite ascending sequence. This is equivalent to the maximum condition. V is a subgroup of the large direct sum of the H_i. If $v \in V$, $v_i \neq 0$ and $v_j = 0$ for all $j > i$, then v_i is called a maximal component of v. A nonzero element of V is positive if each maximal component v_i of v is positive with respect to the partial order on the group H_i. Then $V[\Gamma, H_i]$ is a po-group [4, Th. 2.1]. We shall denote this po-group by $V[\Gamma, (H_i, \leqq_i)]$. The po-group $V[\Gamma, (H_i, \leqq_i)]$ is called the mixed product of po-groups $(H_i, \leqq_i)$. For $x, y \in V[\Gamma, (H_i, \leqq_i)]$, $x \neq y$ let $M_x = \{i \in \Gamma, x_i \neq 0$ and $x_j = 0$ for all $j > i\}$ and let $M_{xy} = \{i \in \Gamma, x_i \neq y_i$ and $x_j = y_j$ for all $j > i\}$.

Unless otherwise stated, in this section $(G, \leqq)$ will always denote an isolated abelian po-group and Γ will be the set of all pairs of convex pure subgroups (G^i, G_i) of $(G, \leqq)$ such that G^i covers G_i (i.e. $G_i \subset G^i$ and for any pure convex subgroup K of G, $G_i \subset K \subseteq G^i$ implies $K = G^i$). We shall frequently identify the pair (G^i, G_i) with i. For i and j in Γ define that $i \leqq j$ if either $G^i = G^j$ and $G_i = G_j$, or $G^i \subseteq G_j$. Then Γ is a po-set [4, p. 148].

For $X \in G^i/G_i$ we define $X > G_i$ if and only if $X \neq G_i$ and nX contains an element $p > 0$ for some positive integer n. Then G^i/G_i is a po-group for each i in Γ [2, p. 22]. For $i \in \Gamma$, G^i/G_i is order isomorphic (notation o-isomorphic) to a subgroup of $(R, \leqq)$ unless it is trivially ordered, and in this case it is isomorphic to a subgroup of the additive group Q of all rational numbers [2, p. 23].

Let $(G, \leqq)$ be an isolated abelian po-group and let Γ be as above. $(G, \leqq)$ is said to be factorially rational if each nontrivially ordered group G^i/G_i, $i \in \Gamma$, is o-isomorphic to a subgroup of $(Q, \leqq)$.

If $g \in G^i \setminus G_i$, $i \in \Gamma$, then i is said to be a value of g. $(G, \leqq)$ is finite valued if each $g \in G$ has only a finite number of values.

An isomorphism φ of G into $V[\Gamma, G^i/G_i]$ is said to be valuation preserving (notation v-isomorphism) provided that it satisfies

(v) $i \in \Gamma$ is a value of $g \in G$ if and only if $(g\varphi)_i$ is a maximal component of $g\varphi$, and in this case $(g\varphi)_i = g + G_i$.

From [2, p. 23] it follows that each v-isomorphism φ of $(G, \leqq)$ into $V[\Gamma, (G^i/G_i, \leqq_i)]$ is an o-isomorphism.

For the remainder of this section let $(G, \leqq)$ be a divisible isolated abelian po-group. Then G is a rational vector space. We need the following result (cf. [4, p. 148], where we find the remark that this result goes back to Banaschewski [1]).

(T) There exists a mapping π of the set of all subspaces of G into itself such that for all subspaces A and B of G

(i) $G = A \oplus \pi(A)$, and

(ii) if $A \subseteq B$, then $\pi(A) \supseteq \pi(B)$.

In the paper of Conrad, Harvey and Holland [4, p. 148—149] there was investigated the mapping φ of G into $V[\Gamma, G^i/G_i]$ such that for $x \in G$ $x\varphi$ is defined as follows: for each $i \in \Gamma$ let $(x\varphi)_i = x_i + G_i$, where $x = x_i + c_i$, $x_i \in G^i$ and $c_i \in \pi(G^i)$. It was proved that φ is a v-isomorphism of $(G, \leqq)$ into $V[\Gamma, (G^i/G_i, \leqq_i)]$.

2. Partially ordered groups with the same convex subgroups

In this section there are studied pairs of isolated abelian po-groups $(H, \leqq)$, $(H, \leqq')$ with the same underlying set and the same group operation such that the system of all convex (directed convex) subgroups of $(H, \leqq)$ coincides with the system of all convex (directed convex) subgroups of $(H, \leqq')$. Such po-groups will be called groups with the same convex (directed convex) subgroups.

2.1. Lemma. *Let $(H, \leqq)$, $(H, \leqq')$ be isolated abelian po-groups with the same convex subgroups. Assume that $a \in H$. Then a is comparable with 0 in $(H, \leqq)$ if and only if a is comparable with 0 in $(H, \leqq')$.*

Proof. Without loss of generality we may assume that $a \geqq 0$. Let A be the subgroup of H generated by the element $2a$ and let $(C(A), \leqq)$ be the convex subgroup of $(H, \leqq)$ generated by A. Since $(H, \leqq)$, $(H, \leqq')$ have the same convex subgroups, $(C(A), \leqq')$ is a convex subgroup generated by A in $(H, \leqq')$. Let P' be the positive cone of $(H, \leqq')$. Then from [5, Chap. II, p. 32] we have $C(A) = (A + P') \cap (A + (-P'))$. Since $a \in C(A)$, we obtain $a = m(2a) + u$, where $u \leqq' 0$, $m \in Z$. From this we get $a = m(2a) + u \leqq' m(2a))$. Thus $0 \leqq' (2m-1)a$. If $(2m-1) \in N$, then $0 \leqq' a$, because $(H, \leqq')$ is isolated. If $(2m-1)$ is a negative integer, then $(1-2m) \in N$. Since $(H, \leqq')$ is isolated, from the relation $0 \leqq' (1-2m)(-a)$ we get $a \leqq' 0$. Thus a is comparable with 0 in $(H, \leqq')$. The sufficiency of the conditions can be verified analogously.

2.2. Proposition. *Let $(G, \leqq)$ and $(G, \leqq')$ be abelian po-groups with the same group operation. Then $(G, \leqq)$ and $(G, \leqq')$ have the same convex subgroups if and only if*

(1) $x, y \in G$ and $0 < y < x$ imply $mx <' y <' nx$ for some $m, n \in Z$ and

(2) $z, t \in G$ and $0 <' z <' t$ imply $kt < z < lt$ for some $k, l \in Z$.

Proof. The conditions are obviously sufficient, we now show their necessity. Let $(G, \leqq)$, $(G, \leqq')$ have the same convex subgroups and let $0 < y < x$ for some $x, y \in G$. Denote by A the subgroup of G generated by the element x. Let $(C(A), \leqq)$ be the convex subgroup of $(G, \leqq)$ generated by A. Since $(G, \leqq)$, $(G, \leqq')$ have the same convex subgroups, $(C(A), \leqq')$ is a convex subgroup generated by A in $(G, \leqq')$. Let P' be the positive cone $(G, \leqq')$. Then from [5, Chap. II, p. 32] we have $C(A) = (A + P') \cap (A + (-P'))$. Since $y \in C(A)$, we obtain $y = mx + u$, $y = nx + v$, where $u \leqq' 0$, $0 \leqq' v$, $m, n \in Z$. From this we get $nx \leqq' y \leqq' mx$. Assertion (2) can be verified analogously.

2.3. Theorem. *Let $(H, \leqq)$, $(H, \leqq')$ be nontrivial isolated divisible abelian groups with the same convex subgroups. Then there exists a po-set Γ and for each $i \in \Gamma$ there exist ordered groups $(C_i, \leqq_i)$ and $(C_i, \leqq'_i)$ such that*

(i) *$(C_i, \leqq_i)$, $(C_i, \leqq'_i)$ have the same group operation and card $C_i > 1$,*

(ii) *the following conditions* (1)—(4) *are fulfilled:*

(1) *there exists a mapping φ of H into $V[\Gamma, C_i]$ such that φ is a v-isomorphism of $(H, \leqq)$ into $V[\Gamma, (C_i, \leqq_i)]$ and also a v-isomorphism of $(H, \leqq')$ into $V[\Gamma, (C_i, \leqq'_i)]$,*

(2) *for each $i \in \Gamma$ we have either* that

(a) *both $(C_i, \leqq_i)$ and $(C_i, \leqq'_i)$ are trivially ordered and each of them is isomorphic to a subgroup of* Q

or (b) *both $(C_i, \leqq_i)$ and $(C_i, \leqq'_i)$ are linearly ordered and each of them is o-isomorphic to a subgroup of $(R, \leqq)$,*

(3) *there exists no element $0 < h \in H$ such that $(h\varphi)_i <' 0$, $0 <' (h\varphi)_j$ for some maximal components $(h\varphi)_i$, $(h\varphi)_j$ of $h\varphi$,*

(4) *there exists no element $0 <' g \in H$ such that $(g\varphi)_k < 0$, $0 < (g\varphi)_l$ for some maximal components $(g\varphi)_k$, $(g\varphi)_l$ of $g\varphi$.*

Proof. If $(H, \leqq)$, $(H, \leqq')$ have the same convex subgroups, then they also have the same pure convex subgroups. Let Γ be the set of all pairs of pure convex subgroups (H^i, H_i) of $(H, \leqq)$ and $(H, \leqq')$ such that H^i covers H_i. For each $i \in \Gamma$, $(H^i/H_i, \leqq_i)$ is o-isomorphic to a subgroup of $(R, \leqq)$ unless it is trivially ordered, and in this case it is isomorphic to a subgroup of Q. The same is valid for $(H^i/H_i, \leqq'_i)$, $i \in \Gamma$ [2, p. 23].

If $i \in \Gamma$ and $(H^i/H_i, \leqq_i)$ is linearly ordered, then there exists $h > 0$, $h \in nX$ for some $n \in N$, $X \in H^i/H_i$, $X \neq H_i$. In view of 2.1 we have $0 <' h$ or $h <' 0$. Thus $H_i <'_i X$ or $X <'_i H_i$. Hence $(H^i/H_i, \leqq'_i)$ is linearly ordered. Thus (2) is valid. Γ can be partially ordered as shown in Section 1. Then the mapping φ of H into $V[\Gamma, H^i/H_i]$ defined as in Section 1 is a v-isomorphism of $(H, \leqq)$ into $V[\Gamma, (H^i/H_i, \leqq_i)]$ and also a v-isomorphism of $(H, \leqq')$ into $V[\Gamma, (H^i/H_i, \leqq'_i)]$, because $(x\varphi)_i$ is determined by group properties of H^i for each $i \in \Gamma$.

(3) and (4) are consequences of 2.1.

It is easy to verify (cf. e.g. [6, Section 1.3]) that if $(G, \leqq)$ is an isolated abelian po-group, then there exists an isolated divisible abelian po-group $(Z(G), \leqq)$

such that $(G, \leqq)$ is a po-subgroup of $(Z(G), \leqq)$ and if $z \in Z(G)$, then there exist $x \in G$ and $m \in N$ such that $mz = x$.

In the present paper $Z(G)$ has the same meaning as in [6, Section 1.3], i.e., $Z(G)$ is the set of all expressions of the form $\frac{x}{n}$, where $x \in G$, $n \in N$, subject to the rules of:

a) equality: $\frac{x}{n} = \frac{y}{k}$ if and only if $kx = ny$,

b) addition: $\frac{x}{n} + \frac{y}{k} = \frac{kx + ny}{nk}$,

c) partial order: for $z \in Z(G)$ we have $z > 0$ if and only if there exists $x \in G$, $x > 0$ such that $z = \frac{x}{n}$ for some $n \in N$.

If C is a subgroup of G, then we can assume that $Z(C)$ is a subgroup of $Z(G)$.

2.4. Lemma. *Let B be a subgroup of a po-group $(A, \leqq)$ and let C be a convex subgroup of $(A, \leqq)$. Then $C \cap B$ is a convex subgroup of $(B, \leqq)$.*

Proof. The assertion is obvious.

2.5. Lemma. *Let $(G, \leqq)$ be an isolated abelian po-group and let C be a convex subgroup of $(G, \leqq)$. Then $Z(C)$ is a convex subgroup of $(Z(G), \leqq)$.*

Proof. Let $0 \leqq y \leqq x$ for some $x \in Z(C)$, $y \in Z(G)$. Then there exist $m, n \in N$ such that $my \in G$, $nx \in C$. Thus $mny \in G$, $mnx \in C$. From the relation $0 \leqq mny \leqq mnx$ and the convexity of C in $(G, \leqq)$ we obtain $mny \in C$. Hence $y \in Z(C)$.

2.6. Lemma. *Let $(G, \leqq)$ be an isolated abelian po-group. Then for each $g \in G$ the equation $g = nx$, where $n \in N$, has at most one solution.*

Proof. Let x_1, x_2, $g \in G$ and let $nx_1 = nx_2 = g$ for some $n \in N$. Then $n(x_1 - x_2) = 0$. Since G is isolated, we have $x_1 - x_2 \geqq 0$. If $x_1 \neq x_2$, we obtain $0 < n(x_1 - x_2)$, a contradiction. Thus $x_1 = x_2$.

As a consequence of 2.6 we obtain

2.7. Lemma. *Let $(G, \leqq)$ be an isolated divisible abelian pogroup. Then for each $g \in G$ the equation $g = nx$, where $n \in N$, has a unique solution.*

2.8. Lemma. *Let $(G, \leqq)$ be an isolated abelian po-goup and let C be a pure convex subgroup of $(G, \leqq)$. Then $Z(C)$ is a pure convex subgroup of $Z(G)$ and $Z(C) \cap G = C$.*

Proof. Let $a \in Z(C)$, $b \in Z(G)$ and $a = nb$ for some $n \in N$. Then there exist $k, l \in N$ such that $ka \in C$, $lb \in G$. Since klb is a solution of the equation $kla = nx$, from 2.6 and the purity of C in G we get $klb \in C$. Thus $b \in Z(C)$. The convexity of $Z(C)$ follows from 2.5. Hence $Z(C)$ is a pure convex subgroup of $Z(G)$.

Clearly $C \subseteq Z(C) \cap G$. Let $a \in Z(C) \cap G$. Then a is a solution of the equation $nx = b$ for some $b \in C$, $n \in N$. From 2.6 and the purity of C in G we get $a \in C$.

Remark. If we do not assume that C is pure in 2.8, then the relation $Z(C) \cap G = C$ need not be valid.

2.9. Lemma. *Let $(G, \leqq)$ be an isolated abelian po-group and let C be a pure convex subgroup of $(Z(G), \leqq)$. Then $C \cap G$ is a pure convex subgroup of $(G, \leqq)$ and $Z(C \cap G) = C$.*

Proof. Let $a \in C \cap G$, $b \in G$ and $a = nb$ for some $n \in N$. Since $a, b \in Z(G)$, from 2.7 and the purity of C in $Z(G)$ we get $b \in C$. Thus $b \in C \cap G$. The convexity of $C \cap G$ follows from 2.4.

Let $x \in C$. Then $nx \in G \cap C$ for some $n \in N$. Thus $x \in Z(G \cap C)$. Hence $C \subseteq Z(G \cap C)$. Let $y \in Z(G \cap C)$. Then $my \in G \cap C$ for some $m \in N$. Since C is pure in $Z(G)$, from 2.7 it follows that $y \in C$. Thus $Z(G \cap C) \subseteq C$.

2.10. Lemma. *Let $(G, \leqq)$ be an isolated abelian po-group and (A^i, A_i) be a pair of convex pure subgroups of $(G, \leqq)$ such that A^i covers A_i. Then $(Z(A^i), Z(A_i))$ is a pair of convex pure subgroups of $(Z(G), \leqq)$ such that $Z(A^i)$ covers $Z(A_i)$.*

Proof. In view of 2.8 it suffices to verify that $Z(A^i)$ covers $Z(A_i)$. Let $Z(A_i) \subseteq B \subseteq Z(A^i)$ for some pure convex subgroup B of $Z(G)$. Then $Z(A_i) \cap G \subseteq B \cap G \subseteq Z(A^i) \cap G$. From 2.8 we also have $Z(A_i) \cap G = A_i$, $Z(A^i) \cap G = A^i$. By 2.9 $G \cap B$ is a pure convex subgroup of G and $Z(G \cap B) = B$. Since A^i covers A_i, we get $G \cap B = A_i$ or $G \cap B = A^i$. Thus $Z(A_i) = B$ or $Z(A^i) = B$. From 2.8 we get $Z(A^i) \neq Z(A_i)$. Hence $Z(A^i)$ covers $Z(A_i)$.

2.11. Corollary. *Let $(G, \leqq)$ be an isolated abelian po-group, $g \in G$ and (A^i, A_i) be a value of g in $(G, \leqq)$. Then $(Z(A^i), Z(A_i))$ is a value of g in $(Z(G), \leqq)$.*

2.12. Lemma. *Let $(G, \leqq)$ be an isolated abelian po-group and let (B^i, B_i) be a pair of convex pure subgroups of $(Z(G), \leqq)$ such that B^i covers B_i. Then $(G \cap B^i, G \cap B_i)$ is a pair of convex pure subgroups of $(G, \leqq)$ such that $G \cap B^i$ covers $G \cap B_i$.*

Proof. In view of 2.9 it suffices to verify that $G \cap B^i$ covers $G \cap B_i$. Let $G \cap B_i \subseteq A \subseteq G \cap B^i$ for some pure convex subgroup A of G. Then $Z(G \cap B_i) \subseteq Z(A) \subseteq Z(G \cap B^i)$. From 2.9 we also have $Z(G \cap B_i) = B_i$, $Z(G \cap B^i) = B^i$. By 2.8 $Z(A)$ is a pure convex subgroup of $Z(G)$ and $Z(A) \cap G = A$. Since B^i covers B_i, we obtain $Z(A) = B^i$ or $Z(A) = B_i$. Thus $A = B^i \cap G$ or $A = B_i \cap G$. From 2.9 we get $G \cap B^i \neq G \cap B_i$. Hence $G \cap B^i$ covers $G \cap B_i$.

2.13. Corollary. *Let $(G, \leqq)$ be an isolated abelian po-group, $g \in Z(G)$ and let*

(B^i, B_i) be a value of g in $(Z(G), \leqq)$. Then there exists $n \in N$ such that $(G \cap B^i, G \cap B_i)$ is a value of ng in $(G, \leqq)$.

Proof. In view of 2.12 it suffices to show that $ng \in G \cap B^i \backslash G \cap B_i$.

If $g \in Z(G)$, $g \in B^i$, then there exists $n \in N$ such that $ng \in G \cap B^i$.

If $ng \in G \cap B_i$, then $g \in Z(G \cap B_i)$. From 2.9 we get $Z(G \cap B_i) = B_i$. Hence $g \in B_i$, a contradiction. Thus $ng \notin G \cap B_i$.

From 2.8 and 2.9 we obtain the following corollary

2.14. Corollary. *If $(H, \leqq)$, $(H, \leqq')$ are isolated abelian po-groups with the same pure convex subgroups, then $(Z(H), \leqq)$, $(Z(H), \leqq')$ have also the same pure convex subgroups and the mapping α of the set P_1 of all pure convex subgroups of $(H, \leqq)$ into the set P_2 of all pure convex subgroups of $(Z(H), \leqq)$ such that $A\alpha = Z(A)$ for each $A \in P_1$ is one-to-one and onto.*

2.15. Lemma. *Let $(H, \leqq)$, $(H, \leqq')$ be isolated abelian po-groups with the same convex subgroups. Then $(Z(H), \leqq)$, $(Z(H), \leqq')$ also have the same convex subgroups.*

Proof. Let C be a convex subgroup of $(Z(H), \leqq)$. Let $0 \leqq' y \leqq' x$ for some $x \in C$, $y \in Z(H)$. Then there exist $m, n \in N$ such that $mx, ny \in H$. Thus $mnx \in C \cap H$. In view of 2.4 $C \cap H$ is a convex subgroup of $(H, \leqq)$. Thus $C \cap H$ is also a convex subgroup of $(H, \leqq')$. Then from $0 \leqq' mny \leqq' mnx$ we get $mny \in C \cap H$. Hence $mny \in C$. Since $0 \leqq' mny$, from 2.1 it follows that $0 \leqq mny$ or $mny \leqq 0$. Since $(Z(H), \leqq)$ is isolated, we infer that $0 \leqq y \leqq mny$ or $0 \leqq -y \leqq mn(-y)$. From this and the convexity of C in $(Z(H), \leqq)$ we obtain $y \in C$. Hence C is a convex subgroup of $(Z(H), \leqq')$.

2.16. Lemma. *Let $(H, \leqq)$, $(H, \leqq')$ be isolated abelian po-groups with the same convex subgroups. Then $(H, \leqq)$, $(H, \leqq')$ have the same directed convex subgroups.*

Proof. Let C be a directed convex subgroup of $(H, \leqq)$. Then C is a convex subgroup of $(H, \leqq')$. Let $y \in C$. Then there exists an element $x \in C$ such that $0 \leqq x$, $y \leqq x$. In view of 2.1 we get $x \leqq' 0$ or $0 \leqq' x$, $y \leqq' x$ or $x \leqq' y$.

1) If $0 \leqq' x$ and $x \leqq' y$ or $x \leqq' 0$ and $y \leqq' x$, then $0, y$ are comparable.

2) If $x \leqq' 0$ and $x \leqq' y$, then $t = y - x$ is an element of C and $0 \leqq' t$, $y \leqq' t$.

3) If $0 \leqq' x$ and $y \leqq' x$, then x is an upper bound of $0, y$ in $(H, \leqq')$. Thus C is a directed convex subgroup of $(H, \leqq')$.

The following example shows that if two isolated abelian po-groups have the same directed convex subgroups, then they need not have the same convex subgroups and if some element is comparable with 0 in one group, then it need not be comparable with 0 in other group.

Example. Let $(G, \leqq)$ be the direct product $(R, \leqq) \oplus (R, \leqq)$ and let $(G, \leqq')$ be the direct product $(R, \leqq) \oplus (R, \leqq \sim)$. Then $(G, \leqq)$, $(G, \leqq')$ have the same directed convex subgroups (cf. Th. 4.3). The $H = \{(x, -x); x \in R\}$ is a convex subgroup of $(G, \leqq)$, but H is not a convex subgroup of $(G, \leqq')$. The element $a = (1,1) \in G$ is comparable with 0 in $(G, \leqq)$, but it is not comparable with 0 in $(G, \leqq')$.

2.17. Lemma. *Let $(G, \leqq)$ be an isolated abelian po-group and let C be a directed convex subgroup of $(Z(G), \leqq)$. Then $C \cap G$ is a directed convex subgroup of $(G, \leqq)$.*

Proof. From 2.4 it follows that $C \cap G$ is a convex subgroup of $(G, \leqq)$. Let $x \in C \cap G$. Since C is a directed subgroup, there exists $y \in C$ such that $0 \leqq y$, $x \leqq y$. Then there exists $n \in N$ such that $ny \in G$. Thus we get $0 \leqq ny$, $x \leqq ny$, $ny \in C \cap G$. Hence $C \cap G$ is a directed convex subgroup of G.

2.18. Lemma. *Let $(G, \leqq)$ be an isolated abelian po-group and let C be a directed convex subgroup of $(G, \leqq)$. Then $Z(C)$ is a directed convex subgroup of $(Z(G), \leqq)$.*

Proof. This follows from 2.5.

2.19. Proposition. Let $(H, \leqq)$, $(H, \leqq')$ be isolated abelian po-groups with the same directed convex subgroups. Then $(Z(H), \leqq)$, $(Z(H), \leqq')$ also have the same directed convex subgroups.

Proof. Let C be a directed convex subgroup of $(Z(H), \leqq)$. Let $0 \leqq' y \leqq' x$ for some $x \in C$, $y \in Z(H)$. Then there exist $m, n \in N$ such that $mx \in H$, $ny \in H$. Thus $mnx \in C \cap H$, $mny \in H$. From 2.17 it follows that $C \cap H$ is a directed convex subgroup of $(H, \leqq)$. Since $(H, \leqq)$ and $(H, \leqq')$ have the same directed convex subgroups, $C \cap H$ is also a directed convex subgroup of $(H, \leqq')$. Then from $0 \leqq' mny \leqq' mnx$ we get $mny \in C \cap H$. Since C is a directed subgroup of $(Z(H), \leqq)$, there exist elements $u, v \in C$ such that $0 \leqq v$, $mny \leqq v$, $u \leqq 0$, $u \leqq mny$. Then we obtain $mny \leqq mnv$, $mnu \leqq mny$. Since $(Z(H), \leqq)$ is an isolated group, we obtain $u \leqq y \leqq v$. From the convexity of C in $(Z(H), \leqq)$ it follows that $y \in C$. Thus C is a convex subgroup of $(Z(H), \leqq')$.

Let $z \in C$. Then there exists $k \in N$ such that $kz \in C \cap H$. Since $C \cap H$ is a directed convex subgroup of $(H, \leqq')$, there exists $t \in C$ such that $0 \leqq' t$, $kz \leqq' t$. Then $kz \leqq' kt$. Since $(Z(H), \leqq')$ is an isolated group, we obtain $z \leqq' t$. Hence C is a directed subgroup of $(Z(H), \leqq')$.

From 2.11, 2.13, 2.14 and 2.15 we obtain

2.20. Corollary: *The hypothesis that $(H, \leqq)$, $(H, \leqq')$ are divisible can be omitted in* 2.3.

3. Lattice ordered grouips with the same convex l-subgroups

In this section there are studied pairs of abelian lattice ordered groups (notation l-groups) $(H, \leqq)$, $(H, \leqq')$ with the same underlying set and the same group operation such that the system of all convex l-subgroups of $(H, \leqq)$

coincides with the system of all convex l-subgroups of $(H, \leqq')$. Such l-subgroups will be called l-groups with the same convex l-subgroups.

A convex l-subgroup A of an abelian l-group $(G, \leqq)$ is said to be regular if it is maximal with respect to not containing some element of G.

A po-set M is called a root system if no pair of incomparable elements of M have a common lower bound.

Let Γ_1 be the set of all pairs of convex l-subgroups (H^i, H_i) of $(H, \leqq)$ and $(H, \leqq')$ such that H^i covers H_i. If $(H, \leqq)$, $(H, \leqq')$ have the same convex l-subgroups, then they also have the same regular l-subgroups. Let $\Gamma = \{i \in \Gamma_1, H_i \text{ is regular}\}$. For i and j in Γ define that $i \leqq j$ if either $H^i = H^j$ and $H_i = H_j$ or $H^i \subseteq H_j$. Then Γ is a root system and $V[\Gamma, (H^i/H_i, \leqq_i)]$, $V[\Gamma, (H^i/H_i, \leqq'_i)]$ are l-groups [4, Th. 2.2, Lemma 4.2].

For each $i \in \Gamma$ $(H^i/H_i, \leqq_i)$, $(H^i/H_i, \leqq'_i)$ are nontrivial linearly ordered groups each of which is o-isomorphic to a subgroup of $(R, \leqq)$ [4, *p.* 143].

In view of (4, Th. 4.2] the mapping φ of H into $V[\Gamma, H^i/H_i]$ defined similarly as in Section 1 (i.e. for $x \in H$ $x\varphi$ is defined as follows: for each $i \in \Gamma$ let $(x\varphi)_i = x_i + H_i$, where $x = x_i + c_i$, $x_i \in H^i$ and $c_i \in \pi(H^i)$) is an l-isomorphism of $(H, \leqq)$ into $V[\Gamma, (H^i/H_i, \leqq_i)]$ and also an l-isomorphism of $(H, \leqq')$ into $V[\Gamma, (H^i/H_i, \leqq'_i)]$.

Thus we have

3.1. Theorem. *Let* $(H, \leqq)$, $(H, \leqq')$ *be nontrivial divisible abelian l-groups with the same convex l-subgroups. Then there exists a root system* Γ *and for each* $i \in \Gamma$ *there exist nontrivial* linearly ordered groups $(C_i, \leqq_i$, $(C_i, \leqq'_i)$ such that
(1) there exists a mapping φ of H into $V[\Gamma, C_i]$ such that φ is an l-isomorphism of $(H, \leqq)$ into the l-group $V[\Gamma, (C_i, \leqq_i)]$ and also an l-isomorphism of $(H, \leqq')$ into the l-group $V[\Gamma, (C_i, \leqq'_i)]$,
(2) for all $i \in \Gamma$ each of the groups $(C_i, \leqq_i)$, $(C_i, \leqq'_i)$ is o-isomorphic to a subgroup of $(R, \leqq)$.

Let $(G, \leqq)$ be an abelian l-group. Then $(Z(G), \leqq)$ is a divisible abelian l-group and $(G, \leqq)$ is an l-subgroup of $(Z(G), \leqq)$ [6, 1.4].

3.2. Lemma. *Let* $(G, \leqq)$, $(G, \leqq')$ *be abelian l-groups with the same convex l-subgroups. Then* $(Z(G), \leqq)$, $(Z(G), \leqq')$ *also have the same convex l-subgroups.*

Proof. This is a consequence of 2.19.

3.3. Corollary. The hypothesis that $(H, \leqq)$, $(H, \leqq')$ are divisible can be omitted in 3.1.

4. A Hahn-type po-group

Let Γ be a po-set and for each $i \in \Gamma$ let $(H_i, \leqq_i)$ be a nonzero linearly or trivially ordered group.

When no misunderstanding is likely to arise we shall omit index i in the notation of the partial order on H_i. We shall denote the neutral elements of all po-groups by 0, because in all cases it will be clear which po-group is considered.

Let $I \subseteq \Gamma$. For each $i \in \Gamma$ let $\leqq'_i = \leqq_i$ if $i \in \Gamma \setminus I$ and let $\leqq'_i = \leqq_i^{\sim}$ if $i \in I$.

Let $V[\Gamma, (H_i, \leqq_i)]$ ($V[\Gamma, (H_i, \leqq'_i)]$) be the mixed product of po-groups $(H_i, \leqq_i)$ ($(H_i, \leqq'_i)$), $i \in \Gamma$. We shall also denote $V(\Gamma, (H_i, \leqq_i)]$ and $V[\Gamma, (H_i, \leqq'_i)]$ by $(V, \leqq)$ or $(V, \leqq')$, respectively.

If $x, y \in V$, $x \neq y$, then $y < x$ ($y <' x$) if and only if $y_i < x_i$ ($y_i <' x_i$) for all $i \in M_{xy}$.

Throughout this section for $a \in V$ let $\bar{a}$ denote the element of V whose components are defined as follows: $\bar{a}_i = 2a_i$ if either $i \in M_a$, $i \in I$, $a_i < 0$ or $i \in M_a$, $i \notin I$, $a_i > 0$ and all other components are zero. From the definition of $\bar{a}$ we infer that $0 \leqq' \bar{a}_i$ for all $i \in \Gamma$. Thus $0 \leqq' \bar{a}$.

4.1. Lemma. *Let C be a directed convex subgroup of $(V, \leqq)$ and let $x \in C$. Then $\bar{x} \in C$.*

Proof. Since C is a directed subgroup of $(V, \leqq)$ there exist elements $a, b \in C$ such that $0 \leqq a$, $x \leqq a$, $3x \leqq a$, $b \leqq 0$, $b \leqq x$, $b \leqq 3x$. We shall show that $\bar{x} \leqq a$.

Let $a \neq \bar{x}$, $i \in M_{a\bar{x}}$. Then $a_i \neq \bar{x}_i$ and $a_j = \bar{x}_j$ for all $j > i$.

1) Let $i \notin M_x$. Then we have $\bar{x}_i = 0$, $a_i \neq 0$.

If $a_j \neq 0$ for some $j > i$, then there exists $k \in M_a$, $k \geqq j$. From $0 \leqq a$ we get $0 < a_k$. Thus $\bar{x}_k = a_k > 0$. Since $\bar{x}_k > 0$ only in the case when $k \in M_x$, $x_k > 0$, $k \notin I$, whereby we get $\bar{x}_k = 2x_k = a_k > 0$. Thus $k \in M_{(3x)a}$. Since $a \geqq 3x$, we obtain $a_k > 3x_k$, which contradicts the relation $2x_k = a_k > 0$.

Therefore $a_j = 0$, $\bar{x}_j = 0$ for all $j > i$. Then $i \in M_a$. Since $0 \leqq a$, we have $a_i > 0 = \bar{x}_i$.

2) Let $i \in M_x$. Then $\bar{x}_j = x_j = a_j = 0$ for all $j > i$.

a) Assume that $x_i > 0$, $i \notin I$. If $a_i = 3x_i$, then we obtain $a_i = 3x_i > 2x_i = \bar{x}_i$. If $a_i \neq 3x_i$, then from the relations $a \geqq 3x$, $i \in M_{(3x)a}$ we get $a_i > 3x_i > 2x_i = \bar{x}_i$.

b) If $x_i < 0$, $i \in I$, then $\bar{x}_i = 2x_i < 0$. Since $a_i \geqq 0$, we get $a_i \geqq 0 > 2x_i = \bar{x}_i$.

c) In other cases $\bar{x}_i = 0$. Since $a_i \neq \bar{x}_i = 0$, from the relation $a \geqq 0$ we obtain $a_i > 0 = \bar{x}_i$.

Thus $\bar{x}_i < a_i$ for all $i \in M_{a\bar{x}}$. Hence $\bar{x} \leqq a$. Analogously we can obtain $b \leqq \bar{x}$. From the convexity of C in $(V, \leqq)$ we have $\bar{x} \in C$.

4.2. Lemma. *Let C be a directed convex subgroup of $(V, \leqq)$ and let $x \in C$. Then there exists $y \in C$ such that $x \leqq' y$, $0 \leqq' y$.*

Proof. From 4.1 we have $\bar{x} \in C$. Since C is a directed subgroup of $(V, \leqq)$, there exists $z \in C$ such that $0 \leqq z$, $x \leqq z$. Then $\bar{z}_j + (\overline{-z})_j \neq 0$ for all $j \in M_z$ and $\bar{z}_i + (\overline{-z})_i \geqq' 0$ for all $i \in \Gamma$. Let $y = \bar{x} + \bar{z} + (\overline{-z})$. Since $\bar{z}, (\overline{-z}) \in C$, then we get $y \in C$ and the relation $y_j \geqq' x_j \geqq' 0$ is valid for all $j \in \Gamma$.

Now we shall prove that $x \leqq' y$.

Let $x \neq y$ and $i \in M_{xy}$. Then $x_i \neq y_i$ and $x_j = y_j$ for all $j > i$.

1) Let $i \notin M_x$. By way of contradiction we shall prove that $x_i = 0$. Suppose $x_i \neq 0$. Then there exists $j > i$, $j \in M_x$.

a) Let $x_j > 0$, $j \in I$.

If $z_j = 0$, then $\bar{z}_j = (\overline{-z})_j = 0$. Since $\bar{x}_j = 0$, we get $x_j > 0 = y_j$, which contradicts the assumption that $x_j = y_j$.

If $z_j \neq 0$ and $j \in M_z$, then $z_j > 0$, $\bar{z}_j = 0$, $(\overline{-z})_j = -2z_j$. Since $\bar{x}_j = 0$, we obtain $y_j = -2z_j < 0 < x_j$, a contradiction.

If $z_j \neq 0$ and $j \neq M_z$, then there exists $k > j$, $k \in M_z$. Then we get $z_k > 0$, $x_k = 0$, $\bar{z}_k + (\overline{-z})_k \neq 0$. From this we obtain $y_k \neq 0 = x_k$, a contradiction.

b) Let $x_j < 0$, $j \in I$.

If $z_j = 0$, then $\bar{z}_j = (\overline{-z})_j = 0$. Since $\bar{x}_j = 2x_j < x_j < 0$, we obtain $y_j = 2x_j < x_j$, a contradiction.

If $z_j \neq 0$ and $j \in M_z$, then $z_j > 0$, $\bar{z}_j = 0$, $(\overline{-z})_j = -2z_j$.From the relations $\bar{x}_j = 2x_j < x_j < 0$, $\bar{z}_j + (\overline{-z})_j = -2z_j < 0$ we get $y_j = 2x_j - 2z_j < x_j$, a contradiction.

If $z_j \neq 0$ and $j \notin M_z$, then there exists $k > j$, $k \in M_z$. Thus $z_k > 0$. Since $x_k = 0$, $\bar{x}_k = 0$, $\bar{z}_k + (\overline{-z})_k \neq 0$, we obtain $y_k \neq 0 = x_k$, a contradiction.

c) Let $x_j > 0$ and $j \notin I$. Then $\bar{x}_j = 2x_j > x_j > 0$.

If $z_j = 0$, then $\bar{z}_j = (\overline{-z})_j = 0$. From this we get $y_j = 2x_j > x_j$, a contradiction.

If $z_j \neq 0$ and $j \in M_z$, then $z_j > 0$. Thus $\bar{z}_j = 2z_j$, $(\overline{-z})_j = 0$. Then we get $y_j = 2x_j + 2z_j > x_j$, a contradiction.

If $z_j \neq 0$ and $j \notin M_z$, then there exists $k > j$, $k \in M_z$. Thus $z_k > 0$. From the relations $x_k = 0$, $\bar{x}_k = 0$, $\bar{z}_k + (\overline{-z})_k \neq 0$ we obtain $y_k \neq 0 = x_k$, a contradiction.

d) Let $x_j < 0$, $j \notin I$. Then $\bar{x}_j = 0$.

If $z_j = 0$, then $\bar{z}_j = 0$, $(\overline{-z})_j = 0$. From this we obtain $y_j = 0 > x_j$, a contradiction.

If $z_j \neq 0$ and $j \in M_z$, then $z_j > 0$. Hence $\bar{z}_j = 2z_j > 0$, $(\overline{-z})_j = 0$. From this we get $y_j = 2z_j > 0 > x_j$, a contradiction.

If $z_j \neq 0$ and $j \notin M_z$, then there exists $k > j$, $k \in M_z$. Then $z_k > 0$. From the relations $x_k = 0$, $\bar{x}_k = 0$, $\bar{z}_k + (\overline{-z})_k \neq 0$ we get $y_k \neq 0 = x_k$, a contradiction.

e) Let x_j be incomparable with 0 in $(H_j, \leqq_j)$. Then $(H_j, \leqq_j)$ is trivially ordered. Suppose that $z_k = 0$ for all $k > j$. If $z_j \neq x_j$, then $j \in M_{xz}$, which contradicts the assumption that $z \geqq x$. If $z_j = x_j \neq 0$, then $j \in M_z$. Thus $z_j, 0$ are incomparable in $(H_j, \leqq_j)$, which contradicts the assumption that $z \geqq 0$. Therefore there exists $k > j$, $k \in M_z$. Thus $z_k > 0$. Then we obtain $x_k = 0$, $\bar{x}_k = 0$, $\bar{z}_k + (\overline{-z})_k \neq 0$. Thus $y_k \neq 0 = x_k$, a contradiction.

Therefore $x_i = 0$. Then from the relations $y_i \neq 0$, $0 \leqq' y_i$ we infer $x_i = 0 <' y_i$.

2) Suppose that $i \in M_x$.

a) If $x_i < 0$, $i \in I$, then $\bar{x}_i = 2x_i$, $x_i >' 0$. From this we have $\bar{x}_i = 2x_i >' x_i$

b) If $x_i < 0$, $i \notin I$, then $\bar{x}_i = 0$, $x_i <' 0$. Thus $\bar{x}_i >' x_i$.
c) If $x_i > 0$, $i \in I$, then $\bar{x}_i = 0$, $x_i <' 0$. Thus $\bar{x}_i >' x_i$.
d) If $x_i > 0$, $i \notin I$, then $\bar{x}_i = 2x_i$, $x_i >' 0$. Hence $\bar{x}_i = 2x_i >' x_i$.

Since $y_i \geqq' \bar{x}_i$ for all $i \in \Gamma$, we obtain $y_i >' x_i$ in all the cases above.

e) Suppose that x_i is incomparable with 0 in $(H_i, \leqq_i)$. Then $(H_i, \leqq_i)$ is trivially ordered. Since $x_i \neq 0$, $x \leqq z$, $0 \leqq z$, then there exists $j > i$, $j \in M_z$. Thus $z_j > 0$, $x_j = 0$, $\bar{z}_j + (\overline{-z})_j \neq 0$. From this we obtain $y_j \neq 0 = x_j$, a contradiction.

Hence $y_i >' x_i$ for all $i \in M_{xy}$. Therefore $y >' x$. From the definition of the element y we have $y \geqq' 0$.

4.3. Theorem. $(V, \leqq)$, $(V, \leqq')$ *have the same directed convex subgroups.*

Proof. Let C be directed convex subgroup of $(V, \leqq)$. From 4.2 we obtain that C is a directed subgroup of $(V, \leqq')$. Thus it suffices to verify that C is a convex subgroup of $(V, \leqq')$.

Let $0 <' y <' x$ for some $y \in V$, $x \in C$. Since $(C, \leqq)$ is a directed subgroup of $(V, \leqq)$ there exist elements $u, v \in C$ such that $2x \leqq v$, $-2x \leqq v$, $0 \leqq v$, $x \leqq v$, $u \leqq -2x$, $u \leqq 2x$, $u \leqq 0$, $u \leqq x$.

We shall prove that $u \leqq y \leqq v$.

Let $y \neq v$ and $i \in M_{yv}$. Then $v_i \neq y_i$ and $v_j = y_j$ for all $j > i$. By contradiction we shall show that $v_j = y_j = x_j = 0$ for all $j > i$.

Suppose that $v_j \neq 0$ for some $j > i$. Then there exists $k \geqq j$, $k \in M_v$. Thus $k \in M_y$.

If $x_l \neq 0$ for some $l > k$, then there exists $m \in M_x$ $m \geqq l$. Thus $x_m >' 0$. Hence $m \in M_{(2x)v}$, $m \in M_{(-2x)v}$. From $v \geqq 2x$, $v \geqq -2x$ we obtain $0 = v_m > 2x_m$, $0 = v_m > -2x_m$, a contradiction.

Suppose that $x_l = 0$ for all $l > k$. Since $v_k \neq 0$, we have $v_k = y_k >' 0$. If $x_k = 0$, then $k \in M_{xy}$. Thus $x_k <' y_k$, a contradiction. If $x_k \neq 0$, then from $x >' 0$ it follows that $x_k >' 0$. Since $x_k \geqq' y_k$, we get $x_k \geqq' y_k = v_k >' 0$. Since $k \in M_v$, $v \geqq 0$, the relation $v_k \geqq 0$ is valid. Thus $k \in \Gamma \setminus I$, hence $0 < v_k = y_k \leqq x_k < 2x_k$. From the relations $k \in M_{(2x)v}$, $v \geqq 2x$ we have $v_k > 2x_k$, a contradiction.

Thus $v_j = 0$ and also $y_j = 0$ for all $j > i$, $j \in \Gamma$. Suppose that $x_j \neq 0$ for some $j > i$. Then there exists $k \geqq j$, $k \in M_x$. Thus $x_k >' 0$. From the relations $k \in M_{(2x)v}$, $k \in M_{(-2x)v}$, $v \geqq 2x$, $v \geqq -2x$ we obtain $0 = v_k > 2x_k$, $0 = v_k > -2x_k$, a contradiction. Hence $x_j = 0$ for all $j > i$.

If y_i, v_i are incomparable in $(H_i, \leqq_i)$, then H_i is trivially ordered. From the relation $0 <' y$ we get $y_i = 0$. Then $v_i \neq 0$, $i \in M_v$, which contradicts the assumption that $v \geqq 0$. If $y_i > v_i$ and $i \in I$, then from $v \geqq 0$ it follows that $y_i > v_i \geqq 0$. Thus $i \in M_y$, $y_i <' 0$, which contradicts the assumption that $y >' 0$.

If $y_i > v_i$ and $i \notin I$, then $y_i > v_i \geqq 0$. From the relations $i \notin I$, $x >' y$, $x_j = y_j$ for all $j > i$ we have $x_i \geqq y_i > v_i \geqq 0$, $i \in M_{xv}$, which contradicts the assumption that $v > x$.

Theorefore $y_i < v_i$ for all $i \in M_{yv}$. Hence $y < v$. Similarly we can prove that $u \leqq y$. From the convexity of C in $(V, \leqq)$ it follows that $y \in C$. Hence C is a convex sugbroup of $(V, \leqq')$.

4.4. Lemma. *Let* $x, y \in (V, \leqq)$ *and let* $0 < y < x$. *Then* $M_{(2x)y} = M_{(-2x)y}$ *and if* $i \in M_{(2x)y}$, *then* $y_j = x_j = 0$ *for all* $j > i$.

Proof. Let $0 < y < x$ for some x, $y \in V$. Then we have $-2x < 0 < y < x < 2x$. Let $i \in M_{(2x)y}$. Then $2x_i > y_i$.

Suppose that $y_j \neq 0$ for some $j > i$. Then there exists $k \geqq j$, $k \in M_y$. Thus $y_k > 0$. Since $i \in M_{(2x)y}$, we have $2x_k = y_k > 0$. From this we obtain $2x_k > x_k > 0$ and $x_l = 0$ for all $l > k$. Since $x > y$, we get $x_k \geqq y_k$. Thus $2x_k > x_k \geqq y_k > 0$, a contradiction.

Therefore $y_j = 0$ for all $j > i$. Then we have $y_i \geqq 0 > -2x_i$. Thus $i \in M_{(-2x)y}$.

Conversely, assume that $i \in M_{(-2x)y}$. Then $-2x_i < y_i$. Suppose that $y_j \neq 0$ for some $j > i$. Then there exists $k \geqq j$, $k \in M_y$. Then $y_k > 0$. Since $i \in M_{(-2x)y}$, we have $-2x_k = y_k > 0$. Thus $-2x_k > -x_k > 0$ and $x_l = y_l = 0$ for all $l > k$. This contradicts the assumption that $x > 0$.

Thus $y_j = 0$ for all $j > i$. Then also $x_j = 0$ for all $j > i$. If $y_i = 0$, then $-2x_i \neq 0$. Therefore $2x_i \neq y_i$. If $y_i \neq 0$, then $y_i > 0$. Assume that $2x_i = y_i$. Then $y_i = 2x_i > x_i > 0$, which contradicts the assumption that $x > y$. Thus $2x_i \neq y_i$. Hence $i \in M_{(2x)y}$.

Remark. Lemma 4.4 is true for an arbitrary mixed product of po-groups (i.e., need not suppose that for each $i \in \Gamma$, H_i is linearly ordered or trivially ordered).

Let $\Gamma_l = \{i \in \Gamma;\ H_i \text{ is linearly ordered}\}$.

4.5. Theorem. $(V, \leqq)$, $(V, \leqq')$ *have the same convex subgroups if and only if there are no incomparable elements* i, j *in* Γ_l *such that* $i \in \Gamma \setminus I$, $j \in I$.

Proof. 1) We first shall prove the sufficiency of the conditions. Note that if a, $b \in V$ and $i, j \in M_{ab}$, $i \neq j$, then i, j are incomparable in Γ.

Suppose that C is a convex subgroup of $(V, \leqq')$. Let $0 < y < x$ for some $y \in V$, $x \in C$. Then we have $-2x < 0 < y < x < 2x$. From 4.4 we have $M_{(2x)y} = M_{(-2x)y}$. By the assumptions we obtain $M_{(2x)y} \subseteq I$ or $M_{(2x)y} \subseteq \Gamma \setminus I$.

If $M_{(2x)} \subseteq I$, then $2x_i <' y_i <' -2x_i$ for all $i \in M_{(2x)y} = M_{(-2x)y}$. Thus $2x <' y <' -2x$.

If $M_{(2x)y} \subseteq \Gamma \setminus I$, then $-2x_i <' y_i <' 2x_i$ for all $i \in M_{(2x)y} = M_{(-2x)y}$. Thus $-2x <' y <' 2x$.

From the convexity of C in $(V, \leqq')$ it follows that $y \in C$. Hence C is a convex subgroup of $(V, \leqq)$.

2) Suppose that there exist incomparable elements i, j in Γ_l such that $i \in \Gamma \setminus I$, $j \in I$. Since H_i, H_j are nontrivial linearly ordered groups, then there exist elements $a \in H_i$, $b \in H_j$ such that $a >_i 0$, $b <_j 0$.

Let v be the element of V such that $v_i = 2a$, $v_j = 2b$ and $v_k = 0$ for all $k \in \Gamma \setminus \{i, j\}$. Let $A = \{mv; m \in Z\}$. Clearly A is a subgroup of V.

Let $0 \leqq y \leqq x$ for some $y \in V$, $x \in A$. Then $x_i = m(2a)$, $x_j = m(2b)$, where $m \in Z$. Since $0 \leqq x$, we get $x_i = m(2a) \geqq 0$, $x_j = m(2b) \geqq 0$. From this we have $m = 0$. Thus $x_k = 0$ for all $k \in \Gamma$. Suppose that $y_k \neq 0$ for some $k \in \Gamma$. Then there exists $l \geqq k$, $l \in M_y$. Since $y \geqq 0$, we obtain $y_l > 0 = x_l$, which contradicts the assumption that $x \geqq y$. Therefore $y_k = 0$ for all $k \in \Gamma$ and hence $y \in A$. Thus A is a convex subgroup of $(V, \leqq)$.

Let z be the element of V such that $z_i = a$, $z_j = b$ and $z_k = 0$ for all $k \in \Gamma$ $\{i, j\}$. Let t be the element of V such that $t_i = 2a$, $t_j = 2b$ and $t_k = 0$ for all $k \in \Gamma \setminus \{i, j\}$. Then $z \notin A$, $t \in A$. From $0 <' z_i <' 2a$, $0 <' z_j <' 2b$, $M_{zt} = \{i, j\}$ we get $0 <' z <' t$. Since $z \notin A$, A is not a convex subgroup of $(V, \leqq')$.

5. Isolated factorially rational groups and isolated finite valued groups

5.1. Lemma. *Let Q_1 be a nontrivial subgroup of the additive group Q of all rational numbers. Then there exist only two different linear orders on Q_1, which are dual to each other.*

Proof. Let $\leqq_1$ be a linear order on Q_1 and let $P(Q_1)$ be the positive cone of $(Q_1, \leqq_1)$. Then $P(Q_1)$ is the positive cone of a partial order on Q. Since Q is a torsion-free group, from [5, Chap. III, Coll. 13] it follows that each partial order on Q can be extended to a linear order on Q. Since there exist only two different linear orders on Q which are dual to each other [8, Chap. II, Sec. 2, Proposition 1], the same holds for Q_1.

5.2. Lemma. *Let $(G, \leqq)$, $(G, \leqq')$ be nontrivial linearly ordered groups with the same group operation and let $(G, \leqq')$ be o-isomorphic to a subgroup of $(Q, \leqq)$. Then $\leqq' = \leqq$ or $\leqq' = \leqq^{\sim}$.*

Proof. This follows from 5.1.

5.3. Lemma. *Let $(G, \leqq)$ be a nontrivial linearly ordered abelian group and let $a \in G$, $a > 0$. Then $(G, \leqq)$ is o-isomorphic to a subgroup of $(Q, \leqq)$ if and only if for each $b \in G$ there exist elements $m, n \in Z$, $n \neq 0$ such that $ma = nb$.*

The proof is obvious.

5.4. Theorem. *Nontrivial isolated factorially rational divisible abelian groups $(H, \leqq)$, $(H, \leqq')$ have the same convex subgroups if and only if there exists a po-set Γ and for each $i \in \Gamma$ there exist linearly or trivially ordered groups $(C_i, \leqq_i)$, $(C_i, \leqq'_i)$ with the same group operation such that*
(1) there exists a mapping φ of H into $V[\Gamma, C_i]$ such that φ is an o-isomorphism of $(H, \leqq)$ into $V[\Gamma, (C_i, \leqq_i)]$ and also an o-isomorphism of $(H, \leqq')$ into $V[\Gamma, (C_i, \leqq'_i)]$,

(2) *for each* $i \in \Gamma$ $\leqq_i' = \leqq_i$ *or* $\leqq_i' = \leqq_i^{\sim}$,
(3) *there exists no element* $0 < h \in H$ *such that* $(h\varphi)_i >' 0$, $(h\varphi)_j <' 0$ *for some maximal components* $(h\varphi)_i$, $(h\varphi)_j$ *of* $h\varphi$,
(4) *there exists no element* $0 <' g \in H$ *such that* $(g\varphi)_k > 0$, $(g\varphi)_l < 0$ *for some maximal components* $(g\varphi)_k$, $(g\varphi)_l$ *of* $g\varphi$.

Proof. In order to prove the necessity of the conditions in view of 2.3 it suffices to show that for each $i \in \Gamma$ $\leqq'_i = \leqq_i$ or $\leqq_i' = \leqq_i^{\sim}$.

Since $(H, \leqq)$, $(H, \leqq')$ are factorially rational, the required relations follow from 5.2.

Now we show the sufficiency of the conditions. Let A be a convex subgroup of $(H, \leqq')$ and let $0 < y < x$ for some $y \in H$, $x \in A$. From this we get $-2x < y < 2x$ and $0 < y\varphi < x\varphi$. Then $(-2x)\varphi < y\varphi < (2x)\varphi$. In view of 4.4 we have $M_{(2x)\varphi y\varphi} = M_{(-2x)\varphi x\varphi}$. Thus $((-2x)\varphi)_i < (y\varphi)_i < ((2x)\varphi)_i$ for each $i \in M_{(2x)\varphi y\varphi}$. From (2) and (3) it follows that $(-2x)\varphi <' y\varphi <' (2x)\varphi$ or $(-2x)\varphi >' y\varphi >' (2x)\varphi$. Thus $-2x <' y <' 2x$ or $-2x >' y >' 2x$. From the convexity of A in $(H, \leqq')$ we infer that $y \in A$. Hence A is a convex subgroup of $(H, \leqq)$. Similarly we can obtain, that if B is a convex subgroup of $(H, \leqq)$, then B is a convex subgroup of $(H, \leqq')$.

5.5 Lemma. *Let* $(G, \leqq)$ *be an isolated factorially rational abelian group. Then* $(Z(G), \leqq)$ *is an isolated divisible factorially rational abelian group.*

Proof. Let (A^i, A_i) be a pair of pure convex subgroups of $(Z(G), \leqq)$ such that A^i covers A_i and $(A^i/A_i, \leqq)$ is nontrivially ordered. From 2.12 we have that $(G \cap A^i, G \cap A_i)$ is a pair of pure convex subgroups of $(G, \leqq)$ such that $G \cap A^i$ covers $G \cap A_i$.

Let $a + A_i$, $b + A_i \in A^i/A_i$, $a + A_i > A_i$. Then $na + d > 0$ form some $n \in N$, $d \in A_i$. Since $a \in A^i \backslash A_i$, $b \in A^i$, $d \in A_i$, we get that $ka \in G \cap A^i \backslash G \cap A_i$, $ld \in A_i \cap G$, $mb \in G \cap A^i$ for some k, l, $m \in N$. Then $kl(na + d) = nlka + kld > 0$. Thus $nlka + G \cap A_i > G \cap A_i$. Hence $G \cap A^i/G \cap A_i$ is nontrivially ordered. Since $(G, \leqq)$ is factorially rational, from 5.3 we get $rnlka + G \cap A_i = smb + G \cap A_i$ for some $r, s \in Z$, $s \neq 0$. Then $rnlka + g_1 = smb + g_2$ for some $g_1, g_2 \in$ $\in G \cap A_i \subseteq Z(G \cap A_i)$. From this we get $rnlka + Z(G \cap A_i) = smb + Z(G \cap A_i)$. From 2.9 and 5.3 it follows that $(A^i/A_i, \leqq)$ is o-isomorphic to a subgroup of $(Q, \leqq)$.

From 5.5 we obtain the following corollary

5.6. Corollary. *The hypothesis that* $(H, \leqq)$, $(H, \leqq')$ *are divisible can be omitted in* 5.4.

5.7. Theorem. *Nontrivial isolated finite valued abelian groups* $(H, \leqq)$, $(H, \leqq')$ *have the same convex subgroups if and only if there exists a po-set* Γ *and for each* $i \in \Gamma$ *there exist ordered groups* $(C_i, \leqq_i)$ *and* $(C_i, \leqq_i')$ *such that*

(i) $(C_i, \leqq_i)$, $(C_i, \leqq_i')$ *have the same group operation and* card $C_i > 1$,

(ii) *The following conditions* (1)—(4) *are satisfied*:

(1) *there exists a mapping* φ *of* H *into* $V[\Gamma, C_i]$ *such that* φ *is a v-isomorphism of* $(H, \leqq)$ *into* $V[\Gamma, (C_i, \leqq_i)]$ *and also a v-isomorphism of* $(H, \leqq')$ *into* $V[\Gamma, (G_i, \leqq_i')]$,

(2) *for each* $i \in \Gamma$ *we have either that*

(a) *both* $(C_i, \leqq_i)$ *and* $(C_i, \leqq_i')$ *are trivially ordered and each of them is isomorphic to a subgroup of* Q

or (b) *both* $(C_i, \leqq_i)$ *and* $(C_i, \leqq_i')$ *are linearly ordered and each of them is o-isomorphic to a subgroup of* $(R, \leqq)$,

(3) *there exists no element* $0 < h \in H$ *such that* $(h\varphi)_i >' 0$, $(h\varphi)_j <' 0$ *for some maximal components* $(h\varphi)_i$, $(h\varphi)_j$ *of* $h\varphi$,

(4) *there exists no element* $0 <' g \in H$ *such that* $(g\varphi)_k > 0$, $(g\varphi)_l < 0$ *for some maximal components* $(g\varphi)_k$, $(g\varphi)_l$ *of* $g\varphi$.

Proof. In view of 2.3 and 2.20 it suffices to show the sufficiency of the conditions.

Let A be a convex subgroup of $(H, \leqq')$ and let $0 < y < x$ for some $y \in H$, $x \in A$. Then $0\ y\varphi < x\varphi$, $-2x < y < 2x$, $(-2x)\varphi < y\varphi < (2x)\varphi$.

Let $((2x)\varphi)_i = 0$ for some $i \in M_{(2x)\varphi, y\varphi}$. Then also $((-2x)\varphi)_i = 0$. In view of Lemma 4.4 from $(-2x)\varphi < y\varphi < (2x)\varphi$ we have $0 <_i (y\varphi)_i$, $(y\varphi)_i <_i 0$, a contradiction.

Hence $((2x)\varphi)_i \neq 0$ for each $i \in M_{(2x)\varphi, y\varphi}$. Since for each $i \in M_{(2x)\varphi, y\varphi}$ $(C_i, \leqq_i')$ is o-isomorphic to a subgroup of $(R, \leqq)$, for each $i \in M_{(2x)\varphi, y\varphi}$ there exists $n_i \in Z$ such that $n_i((-2x)\varphi)_i <_i' (y\varphi)_i <_i' n_i((2x)\varphi)_i$.

Since φ is a v-isomorphism, i is a value of $2x - y$ if and only if $i \in M_{(2x-y)\varphi}$. Then from the relation $M_{(2x-y)\varphi} = M_{(2x)\varphi, y\varphi}$ and from the fact that $(H, \leqq')$ is finite valued we obtain that $M_{(2x)\varphi, y\varphi}$ is a finite set. Then there exists $n \in Z$ such that $n((-2x)\varphi)_i <_i' (y\varphi)_i <_i' n((2x)\varphi)_i$ for all $i \in M_{(2x)\varphi, y\varphi} = M_{(-2x)\varphi, y\varphi}$. Thus $n((-2x)\varphi) <' y\varphi <' n((2x)\varphi)$ and hence $n(-2x) <' y <' n(2x)$. From the convexity of A in $(H, \leqq')$ we get $y \in A$. Hence A is a convex subgroup of $(H, \leqq')$. Analogously we can prove that if B is a convex subgroup of $(H, \leqq)$, then B also is a convex subgroup of $(H, \leqq')$.

5.8. Corollary. *Nontrivial abelian linearly ordered groups* $(H, \leqq)$, $(H, \leqq')$ *have the same convex subgroup if and only if there exists a linearly ordered set* Γ *and for each* $i \in \Gamma$ *there exist nontrivial linearly ordered groups* $(C_i, \leqq_i)$, $(C_i, \leqq_i')$ with *the same group operation such that*

(1) *there exists a mapping* φ *of* H *into* $V[\Gamma, C_i]$ *such that* φ *is a v-isomorphism of* $(H, \leqq)$ *into* $V[\Gamma, (C_i, \leqq_i)]$ *and also a v-isomorphism of* $(H, \leqq')$ *into* $V[\Gamma, (C_i, \leqq_i')]$,

(2) *for all* $i \in \Gamma$ *each of the groups* $(C_i, \leqq_i)$, $(C_i, \leqq_i')$ *is o-isomorphic to a subgroup of* $(R, \leqq)$.

Proof. Since the set of all convex subgroups of a linearly ordered abelian group is lilnearly ordered by inclusion [5, p. 80], each of its nonzero elements has only one value.

The necessity of the conditions in view of 5.7 follows from the fact that the set Γ in Theorem 5.7 is linearly ordered in the case when $(H, \leqq)$, $(H, \leqq')$ are linearly ordered (see the description of Γ in the proof of Theorem 2.3).

In view of 5.7 sufficiency of the conditions follows from the fact that for each $h \in H$, $h \neq 0$, $h\varphi$ has only one maximal component.

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ПАРЫ ЧАСТИЧНО УПОРЯДОЧЕННЫХ ГРУПП С ОДИНАКОВЫМИ ВЫПУКЛЫМИ ПОДГРУППАМИ

Milan Jasem

Резюме

В статье исследуются пары изолированных абелевых групп $(H, \leqq)$ и $(H, \leqq')$ определённых на одном и том же множестве с одной и той же групповой операцией, причём система всех выпуклых подгрупп $(H, \leqq)$ совпадает со системой всех выпуклых подгрупп $(H, \leqq')$.