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LATTICE BETWEENNESS RELATION AND A GENERALIZATION OF KÖNIG'S LEMMA

JARMILA HEDLÍKOVÁ* — TIBOR KATRIŇÁK**

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ABSTRACT. A tree is a partially ordered set $(T, \leq)$ such that for every $x \in T$, the set $\{y \in T \mid y < x\}$ is well-ordered. Equivalently, a tree is a transitive α-partite König graph G for some ordinal α. König's lemma states that every transitive ω-partite König graph G with finite parts contains an ω-frame. We present an extension of König's lemma which has the origin in a characterization of lattices by a ternary relation (the lattice betweenness relation) given by M. Kolibiar. Our generalization of König's lemma states that for every up-directed partially ordered set S, each transitive S-partite König graph G with sufficiently many finite parts contains an S-frame. As an example, we apply this result in lattice theory.

0. Introduction and preliminaries

There are various attempts to generalize the famous König's lemma [5]. For instance, E. C. Milner and N. Sauer [6] have proved two infinitary graph-theoretical variants of this result. For the set-theoretical purposes, one needs generalizations given in B. Balcar and P. Štěpánek [1]. A version of König's lemma which is applicable in computer science is used by W. Wechler [7].

The aim of this note is to present an extension of König's lemma as a theorem which has the origin in M. Kolibiar's [4] characterization of lattices in terms of a ternary relation — the lattice betweenness relation. In the first part of this

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paper, our theorem is presented in a graph-theoretical form, which we regard as a more convenient one. In the second part, we shall show how our version of König's lemma works in lattice theory (see [2], [3], [4]).

First we need some preliminary definitions and facts.

Let G be a graph with the vertex set $V(G) = V$, and the edge set $E(G) = E \subseteq V \times V$. If x, y are vertices in G and $(x,y) \in E$, we write also simply xEy. A graph G is called *transitive* if for all vertices $x, y, z \in V$, xEy and yEz implies xEz.

Let S be a partially ordered set. (We shall use two particular cases, when S is an up-directed partially ordered set, or when S is a well-ordered set of type α, where α is an ordinal.)

A graph G is called *S-partite* if there is a partition of the vertex set V into S pairwise disjoint non-empty sets V_i, $i \in S$, such that the edge set

$$E \subseteq \bigcup (V_i \times V_j \mid i, j \in S \text{ and } i < j).$$

In what follows, G will always denote such an S-partite (or, α-partite) graph. If $i \in S$, the set V_i is called the ith *part* of G. Let us observe that a transitive S-partite graph G is a partially ordered set with E as a (strict) partial order on the set V.

An S-partite graph G is called *König*, or G is said to have the *König property* if for all $i, j \in S$ with $i < j$ and for every $y \in V_j$ there exists a unique $x \in V_i$ such that xEy. (Let us notice that in [6], the same name is used in an α-partite graph G for the following weaker property. For every ordinal i with $i + 1 < \alpha$ and $y \in V_{i+1}$ there is some $x \in V_i$ such that xEy. However, for our purposes it is necessary to use the above stronger condition.)

Let G be an S-partite graph. If $J \subseteq S$, a *J-frame* in G is a function $f\colon J \to V$ such that for every $i \in J$, $f(i) \in V_i$, and for all $i, j \in J$, $i < j$ implies $f(i)Ef(j)$. Let us observe that the J-frames f in G, where $J \subseteq S$ is a chain, are in one-one correspondence with the complete subgraphs of G. Namely, for every such a J-frame f in G, the set $\{f(i) \mid i \in J\}$ of vertices of G induces a complete subgraph of G. And conversely, if U is a set of vertices of G inducing a complete subgraph of G, then for every $i \in S$ there is at most one $x \in U$ with $x \in V_i$, the set $J = \{i \in S \mid x \in V_i \text{ for some } x \in U\}$ is a chain in S, and the function $f\colon J \to V$, such that $f(i)$ is the unique vertex of U belonging to V_i, is a J-frame in G. (In [6], for an α-partite graph G and an ordinal β with $\beta \le \alpha$, a *β-path* in G is defined as a function $f\colon \beta \to V$ such that $f(i) \in V_i$ for every i, $i < \beta$, and for all i, j with $i < j < \beta$ there exists k such that $i \le k < j$ and $f(k)Ef(j)$. Let us note that every β-frame in G is a β-path in G, but not conversely. However, every β-path in G is a β-frame in G whenever G is transitive.)

LEMMA 1. *Let G be a transitive S-partite König graph. Let $i, j, k \in S$, and $x \in V_i$, $y \in V_j$ and $z \in V_k$. Assume that $i < j < k$ in S. Then xEz and yEz implies xEy.*

P r o o f. Since G is König, $i < j$ and $y \in V_j$, there is a unique $x_1 \in V_i$ such that x_1Ey. By transitivity of G, x_1Ey and yEz implies x_1Ez. Because G is König and $i < k$, xEz and x_1Ez implies $x = x_1$. Therefore xEy. □

LEMMA 2. *Let G be a transitive S-partite König graph. If $J \subseteq S$, $i \in S$, $x \in V_i$ and $j \leq i$ for all $j \in J$, then there exists a unique $J \cup \{i\}$-frame f in G with $f(i) = x$.*

P r o o f. Since G is König, for every $j \in J$ with $j < i$, there exists a unique $x_j \in V_j$ such that x_jEx. Thus the only way how to define f is the following one:

$$f(j) = \begin{cases} x_j & \text{if } j \in J \setminus \{i\}, \\ x & \text{if } j = i. \end{cases}$$

To show that f is a $J \cup \{i\}$-frame in G, let $j, k \in J \setminus \{i\}$ with $j < k$. Then from $j < k < i$, x_jEx and x_kEx, by Lemma 1, it follows that x_jEx_k. □

COROLLARY. *Let G be a transitive S-partite König graph. If i is a greatest element in S and $x \in V_i$, then there exists a unique S-frame f in G with $f(i) = x$.*

Example.

A partially ordered set $(T, \leq)$ is called a *tree* if for every $x \in T$, the set $\{y \in T \mid y < x\}$ is well-ordered.

Let S be an up-directed partially ordered set, and let G be a transitive S-partite König graph. Then G is a tree if and only if S is well-ordered.

Indeed, if S is a well-ordered set, $x \in V$, and if U is a nonempty subset of the set $\{y \in V \mid yEx\}$, then $x \in V_i$ for some $i \in S$, for every $y \in U$ there is $i(y) \in S$ with $y \in V_{i(y)}$, and the set $J = \{i(y) \mid y \in U\}$ has a smallest element. Let $z \in U$ be such that $i(z)$ is the smallest element of J. For every $y \in U$, $i(y) < i$, hence by Lemma 2, there exists a unique

$$J \cup \{i\}\text{-frame } f \text{ in } G \qquad \text{with} \quad f(i) = x.$$

Since G is König, $f(i(y)) = y$ for every $y \in U$. It follows that z is the smallest element in U. Thus the set $\{y \in V \mid yEx\}$ is well-ordered in G, and therefore G is a tree.

Conversely, assume that S is not well-ordered.

If S is not a chain, then there are non-comparable $i, j \in S$, and since S is up-directed, there is $k \in S$ with $i, j < k$. Choose an arbitrary vertex $x \in V_k$. As G is König, there exist (unique)

$$y \in V_i \text{ and } z \in V_j \qquad \text{such that} \quad yEx \text{ and } zEx .$$

G is an S-partite graph, and i, j are not comparable, hence neither yEz, nor zEy, which means that y, z are not comparable in G. Thus the set $\{t \in V \mid tEx\}$ is not a chain in G, and therefore G is not a tree.

If S is a chain, then there is a non-empty subset P of S which has not a smallest element. Choose $i \in P$ and $x \in V_i$, then, by Lemma 2, there is a (unique)

$$\{j \in P \mid j \leq i\}\text{-frame } f \text{ in } G \qquad \text{with} \quad f(i) = x .$$

The set

$$\{f(j) \mid j \in P \text{ and } j < i\}$$

has no smallest element, and thus the set $\{y \in V \mid yEx\}$ is not well-ordered in G. Therefore G is not a tree. □

From what is known about trees (cf. [1]), it is now clear that a tree can be equivalently defined as a transitive α-partite König graph G for some ordinal α.

The well-known lemma of D. König may be stated as follows.

KÖNIG'S LEMMA. *If $G = \Big(\bigcup_{i<\omega} V_i, E\Big)$ is a transitive ω-partite König graph, and if each V_i is finite, then G contains an ω-frame.*

We shall generalize this lemma in such a way that instead of the ordinal ω, an arbitrary up-directed partially ordered set S is assumed, and instead of the assumption that each V_i is finite, it suffices to suppose that sufficiently many V_i's are finite in the sense that there exists a cofinal subset P of S such that for every $i \in P$, the set V_i is finite (see Theorem 1 below). In particular, König's lemma is true if the ordinal ω is replaced by an arbitrary ordinal α, and the assumption that each V_i is finite is replaced by the condition that there exists a cofinal subset P of α such that for every $i \in P$, the set V_i is finite.

Let us note that the generalizations of König's lemma in [1], [6] and [7] work, in fact, with α-partite graphs, where α is an ordinal. In [1], for instance, the ordinal ω is replaced by any limit ordinal α, and the assumption that each V_i is finite is replaced by the condition that for each V_i, $|V_i| < \beta$, where β is a fixed ordinal with $\beta < \mathrm{cf}(\alpha)$. (Recall that $\mathrm{cf}(\alpha)$ is the minimal ordinal which is the order type of some cofinal subset of α.)

1. A generalization of König's lemma

We begin by considering an up-directed partially ordered set S and an S-partite graph G which is König and transitive.

LEMMA 3. *Let g be a J-frame in G for some cofinal subset J of S. Then g has a unique extension f to an S-frame in G.*

P r o o f. First observe that the set J itself is up-directed. Suppose that $i < j$, where $i \in S$ and $j \in J$. Then there exists a unique vertex $x_{i(j)} \in V_i$ such that $x_{i(j)} E g(j)$. Thus, the only possibility to define the function f is as follows:

$$f(i) = \begin{cases} g(i) & \text{for } i \in J\,, \\ x_{i(j)} & \text{for } i \in S \setminus J \text{ and some } j \in J \text{ with } i < j\,. \end{cases}$$

We have to show that f is correctly defined. Let $i \in S \setminus J$, and let $j, k \in J$ with $i < j$ and $i < k$. Then

$$x_{i(j)} E g(j) \quad \text{and} \quad x_{i(k)} E g(k)\,.$$

Since J is up-directed, there exists $l \in J$ with $j, k \leq l$. Using the transitivity of G we get

$$x_{i(j)} E g(l) \quad \text{and} \quad x_{i(k)} E g(l)\,.$$

Therefore $x_{i(j)} = x_{i(k)}$, as G is König. Thus the value $f(i)$ is correctly defined.

It remains to verify that f is an S-frame in G. Take $i, j \in S$ with $i < j$. Two cases can occur: 1) $j \in J$, 2) $j \notin J$.
Evidently,

$$\big(f(i), f(j)\big) = \big(f(i), g(j)\big) \in E$$

in the first event. In the second case, there exists $k \in J$ with $j < k$. By Lemma 1,

$$i < j < k\,, \ \ f(i) E f(k) \text{ and } f(j) E f(k) \implies f(i) E f(j)\,,$$

which completes the proof. □

LEMMA 4. *Let V_i be finite for some $i \in S$. Then there exists $x \in V_i$ such that for every $j \in S$ with $i < j$ there is $y \in V_j$ with xEy.*

P r o o f. Let $V_i = \{x_1, \dots, x_n\}$. Assume to the contrary that for every k, $1 \leq k \leq n$, there exists $j(k) \in S$ with $i < j(k)$ such that $x_k E x$ for no $x \in V_{j(k)}$. Since S is up-directed, there is $s \in S$ with

$$j(1), \dots, j(n) \leq s\,.$$

Take $y \in V_s$. By Lemma 2, there exists a (unique)

$$\big\{i, j(1), \dots, j(n), s\big\}\text{-frame } f \text{ in } G \qquad \text{with} \quad f(s) = y\,.$$

$f(i) = x_k$ for some k, $1 \leq k \leq n$. Then $x_k E f\big(j(k)\big)$, since $i < j(k)$, which is a contradiction. □

THEOREM 1. *Let S be an up-directed partially ordered set, and let G be a transitive S-partite König graph. Suppose that there exists a cofinal subset P of S such that for every $i \in P$ the set V_i is finite. Then there exists an S-frame in G.*

P r o o f. By Lemma 3, it is enough to show that there exists a P-frame in G. First we introduce two new concepts.

Let g be a J-frame in G for some $J \subseteq S$. Let $x \in V_i$ for some $i \in S$. Then we say that x is *assimilated* by g if for every $j \in J$ with $j < i$, we have $g(j)Ex$. Evidently, $g(i)$ is assimilated by g for every $i \in J$. Let us observe that if $i, k \in S$, $k < i$, $x \in V_i$, $y \in V_k$, yEx and x is assimilated by g, then y is assimilated by g, too. Indeed, if $j \in J$ and $j < k$, then by Lemma 1,

$$j < k < i\,,\ \ g(j)Ex \ \text{and}\ yEx \implies g(j)Ey\,.$$

Let g be a J-frame in G for some $J \subseteq P$. Then g is said to be *normal* if for every $i \in P$ there exists $x \in V_i$ such that g assimilates x.

Consider now the family N of all normal J-frames in G, where $J \subseteq P$. Clearly, N is non-empty by Lemma 4. N is partially ordered by the set-inclusion. We claim that N has a maximal element.

Really, let C be a non-empty set, and let g_k, $k \in C$, be a normal J_k-frame in G, where $J_k \subseteq P$. Assume that $\{g_k \mid k \in C\}$ is a chain in N. Consider $g = \bigcup(g_k \mid k \in C)$. Clearly, g is a J-frame in G, where $J = \bigcup(J_k \mid k \in C) \subseteq P$. Assume to the contrary that g is not normal. Then there exists $i \in P$ such that for every $x \in V_i$ there is $j(x) \in J$ with

$$j(x) < i \quad \text{and} \quad \big(g\big(j(x)\big), x\big) \notin E\,.$$

Since $\{g_k \mid k \in C\}$ is a chain, and V_i is a finite set, there is $k \in C$ such that $g\big(j(x)\big) = g_k\big(j(x)\big)$ for all $x \in V_i$. Because g_k is a normal J_k-frame in G, we obtain that there exists $x \in V_i$ with $g_k\big(j(x)\big)Ex$, which is a contradiction. Thus, g is a normal J-frame in G. By Zorn's lemma, there exists a maximal element of N, i.e., a maximal normal J-frame in G for some $J \subseteq P$, as claimed. Let us denote it by g. We want to show that $J = P$.

Suppose to the contrary that there exists $i \in P \setminus J$. Since g is normal, there is an element in V_i assimilated by g. Let $\{x_1, \dots, x_n\}$ be the set of all elements of V_i assimilated by g. For every r, $1 \le r \le n$, denote by g_r the extension of g defined as follows:

$$g_r(j) = \begin{cases} g(j) & \text{for } j \in J\,, \\ x_r & \text{for } j = i\,. \end{cases}$$

No g_r, $1 \le r \le n$, is a normal $J \cup \{i\}$-frame in G. If g_r is not a $J \cup \{i\}$-frame in G for some r, $1 \le r \le n$, then there exists $i(r) \in J$ with $i < i(r)$ and $\big(x_r, g\big(i(r)\big)\big) \notin E$. If g_r is a $J \cup \{i\}$-frame in G for some r, $1 \le r \le n$, then

g_r is not normal by maximality of g, hence there exists $i(r) \in P$ such that $i < i(r)$, and $(x_r, y) \notin E$ for every $y \in V_{i(r)}$ which is assimilated by g.

With respect to Corollary, we can assume that S has no greatest element. Since P is up-directed, P has no greatest element, and therefore there is $k \in P$ with $i(1), \dots, i(n) < k$. Choose $y \in V_k$ which is assimilated by g. By Lemma 2, there exists a (unique)

$$\{i, i(1), \dots, i(n), k\}\text{-frame } f \text{ in } G \qquad \text{with} \quad f(k) = y\,.$$

Then $f(i) = x_r$ for some r, $1 \le r \le n$. There exist

$$y_1 \in V_{i(1)}, \dots, y_n \in V_{i(n)} \qquad \text{assimilated by} \quad g$$

such that

$$f\bigl(i(1)\bigr) = y_1, \dots, f\bigl(i(n)\bigr) = y_n\,.$$

The case $(x_r, y_r) \notin E$ is not possible. Thus

$$i(r) \in J\,, \qquad \bigl(x_r, g\bigl(i(r)\bigr)\bigr) \notin E\,, \qquad x_r E y_r\,, \qquad \text{and} \qquad y_r E y\,.$$

Since y is assimilated by g, we get $g\bigl(i(r)\bigr) E y$. Since G is König, $g\bigl(i(r)\bigr) = y$, which is a contradiction. □

To find correlations between the properties used in [6; Theorem 2] and the assumptions of our Theorem 1, it should be observed that König's lemma is true also in the following stronger form. Let S be a partially ordered set. Call an S-partite graph G *weakly König* if for all $i, j \in S$ with $i < j$ and for every $y \in V_j$ there exists $x \in V_i$ such that xEy. If $G = \Bigl(\bigcup\limits_{i<\omega} V_i, E\Bigr)$ is a transitive ω-partite weakly König graph, and if each V_i is finite, then G contains an ω-frame.

To generalize this form of König's lemma, the following definitions were introduced in [6]. Let α be an ordinal, and let $G = \Bigl(\bigcup\limits_{i<\alpha} V_i, E\Bigr)$ be an α-partite graph. G is said to be "*König*" (here we use inverted commas to distinguish the same name used for two different notions) if for every ordinal i with $i+1 < \alpha$ and $y \in V_{i+1}$ there is some $x \in V_i$ such that xEy. Let us note that G is "König" whenever G is weakly König. G is said to be *back-connected* if for every i, $i < \alpha$, and for every $x \in V_i$ there exists an $(i+1)$-path f in G such that $f(i) = x$. This is stronger than being "König". Let us observe that if G is an ω-partite graph, then G is "König" if and only if G is back-connected. Then König's lemma may be formulated as follows. If $G = \Bigl(\bigcup\limits_{i<\omega} V_i, E\Bigr)$ is a back-connected ω-partite graph, and if each V_i is finite, then G contains an ω-path.

An α-partite graph G is said to be *narrow* if for every ordinal i, $i < \alpha$, there exists an ordinal j with $i \le j < \alpha$ such that either V_j is finite, or whenever

$U \subseteq V_{j+1}$ and $|U| \le \max(|\alpha|, |V_i|)$, then there exists $y \in V_j$ with $y \times U \subseteq E$. The generalization is as follows. If $G = \left(\bigcup_{i<\alpha} V_i, E \right)$ is a back-connected, narrow, α-partite graph, then G contains an α-path.

The following simple examples show that Theorem 1 is not valid under some weaker assumptions.

Examples.

(**1**) Let $S = \{0, 1, 2\}$ be a three-element partially ordered set given by the relations $0 < 1$ and $0 < 2$. S is not up-directed. Let G be a graph with the four-element vertex set $V = \{x, y, z, t\}$ and the edge set $E = \{(x, z), (y, t)\}$. Put

$$V_0 = \{x, y\}, \quad V_1 = \{z\}, \quad V_2 = \{t\}.$$

Then G is a transitive S-partite König graph. There is no S-frame in G.

(**2**) Let $S = \{0, 1, 2\}$ be endowed with the natural order. Let G be a graph with the four-element vertex set $V = \{x, y, z, t\}$ and the edge set $E = \{(x, z), (y, t), (z, t)\}$. Put

$$V_0 = \{x, y\}, \quad V_1 = \{z\}, \quad V_2 = \{t\}.$$

Then G is an S-partite König graph which is not transitive. There is no S-frame in G.

(**3**) Let $S = \{0, 1, 2, 3\}$ be a four-element lattice with the smallest element 0, the greatest element 3 and with 1 and 2 non-comparable. Let G be a graph with the five-element vertex set $V = \{x, y, z, t, u\}$ and the edge set

$$E = \{(x, z), (x, u), (z, u), (y, t), (y, u), (t, u)\}.$$

Put the parts of G as follows:

$$V_0 = \{x, y\}, \quad V_1 = \{z\}, \quad V_2 = \{t\}, \quad V_3 = \{u\}.$$

Then G is a transitive S-partite graph with the following property. For every $i, j \in S$ with $i < j$ and for every $w \in V_j$ there exists $v \in V_i$ such that vEw. This means that G is weakly König, but G is not König. There is no S-frame in G.

Let us note that in our paper [3], there is, in fact, an example of G, a transitive S-partite König graph, where S is an up-directed partially ordered set such that there is no S-frame in G. By Theorem 1, for every cofinal subset P of S there is $i \in P$ such that the set V_i is not finite. This example is examined more detailed in the next section.

2. Lattices

As we mentioned earlier, the motivation for Theorem 1 came from K o l i - b i a r 's characterization of lattices by the (ternary) betweenness relation [4]. First, let us recall some concepts and results. We shall use the notation from [3].

A *ternary relation* R on a set M is a subset of $M \times M \times M$. For $a, b, c \in M$, we shall write abc instead of $(a, b, c) \in R$. Also, we shall say that a ternary relation abc on the set M is given. For $a, b \in M$, we define $\langle a, b \rangle$ as the set $\{c \in M \mid acb\}$. This set will be called a *segment* on M. A subset K of M is said to be *closed* if $\langle a, b \rangle \subseteq K$ for every $a, b \in K$. Since the intersection of any system of closed subsets of M is again closed, we can introduce a closure operation $^-$ on the subsets of M as follows: K^- is the intersection of all closed subsets of M containing K.

Now we can formulate the following K o l i b i a r 's conditions (cf. [4]):

(A) For any $a, b, c \in M$ there are $d, e \in M$ such that
$\{a, b, c\} \subseteq \langle d, e \rangle = \langle d, e \rangle^-$.

(B) For any elements $a, b, c \in M$,
$\langle a, b \rangle^- \cap \langle b, c \rangle^- \cap \langle c, a \rangle^- \neq \emptyset$.

(C) If $a, b, c \in M$, then abc if and only if
$\langle a, b \rangle^- \cap \langle c, b \rangle^- = \{b\}$.

(F) The closed segments on M can be "oriented" in the following sense: There exists a mapping assigning to every closed segment H a pair $(a_H, b_H) \in M \times M$ such that $H = \langle a_H, b_H \rangle$, and for all closed segments H, K the following holds:
If $H \subseteq K$ and $\langle a_K, b_H \rangle$ is closed, then $a_K a_H b_H$.

Having a lattice L, one can define a ternary relation abc (the *betweenness* relation) on L as follows:

$$abc \iff (a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c).$$

In [4], it was shown that the betweenness relation satisfies the conditions (A), (B), (C) and (F). Conversely, M. K o l i b i a r [4] proved that if there is a ternary relation abc on a set M satisfying (A), (B), (C) and (F), then lattice operations can be defined on M such that the corresponding betweenness relation on M and the given ternary relation abc on M coincide.

We are now in a position to formulate our goals in this section. Our first task is to establish the fact that for every set M with a ternary relation abc satisfying (A), (B) and (C), a partially ordered set $CS(M)$ and a $CS(M)$-partite graph G can be assigned in a natural way, such that $CS(M)$ is up-directed and G is König and transitive. This enables us to reformulate the main K o l i b i a r 's result from [4] in terms of graphs. Eventually, in accordance with this approach, we shall apply Theorem 1.

Let us consider a ternary relation abc on a set M satisfying the conditions (A), (B) and (C). Let $CS(M)$ denote the set of all closed segments on M partially ordered by the set-inclusion. By virtue of condition (A), $CS(M)$ is up-directed. An ordered pair $(a,b) \in M \times M$ is called a *base* of a closed segment H if $\langle a,b\rangle = H$. The set of all bases of H will be denoted by $\mathrm{Fund}(H)$. In addition, if $H, K \in CS(M)$ with $H \subseteq K$ and $(a,b) \in \mathrm{Fund}(H)$, $(c,d) \in \mathrm{Fund}(K)$, then the bases (a,b) and (c,d) are said to *have the same orientation*, whenever cab is true.

Now we shall give the description of a $CS(M)$-partite graph G assigned to M. The vertex set $V(G) = V$ of G is defined as follows:

$$V = \bigcup\big(\mathrm{Fund}(H) \mid H \in CS(M)\big).$$

(It can be easily verified that the family $\{\mathrm{Fund}(H) \mid H \in CS(M)\}$ forms a partition of V.) Eventually, the edge set $E(G) = E$ of G comprises all pairs of distinct bases having the same orientation.

LEMMA 5. *If M is a set with a ternary relation abc satisfying* (A), (B) *and* (C), *then for all $a, b, c, d \in M$ the following is true:*

(i) $\langle a,b\rangle = \langle b,a\rangle$,
(ii) $\langle a,a\rangle^- = \{a\}$,
(iii) $a, b \in \langle a,b\rangle$,
(iv) *if abc and $d \in \langle a,b\rangle^-$, then dbc,*
(v) *if $\langle a,c\rangle$ is closed and abc, then $\langle a,b\rangle$ is closed,*
(vi) *if abc and acb, then $b = c$.*

P r o o f. A proof of the statements (i)–(v) can be found in [4]. Cf. [4; 4.3.2, 4.3.4–4.3.7]. Condition (vi) follows, e.g., by (C) and (iii). □

THEOREM 2. *Let M be a set with a ternary relation abc satisfying* (A), (B) *and* (C). *Then the assigned $CS(M)$-partite graph G is König and transitive.*

P r o o f. First we prove that G is König. Assume that $H, K \in CS(M)$ with $H \subset K$. Let $(x,y) \in \mathrm{Fund}(H)$ and $(c,d) \in \mathrm{Fund}(K)$. We claim that there exists $(a,b) \in \mathrm{Fund}(H)$ such that (a,b) and (c,d) have the same orientation. Really, by (B) there exist elements

$$a \in \langle c,x\rangle^- \cap \langle c,y\rangle^- \cap H \qquad \text{and} \qquad b \in \langle d,x\rangle^- \cap \langle d,y\rangle^- \cap H.$$

Therefore, $\langle a,b\rangle \subseteq H$. Moreover, dxc and $b \in \langle d,x\rangle^-$ implies bxc by Lemma 5. By the same argument, axb follows from cxb and $a \in \langle c,x\rangle^-$. Similarly, we obtain ayb. Now, $\langle b,c\rangle$ is a closed segment by Lemma 5 and the fact that

$b \in \langle c, d \rangle$. From $x \in \langle b, c \rangle$ it follows that $a \in \langle c, x \rangle^- \subseteq \langle b, c \rangle$, which implies cab. Moreover, $\langle a, b \rangle$ is closed and $x, y \in \langle a, b \rangle$, hence

$$H = \langle x, y \rangle \subseteq \langle a, b \rangle \subseteq H ,$$

which means that $\langle a, b \rangle = H$. Evidently, the bases (a, b) and (c, d) have the same orientation, as claimed.

Suppose that there exists another base $(u, v) \in \mathrm{Fund}(H)$ with the same orientation as (c, d). Then by Lemma 5, we get successively:

$$bac \text{ and } bua \implies uac ; \qquad cuv \text{ and } uav \implies auc .$$

By Lemma 5, cau and cua implies $a = u$, and then from abv and avb we similarly obtain $b = v$. Thus $(u, v) = (a, b)$, and therefore G is König.

It remains to prove that G is transitive. Let $H, K, N \in CS(M)$ with $H \subset K \subset N$, and let $(a, b) \in \mathrm{Fund}(H)$, $(c, d) \in \mathrm{Fund}(K)$ and $(e, f) \in \mathrm{Fund}(N)$. Let cab and ecd be true, i.e., (a, b), (c, d) and (c, d), (e, f) have the same orientation. Since G is König, there exists $(a', b') \in \mathrm{Fund}(H)$ such that (a', b') and (e, f) have the same orientation, that means, $ea'b'$. Now, ecd and $ca'd$ implies eca' by Lemma 5. It follows that $ca'b'$, again by Lemma 5, as $ea'b'$. This means that (a', b') and (c, d) have the same orientation. Because G is König, $(a, b) = (a', b')$, and thus (a, b), (e, f) have the same orientation. The proof is complete. □

Let us observe that if M is a set with a ternary relation abc satisfying (A), (B) and (C) containing more than one element, then $CS(M)$ is not a chain. Even more is true, for every $H \in CS(M)$ containing more than one element, the set $\{K \in CS(M) \mid K \subset H\}$ is not a chain. This is caused by the fact that the set $\{\langle a, a \rangle \mid a \in M\}$ is an anti-chain in $CS(M)$.

COROLLARY. *Let M be a set with a ternary relation abc satisfying the conditions* (A), (B) *and* (C). *Then M satisfies condition* (F) *if and only if there is a $CS(M)$-frame in the assigned graph G.*

P r o o f . It suffices to observe that condition (F) can be formulated without the assumption "$\langle a_K, b_H \rangle$ is closed". This follows from Lemma 5. □

In [3], it was proved that the conditions (A), (B), (C) and (F) are independent. Thus, an example of a set M with a ternary relation abc satisfying (A), (B) and (C), but not (F) (cf. [3; Example 4]) provides simultaneously an example of a transitive $CS(M)$-partite König graph G which does not contain any $CS(M)$-frame. By Theorem 1, for every cofinal subset P of $CS(M)$ there is $H \in P$ such that the set $\mathrm{Fund}(H)$ is not finite.

Now, as a consequence of Theorem 1 and 2 we have:

THEOREM 3. *Let M be a set with a ternary relation abc satisfying* (A), (B) *and* (C). *If there exists a cofinal subset P of $CS(M)$ such that for every $H \in P$ the set* Fund(H) *is finite, then M satisfies condition* (F).

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