Sergei A. Gurchenkov

Every l-variety satisfying the amalgamation property is representable

Mathematica Slovaca, Vol. 47 (1997), No. 3, 221--229

Persistent URL: http://dml.cz/dmlcz/129405

Math. Slovaca, 47 (1997), No. 3, 221–229

Mathematica Slovaca

EVERY l-VARIETY SATISFYING THE AMALGAMATION PROPERTY IS REPRESENTABLE

SERGEI A. GURCHENKOV

(*Communicated by Tibor Katriňák*)

ABSTRACT. We show that every l-variety satisfying the amalgamation property is representable. Furthermore, we construct an infinite set of varieties of weakly abelian l-groups which fail the amalgamation property.

Introduction

A variety $\mathcal{G}$ of l-groups is said to satisfy the *amalgamation property* in the l-variety $\mathcal{M}$ if, first, $\mathcal{G} \subseteq \mathcal{M}$, and, second, if for any l-groups $A, B, C \in \mathcal{G}$ and embeddings $\sigma\colon A \to B$, $\mu\colon A \to C$ there exist an l-group $D \in \mathcal{M}$ and embeddings $\phi\colon B \to D$, $\psi\colon C \to D$ such that $\phi\sigma = \psi\mu$. The quintuple (A, B, C, σ, μ) is called a *V-formation* in $\mathcal{G}$, and the triple (ϕ, ψ, D) is called an *amalgamation* in $\mathcal{M}$ of this V-formation. An l-group A, $A \in \mathcal{G}$, is said to be an amalgamation base for $\mathcal{G}$ in $\mathcal{M}$ if every V-formation (A, B, C, σ, μ) in $\mathcal{G}$ has an amalgamation (ϕ, ψ, D) in $\mathcal{M}$.

The *amalgamation class* of $\mathcal{G}$ in $\mathcal{M}$, $\mathrm{Amal}_{\mathcal{M}}(\mathcal{G})$, is the class consisting of all amalgamation bases for $\mathcal{G}$ in $\mathcal{M}$. For the case $\mathcal{G} = \mathcal{M}$ all these definitions are usual (see Powell, Tsinakis [9; p. 308]). In this article, we use the following notation

$\mathcal{L}$ – variety of all l-groups,
$\mathcal{R}$ – variety of all representable l-groups,
$\mathcal{N}$ – variety of all normal-valued l-groups,
$\mathcal{W}_a$ – variety of all weakly abelian l-groups,
$\mathcal{A}$ – variety of all abelian l-groups,
$\mathcal{S}_p$ – Scrimger l-variety for prime p,
$\mathcal{N}_n$ – variety of all nilpotent of class $\leq n$ l-groups,

AMS Subject Classification (1991): Primary 06F15.
Key words: l-variety, amalgamation.

$\mathcal{M}^+$, $\mathcal{M}^-$ – solvable non-nilpotent representable covers of $\mathcal{A}$,
$A\lambda B$ – semidirect extension of a group B by a group A,
$A\overrightarrow{\lambda}B$ – lexicographic semidirect extension of an l-group B by a totally ordered group A,
$\mathbb{Z}$ – additive group of integers,
P_n $= \{1, 2, \dots, n\}$.

We recall the main results in the theory of l-groups connected with the amalgamation property.

The variety $\mathcal{A}$ satisfies the amalgamation property (Pierce [7]).

If an l-variety $\mathcal{M}$ contains $\mathcal{S}_p$ for some prime p, then $\mathcal{M}$ fails the amalgamation property (Pierce [6]).

If a representable l-variety $\mathcal{M}$ contains one of the l-varieties $\mathcal{M}^+$, $\mathcal{M}^-$, then $\mathcal{M}$ fails the amalgamation property (Powell, Tsinakis [9], or Powell, Tsinakis [10]).

The l-variety $\mathcal{W}_a$ of all weakly abelian l-groups fails the amalgamation property (Glass, Saracino, Wood [1]).

Every totally ordered archimedian l-group belongs to Amal($\mathcal{L}$) (Pierce [6]).

Amal($\mathcal{N}_n$), for $n > 1$, does not contain non-trivial totally ordered abelian groups (Powell, Tsinakis [8]).

The following important questions related to the amalgamation property in l-varieties remain open.

1. Which l-varieties satisfy the amalgamation property? (See Powell, Tsinakis [9] and [11].)

2. Is $\mathcal{A}$ the only non-trivial l-variety satisfying the amalgamation property? (See Powell, Tsinakis [9] and [11].)

3. Which l-varieties have $\mathbb{Z}$ in their amalgamation class? (See [11].)

The purpose of this article is to establish the following results related to the aforementioned questions.

1. If a non-representable l-variety $\mathcal{M}$ contains $\mathbb{Z}$ in its amalgamation class, then $\mathcal{M}$ includes the variety $\mathcal{N}$ of normal-valued l-groups.

2. If an l-variety satisfies the amalgamation property, then it is representable.

3. If an l-variety $\mathcal{M}$ includes an l-variety $\mathcal{A}^2 \cap \mathcal{W}_a$, then $\mathcal{M}$ fails the amalgamation property.

Preliminaries

DEFINITION 1. We say that an l-group G has *finite conjugate-orthogonal rank* n (and we write $\mathrm{co}(G) = n$) if there are elements $g, a \in G$, $g > e$, $a > e$, such that $a \wedge g^{-i}ag^{i} = e$ for $i \in P_n$, and for every elements $x, y \in G$, $x > e$, $y > e$, the following implication holds:

$$\left(x \wedge y^{-i}xy^{i} = e \text{ for } i \in P_n\right) \implies \left(x \wedge y^{-(n+1)}xy^{n+1} \neq e\right).$$

DEFINITION 2. An l-variety $\mathcal{M}$ is said to have a *finite conjugate-orthogonal rank* n, denoted $\mathrm{co}(\mathcal{M}) = n$, provided every l-group $H \in \mathcal{M}$ satisfies $\mathrm{co}(H) \leq n$, and there exists $G \in \mathcal{M}$ with $\mathrm{co}(G) = n$.

DEFINITION 3. An l-variety $\mathcal{M}$ is said to have an *infinite conjugate-orthogonal rank*, denoted $\mathrm{co}(\mathcal{M}) = \infty$, provided for every integer n there exists $G \in \mathcal{M}$ with $\mathrm{co}(G) \geq n$.

The proof of the following lemma may be found, for example, in [3].

LEMMA 1. *For any variety of l-groups $\mathcal{M}$, if $\mathrm{co}(\mathcal{M}) = \infty$, then $\mathcal{M} \supseteq \mathcal{A}^2$.*

The ideas used in the proof of the following lemma are due to Kopytov, Gurchenkov [4].

LEMMA 2. *Let G be a normal-valued l-group with $\mathrm{co}(G) = n$, where $n \geq 1$. Let $g \in G$, and the non-trivial convex l-subgroup H of G satisfy the conditions $H \cap g^{-i}Hg^{i} = E$ for $i \in P_n$. Then the l-subgroup $l(g^{n+1}, H)$ of the l-group G is representable.*

Proof. Let p denote the integer $n+1$. Consider the l-subgroups

$$X = l\left(\{g^{-jp}Hg^{jp},\ j \in \mathbb{Z}\}\right),$$
$$Y = g^{-1}Hg \times_l g^{-2}Hg^{2} \times_l \cdots \times_l g^{-n}Hg^{n}$$

of the l-group G. We claim that $X \cap Y = E$. Firstly we verify by induction on m, $m \geq 0$, that

$$g^{-mp}Hg^{mp} \cap Y = E. \tag{1}$$

For $m = 0$ condition (1) follows by assumption. Suppose next by induction hypothesis that condition (1) is true for all $m < k$, and for $m = k$ condition (1) fails, that is, $g^{-kp}Hg^{kp} \cap Y \neq E$. Then necessarily exists an element a, $e < a \in g^{-(k-1)p}Hg^{(k-1)p}$, such that $g^{-p}ag^{p} \in g^{-kp}Hg^{kp} \cap Y$. It follows by induction hypothesis that

$$g^{-(k-1)p}Hg^{(k-1)p}\cap Y=E\,,$$

and (2)

$$a\wedge g^{-p}ag^{p}=e\,.$$

Conditions (1) and (2) above yield $H\cap g^{-i}Hg^{i}=E$, $i\in\{1,\dots,n\}$,

$$\begin{aligned}&g^{-(k-1)p}Hg^{(k-1)p}\cap g^{-(k-1)p}g^{-i}Hg^{i}g^{(k-1)p}\\&=g^{-(k-1)p}Hg^{(k-1)p}\cap\left[g^{-(k-1)p}Hg^{(k-1)p}\right]^{g^{i}}=E\,,\qquad i\in\{1,\dots,n\}\,,\end{aligned}$$

and hence,

$$a\wedge g^{-i}ag^{i}=e\qquad\text{for}\quad i\in\{1,\dots,n\}\,. \tag{3}$$

But (2), (3) contradict $\mathrm{co}(G)=n$. This establishes condition (1). In the same way, we can prove that (1) is true in the case $m\leq 0$. It is easy to see that

$$X\cap g^{-i}Xg^{i}=E\,,\qquad i\in\{1,\dots,n\}\,, \tag{4}$$

and that a convex l-subgroup X is g^{p}-invariant (i.e., $g^{-p}Xg^{p}=X$). Let S be any non-trivial polar of an l-subgroup X. Suppose that $g^{-p}Sg^{p}\neq S$. It follows from the definition of the polar there exists a set M, $M\subseteq X$, such that $S=M^{\perp}$. Suppose the element a, $e<a\in S$, exists such that $g^{-p}ag^{p}\in g^{-p}Sg^{p}\cap M$. Then, $a\wedge a^{-p}ag^{p}=e$. But $a\in X$ and condition (4) is true, that contradicts $\mathrm{co}(G)=n$.

Thus for every polar $S\subseteq X$, we have $g^{-p}Sg^{p}=S$. Let us prove that X is representable. Suppose that X is not representable. Then there exist elements a, b, $e<a,b\in X$, such that $a\wedge b^{-1}ab=e$. Let us consider elements f, $y=gb$ in an l-group G. We have $y^{k}=(gb)^{k}=g^{k}b^{g^{k-1}}b^{g^{k-2}}\dots b^{g}b$. For $k\in P_{n}$, by condition (4) and $a,b\in X$, it follows that $\left[a^{g^{k}},b^{g}\right]=\left[a^{g^{k}},b^{g^{2}}\right]=\dots=\left[a^{g^{k}},b^{g^{k-1}}\right]=e$, hence $a\wedge y^{-k}ay^{k}=a\wedge g^{-k}ag^{k}=e$, $a\wedge y^{-p}ay^{p}=a\wedge(gb)^{-1}g^{-n}ag^{n}gb=a\wedge\left[g^{-(n+1)}ag^{n+1}\right]^{b}=e$ (as we established earlier, $(a^{\perp})^{g^{n+1}}=a^{\perp}$). This contradicts $\mathrm{co}(G)=n$ and establishes that X is representable. It follows by the condition of the lemma $G\in\mathcal{N}$. For every element $a\in X$ we have $|a|^{g}\wedge|a|=e$, and, in the l-variety $\mathcal{N}$, the identity $|[x,y]|\ll|x|\vee|y|$ is true. Hence we immediately have in G that $|a|\leq|a^{-1}a^{g}|=|a||a^{g}|\ll|a|\vee|g|$. Thus $|a|\ll|g|$, and an l-subgroup $l(g^{p},X)$ admits a representation $l(g^{p},X)=\langle g^{p}\rangle\overrightarrow{\lambda}X$. It is easy to see that any polar in $l(g^{p},X)$ is a polar in X. As we established earlier, any polar in an l-subgroup X is normal in $\langle g^{p}\rangle\overrightarrow{\lambda}X$, and hence the l-subgroup $l(g^{p},X)$ is representable. The proof is now completed. □

LEMMA 3. *Let* $\mathrm{var}_l(G) \supseteq \mathcal{A}^k$ *for some* $k \geq 2$, *and* $\mathrm{Amal}(\mathrm{var}_l(G)) \ni \mathbb{Z}$. *Then* $\mathcal{N} \subseteq \mathrm{var}_l(G)$.

Proof. It is well known that $\mathcal{A}^k = \mathrm{var}_l(\mathrm{wr}^k \mathbb{Z})$ (see Holland, Glass, McCleary [5]). Let $\langle a \rangle$, $\langle a_i \rangle$, $i = 1, \dots, k+1$, be an infinite cyclic groups, where $a > e$, $a_i > e$, $i = 1, \dots, k+1$, and let $B = \big(\dots\big(\langle a_2 \rangle \,\mathrm{wr} \langle a_3 \rangle\big)\,\mathrm{wr} \dots \dots\big)\,\mathrm{wr}\langle a_{k+1} \rangle$, $C = \langle a_1 \rangle \,\mathrm{wr} \langle a \rangle$. Consider a V-formation $(\mathbb{Z}, B, C, \sigma, \mu)$, where $\sigma(1) = a_2$, $\mu(1) = a$. It follows from the conditions of the lemma that there exist $D \in \mathrm{var}_l(G)$ and embeddings $\phi\colon B \to D$, $\psi\colon C \to D$ such that $\phi\sigma(1) = \psi\mu(1)$. Let $b_1 = \psi(a_1)$, $b_2 = \phi\sigma(1) = \psi\mu(1)$, and $b_i = \phi(a_i)$ for $i \geq 3$. It is easy to see that for the elements $b_1, \dots, b_{k+1}$ in an l-group D the following conditions are true:

$$b_{k+1} \gg b_k \gg \dots \gg b_2 \gg b_1 > e\,,$$
$$b_i \wedge b_j^{-s} b_i b_j^s = e\,, \qquad 1 \leq i < j \leq k+1\,, \quad s \in \mathbb{Z}.$$

Hence, the l-subgroup $l(b_1, b_2, \dots, b_{k+1})$ is l-isomorphic to a wreath product $\mathrm{wr}^{k+1} \mathbb{Z}$. Thus, for every l-group G that satisfies the conditions of the lemma, we immediately have the inclusion $\mathrm{var}_l(G) \supseteq \mathcal{A}^s$ for every $s \in \mathbb{N}$. It is well known (see Holland, Glass, McCleary [5]) that $\mathcal{N} = \bigcup_{s=1}^{\infty} \mathcal{A}^s$, and therefore $\mathrm{var}_l(G) \supseteq \mathcal{N}$. The proof is completed. □

LEMMA 4. *If* $\mathrm{Amal}(\mathcal{M}) \ni \mathbb{Z}$ *and* $\mathrm{co}(\mathcal{M}) \geq 1$, *then* $\mathrm{co}(\mathcal{M}) = \infty$.

Proof. Suppose that $\mathrm{co}(\mathcal{M}) = n$ for some $n \in \mathbb{N}$. Let a, b be the elements of any l-group $G \in \mathcal{M}$ such that $a \wedge g^{-i} a g^i = e$ for $i \in P_n$. Let H denote a convex l-subgroup of G generated by the set $\{a\}$. It is easy to see that this implies the conditions $H \cap g^{-i} H g^i = E$ for $i \in P_n$. It follows from Lemma 2 that the l-subgroup $B = l(g, X)$ of the l-group G, where $X = l\big(\{g^{-ip} H g^{ip}, i \in \mathbb{Z},\ p = n+1\}\big)$, admits representation $B = \langle g \rangle \overrightarrow{\lambda} (X \times_l X^g \times_l \dots \times_l X^{g^n})$. Let $\hat{B} = \langle \hat{g} \rangle \overrightarrow{\lambda} \big(\hat{X} \times_l \hat{X}^{\hat{g}} \times_l \dots \times_l \hat{X}^{\hat{g}^n}\big)$ denote an l-isomorphic copy of an l-group B. Consider a V-formation $(\mathbb{Z}, B, \hat{B}, \sigma, \mu)$, where $\sigma(1) = a$, $\mu(1) = \hat{g}$. It follows from the conditions of the lemma that there exists an amalgamation (ϕ, ψ, D). Let $b = \phi\sigma(1) = \phi(a) = \psi\mu(1) = \psi(\hat{g})$, $c = \phi(g)$, and $f = \psi(\hat{a})$. It is easy to see that, in an l-group D, we have conditions

$$c \gg b \gg f > e\,; \qquad b \wedge c^{-i} b c^i = e\,, \quad f \wedge b^{-i} f b^i = e\,, \quad i \in \{1, \dots, n\}\,. \tag{5}$$

Let A be a convex l-subgroup of the l-group D generated by the element b. From (5), we immediately have $A \cap c^{-i} A c^i = E$, $i \in P_n$. It follows from Lemma 2 that the l-subgroup A is representable, but $b, f \in A$ and $f \wedge b^{-1} f b = e$. This contradiction establishes the proof of the lemma. □

LEMMA 5. *Let* $\mathrm{co}(\mathcal{M}) = n \geq 1$. *There exists an* l*-group* $G \in \mathcal{M}$ *such that* $G = l(X,g)$, $g > e$, X *is convex in* G, $X \cap g^{-i}Xg^{i} = E$, $g^{-(n+1)}Xg^{n+1} = X$, $i = 1,\dots,n$, *and* $l(X,g^{n+1}) \in \mathcal{R}$.

Proof. Consider any l-group $A \in \mathcal{N}$ with $\mathrm{co}(A) = n$. Then A has elements $a, g > e$ such that $a \wedge g^{-i}ag^{i} = e$ for $i \in P_n$. Let H denote a convex l-subgroup of the l-group A generated by the element a. It is easy to see that $H \cap g^{-i}Hg^{i} = E$, $i \in P_n$, and the l-subgroup $G = l(X,g)$ where $X = l(\{g^{-j(n+1)}Hg^{j(n+1)},\ j \in \mathbb{Z}\})$ has the necessary properties (it follows from Lemma 2). The proof is completed. □

PROPOSITION 1. *Let* $\mathrm{co}(\mathcal{M}) = n \geq 1$. *Then* $\mathcal{M}$ *fails the amalgamation property in* $\mathcal{L}$.

Proof. It follows from Lemma 5 that there exists an l-group G which admits representation

$$G = \langle g\rangle \overrightarrow{\lambda}(X_0 \times_l X_1 \times_l X_2 \times_l \cdots \times_l X_n),$$

where $X_0 = X$, $X_i = g^{-i}Xg^{i}$, $i = 1,\dots,n$, and $l(X,g^{n+1}) \in \mathcal{R}$. Let G_0 denote an l-subgroup $\langle g^{n+1}\rangle \overrightarrow{\lambda}(X \times_l X^{g} \times_l X^{g^2} \times_l \cdots \times_l X^{g^n})$ of the l-group G. Let G_1 denote an l-group $\hat{G}_0 \times_l \hat{G}_1 \times_l \hat{G}_2 \times_l \cdots \times_l \hat{G}_n$, where $\hat{G}_i = \langle g_i\rangle \overrightarrow{\lambda} X_i$, and for $h \in X_i$, $h^{g_i} = h^{g^{n+1}}$, $i = 0,1,\dots,n$. It is easy to see that $G_0, G_1 \in \mathcal{M}$. Consider the embeddings $\sigma\colon G_0 \to G_1$, $\mu\colon G_0 \to G$ defined as follows: $\sigma(h) = h$, $\mu(h) = h$ for $h \in X_i$, $i = 0,1,\dots,n$, $\mu(g^{n+1}) = g^{n+1}$, and $\sigma(g^{n+1}) = g_0g_1\cdots g_n$. We show that the V-formation $(G_0, G_1, G, \sigma, \mu)$ has not the amalgamation in $\mathcal{L}$. Suppose it is not the case, and let (ϕ, ψ, D) be an amalgamation for $(G_0, G_1, G, \sigma, \mu)$. Let us use the following notation in an l-group D: $\phi\sigma(h) = \psi\mu(h) = a_h$ for $h \in X_i$, $i = 1,\dots,n$, $\psi(g_i) = b_i$, $i = 1,\dots,n$, $\phi(g) = b$. Then the following conditions are true in the l-group D: $a_h \wedge b^{-i}a_hb^{i} = e$, where $h \in X$, $h > e$, $i = 1,\dots,n$; $b^{n+1} = b_0b_1\dots b_n$, where $b_i \in \phi(\hat{G}_i)$, and $\phi(\hat{G}_i) \cap \phi(\hat{G}_j) = E$ for $i \neq j$, $i,j \in \{0,1,\dots,n\}$. It is easy to see that $e < b < b^{n+1}$, and hence, the element b may be written in the form $b = f_0f_1\cdots f_n$ for some $f_i \in H_i$, where H_i denotes a convex l-subgroup of D generated by the l-subgroup $\phi(\hat{G}_i)$, $i = 0,1,\dots,n$. Since $(\hat{G}_0) \cap [\phi(\hat{G}_1) \times_l \phi(\hat{G}_2) \times_l \cdots \times_l \phi(\hat{G}_n)] = E$, it follows that $H_0 \cap (H_1 \times_l H_2 \times_l \cdots \times_l H_n) = E$. Since $f_0 \in H_0$, $f_1f_2\cdots f_n \in H_1H_2\dots H_n$ for every element h, $e < h \in X$, we have $a_h^{b} = \psi(h)^{\psi(g)} \in \psi(X^{g}) = \psi(X_1) \subseteq H_1$. This contradicts $H_0 \cap H_1 = E$, and completes the proof of the proposition. □

COROLLARY 1. (Pierce [6]) *If an* l*-variety* $\mathcal{M}$ *contains* $\mathcal{S}_p$ *for some prime* p, *then* $\mathcal{M}$ *fails the amalgamation property.*

Remark. There exist non-representable l-varieties $\mathcal{M}$ such that $\mathcal{M} \cap \mathcal{S}_p = \mathcal{A}$ for every p.

The main results

THEOREM 1. *Let an l-variety $\mathcal{M}$ satisfy $\mathcal{M} \cap \mathcal{R} \neq \mathcal{M}$ and $\mathrm{Amal}(\mathcal{M}) \ni \mathbb{Z}$. Then $\mathcal{M} \supseteq \mathcal{N}$.*

P r o o f. If $\mathrm{co}(\mathcal{M}) = \infty$, then it follows from Lemmas 1 and 3 that $\mathcal{M} \supseteq \mathcal{A}^2$ and $\mathcal{M} \supseteq \mathcal{N}$. Let $\mathrm{co}(\mathcal{M}) = n < \infty$. Note that $n \geq 1$ since $\mathcal{M} \cap \mathcal{R} \neq \mathcal{M}$. Thus, by Lemma 4, $\mathrm{co}(\mathcal{M}) = \infty$, in contradiction with the assumption. □

THEOREM 2. *Every non-representable l-variety fails the amalgamation property.*

P r o o f. Let $\mathcal{M}$ be a non-representable l-variety satisfying the amalgamation property. In particular, $\mathrm{Amal}(\mathcal{M}) \ni \mathbb{Z}$. It follows from Theorem 1 that $\mathcal{M} \supseteq \mathcal{N}$. It is well known that $\mathcal{N}$ contains some l-varieties $\mathcal{G}$ with $\mathrm{co}(\mathcal{G}) = n \geq 1$. It follows from Proposition 1 that the l-variety $\mathcal{G}$ fails the amalgamation property in $\mathcal{L}$, so $\mathcal{M}$ fails the amalgamation property. The proof is completed. □

Let W denote a group $\mathrm{gr}\big(a, b \parallel [b^{-i}ab^i, b^{-j}ab^j] = e,\ j, i \in \mathbb{Z}\big)$. It is easy to see that $W \cong \mathbb{Z} \,\mathrm{wr}\, \mathbb{Z}$. It is well known the group W admits total orders and weakly abelian total orders (see, for example, G u r c h e n k o v [2]). Let P denote one such order. Let T denote a subgroup $\mathrm{gr}\big(\{b^{-i}ab^i,\ i \in \mathbb{Z}\}\big)$ of the group W with a total order induced on T by the total order P of the group W. Let $A = T \overrightarrow{\times} \langle c \rangle$ be a lexicographic product of an infinite cyclic group $\langle c \rangle$, $c > e$, and a totally ordered group T.

Now we define two automorphisms α, β of the group A as follows:

$$c^\alpha = c\,, \qquad a_n^\alpha = a_{n+1}\,, \quad n \in \mathbb{Z}\,,$$

$$c^\beta = c\,, \qquad a_n^\beta = \begin{cases} a_n c & \text{if } n \equiv 0 \pmod{p}\,, \\ a_n & \text{if } n \not\equiv 0 \pmod{p}\,, \end{cases}$$

where a_n denotes the element $b^{-n}ab^n$, $n \in \mathbb{Z}$. It is easy to see that the automorphisms α, β preserve the total order on A. Let $\mathrm{Aut}\, A$ denote the group of order-preserving automorphisms of the abelian totally ordered group A. Since

$$\begin{aligned} a_n^{\beta^{-1}\alpha^p\beta} &= (a_n c^{-1})^{\alpha^p\beta} \\ &= \begin{cases} (a_{n+p}c^{-1})^\beta = a_{n+p} & \text{if } n \equiv 0 \pmod{p}\,, \\ (a_n)^{\alpha^p\beta} = (a_{n+p})^\beta = a_{n+p} & \text{if } n \not\equiv 0 \pmod{p}\,, \end{cases} \end{aligned}$$

and $a_n^{\alpha^p} = a_{n+p}$, then $\beta^{-1}\alpha^p\beta = \alpha^p$, but $\beta^{-1}\alpha\beta \neq \alpha$ in the group $\operatorname{Aut} A$. Set $g = \alpha^p$, $\gamma = \beta^{-1}\alpha\beta$ in the group $\operatorname{Aut} A$.

Consider the totally ordered groups $G_3 = \langle g\rangle \overrightarrow{\lambda} A$, $G_2 = \langle\alpha\rangle \overrightarrow{\lambda} A$, $G_1 = \langle\gamma\rangle \overrightarrow{\lambda} A$, where $g > e$, $\alpha > e$, $\gamma > e$. It is easy to see that $G_i \in \mathcal{R}$ and $G_i \in \mathcal{W}_a$ if P is weakly abelian, $i = 1,2,3$. Define embeddings μ, σ, $\sigma\colon G_3 \to G_1$, $\mu\colon G_3 \to G_2$ as follows:

$$\mu(c) = c\,, \quad \mu(a) = a\,, \quad \mu(g) = \alpha^p\,, \qquad \sigma(c) = c\,, \quad \sigma(a) = a\,, \quad \sigma(g) = \gamma^p\,.$$

Suppose there exists an amalgamation (ϕ,ψ,D) in $\mathcal{R}$ for the V-formation $(G_3, G_1, G_2, \sigma, \mu)$. Let $\hat{a} = \phi(a) = \psi(a)$ for $a \in A$, $\phi(\gamma) = \hat{\gamma}$, $\psi(\alpha) = \hat{\alpha}$. We have, in the l-group D, $\hat{\alpha}^p = \hat{\gamma}^p$, but $\hat{\gamma}^{-1}\hat{a}\hat{\gamma} \neq \hat{\alpha}^{-1}\hat{a}\hat{\alpha}$, thus the l-group D cannot be representable. Thus the V-formation $(G_3, G_1, G_2, \sigma, \mu)$ does not have an amalgamation in $\mathcal{R}$ (in $\mathcal{W}_a$, if P is weakly abelian).

THEOREM 3. *Let $\mathcal{M} \supseteq \mathcal{A}^2 \cap \mathcal{W}_a$. Then $\mathcal{M}$ fails the amalgamation property.*

Proof. If $\mathcal{M} \cap \mathcal{R} \neq \mathcal{M}$, then the result follows from Theorem 2. Let $\mathcal{M} \subseteq \mathcal{R}$. In this case, for the V-formation $(G_3, G_1, G_2, \sigma, \mu)$ the amalgamation (ϕ,ψ,D) in $\mathcal{R}$ exists. As we established earlier, this is impossible. □

REFERENCES

[1] GLASS, A. M. W.—SARACINO, P.—WOOD, C.: *Non-amalgamation of ordered groups*, Math. Proc. Cambridge Philos. Soc. **95** (1984), 191–195.

[2] GURCHENKOV, S. A.: *Varieties of nilpotent lattice ordered groups*, Algebra and Logic **21** (1982), 499–510. (Russian)

[3] GURCHENKOV, S. A.: *Varieties of l-groups with the identity $[x^p, y^p] = e$ have finite basis*, Algebra and Logic **23** (1984), 27–47. (Russian)

[4] GURCHENKOV, S. A.—KOPYTOV, V. M.: *On covers of variety of abelian lattice ordered groups*, Siberian Math. J. **28** (1987), 66–69. (Russian)

[5] HOLLAND, W. CH.—GLASS, A. M. W.—McCLEARY, S.: *The structure of l-group varieties*, Algebra Universalis **10** (1980), 1–20.

[6] PIERCE, K. R.: *Amalgamating abelian ordered groups*, Pacific. J. Math. **43** (1972), 711–723.

[7] PIERCE, K. R.: *Amalgamations of lattice ordered groups*, Trans. Amer. Math. Soc. **172** (1972), 249–260.

[8] POWELL, W. B.—TSINAKIS, C.: *Amalgamations of lattice ordered groups.* In: Ordered Algebraic Structures. (W. B. Powell, C. Tsinakis, eds.) Lecture Notes in Pure and Appl. Math. 99, Marcel Dekker, New York, 1985, pp. 171–178.

[9] POWELL, W. P.—TSINAKIS, C.: *Amalgamations of l-groups.* In: Lattice ordered groups. (A. M. W. Glass, W. Ch. Holland, eds.) Advances and Techniques, D. Reidel, Dordrecht, 1989, pp. 308–327.

[10] POWELL, W. B.—TSINAKIS, C.: *The failure of the amalgamation property for varieties of representable l-groups*, Math. Proc. Cambridge Philos. Soc. **106** (1989), 439–443.

[11] *Problem lists, Ordered Algebraic Structures*, Notices Amer. Math. Soc. **29** (1982), 327.

Received May 18, 1992
Revised July 20, 1995

Chair of Mathematics
Polytechnical Institute
Rubtsovsk
RUSSIA
E-mail: vtuz@rubltd.altai.su