Bernd Kirchheim

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A CHAOTIC FUNCTION WITH ZERO TOPOLOGICAL ENTROPY HAVING A NON-PERFECT ATTRACTOR

BERND KIRCHHEIM

ABSRACT. In the paper there is constructed a continuous mapping f from a compact real interval to itself satisfying the following: The mapping f has zero topological entropy and there is an infinite attractor containing isolated points which are not approximable by periodic points.

In the papers [4], [8] the existence of a continuous mapping f from the unit interval into itself with the following properties is remarked:

a) The function f is of type 2^{∞}, that means f has no cycle of order not a power of 2 and has a cycle of any of the orders 1, 2, 2^2, ...
b) There is a point x such that $w_f(x)$ is infinite and has isolated points. (Here $w_f(x)$ denotes the set of all limit points of the trajectory $\{f^n(x), n \geq 0\}$ generated by x.

Both examples in [4], [8] are not correct. In [3] a function having the described properties a, b) and satisfying moreover the condition that all isolated points of the infinite $w_f(x)$ belong to $\overline{\mathbf{Per}(f)} \setminus \mathbf{Per}(f)$ is constructed. (The existence of such an attractor is not explicitly remarked in [3] but follows from [6] where the following is shown: If $u \in \overline{\mathbf{Per}(f)} \setminus \mathbf{Per}(f)$, then there is some x such that $u \in w_f(x)$ and $w_f(x)$ is infinite.)

However, both examples in [4], [8] should be such that the isolated points of the infinite attractor do not belong to $\overline{\mathbf{Per}(f)}$. In the present paper a construction of such a function is given.

Any function of this type is chaotic in the sense of Li and York and has zero topological entropy. Indeed, from [7] it follows that f is chaotic if there is an infinite $w_f(x)$ and there are two distinct points $u, v \in w_f(x)$, which are non-separable by f-periodic intervals. Let f be a function of type 2^{∞} having an infinite $w_f(x)$ and let u be an isolated point of $w_f(x)$. By [5] there is some point $v \in \mathbf{Der}\, w_f(x)$ (where $\mathbf{Der}$ M is the set of all accumulation points of M) such that no point of $w_f(x)$lies between u and v. Then u and v have a common trajectory, see [5], and therefore are non-separable by f-periodic intervals.

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1. Construction

To avoid the notation of fractional numbers we will work on the interval $I = [0, 69]$. Our construction would also work with slightly different proportions between the used subintervals, however, these proportions cannot be chosen arbitrarily.

Let f be a continuous mapping from I into itself satisfying the conditions:

1(a) $f0) = 36$

1(b) $f(x) \leq 57$ if $x \leq 12$ and $f(x) \geq 12$ if $x \leq 57$

1(c) $f(x) = 69 - x$ for $x \in [57, 69]$.

The new function $\tilde{f}$ is defined by the following conditions:

2(a) $\tilde{f}(0) = 36$, $\tilde{f}(3) = 39$, $\tilde{f}(12) = 42$, $\tilde{f}(27) = 57$, $\tilde{f}(30) = 66$, $\tilde{f}(33) = 69$, $\tilde{f}(36) = 31$ and $\tilde{f}(38) = 20$.

2(b) $\tilde{f}$ is linear on each $[p_i, p_{i+1}]$, $i = 1, \ldots, 7$, where $p_1 < \ldots < p_8$ are the points occurring as arguments of $\tilde{f}$ in the eight statements of 2(a).

2(c) $\tilde{f}(x) = (s \circ f \circ t^{-1})(x)$ for $p_8 \leq x \leq p_9 = 61$, here s or t denote the affine transformation mapping I onto $[8, 31]$ or $[p_8, p_9]$, respectively.

2(d) $\tilde{f}(x) = 69 - x$ for $61 \leq x \leq 69$, see the figure.

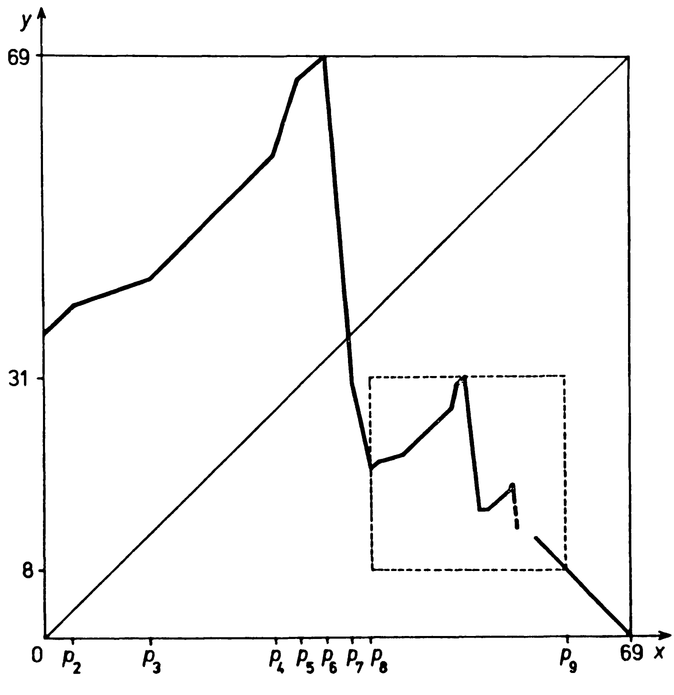

Fig. 1

It is not difficult to show that $\tilde{f}$ maps again I into itself and satisfies 1(a), (b), (c), clearly $\tilde{f}$ is also continuous.

2. Relations between f and $\tilde{f}$

Denote $J = [38, 57]$, then clearly $\tilde{f}([27, 31]) = [57, 67]$, $\tilde{f}([57, 67]) = [2, 12]$ and $\tilde{f}([2, 27]) = J$. If we set $a = 2$, $b = 31$ and $M = [a, b] \cup [57, 67]$, we obtain from the foregoing that for any $x \in M$ there is some $n \leq 3$ such that $\tilde{f}^n(x) \in J$. Furthermore $\tilde{f}([b, p_6]) = [67, 69]$, $\tilde{f}([67, 69]) = [0, 2]$, $\tilde{f}([0, 2]) = [p_7, p_8]$, and $\tilde{f}([p_7, p_8]) \subset M$. Consequently, if $x \notin [p_6, p_7]$, then there is some $n \leq 7$ such that $\tilde{f}^n(x) \in J$.

If $x \in J$, then one of the following cases occurs:

3(a) $f(t^{-1}(x)) \leq 57$ and then $\tilde{f}(x) \in [p_3, p_4]$ and $(t \circ f \circ t^{-1})(x) = \tilde{f}_2(x) \in J$.

3(b) $f(t^{-1}(x)) > 57$ and then $\tilde{f}(x), \tilde{f}^2(x), \tilde{f}^3(x) \notin J$
and $(t \circ f^2 \circ t^{-1})(x) = \tilde{f}^4(x) \in J$.

This can be easily computed using $s(x) = \dfrac{x}{3} + 8$, $t(x) = \dfrac{x}{3} + 38$.

Lemma. *If* x, $f^n(x) \leq 57$, *then* $t(x)$, $t(f^n(x)) \in J$ *and* $\tilde{f}^{2n}(t(x)) = t(f^n(x))$. *Conversely, if* x, $\tilde{f}^n(x) \in J$, *then there exist* $t^{-1}(x)$, $t^{-1}(\tilde{f}^n(x)) \leq 57$ *and* $f^{(n/2)}(t^{-1}(x)) = t^{-1}(\tilde{f}^n(x))$. *In particular,* $\tilde{f}$ *has a cycle of order* $n > 1$ *if and only if* f *has a cycle of order* $n/2$.

P r o o f: The first two statements immediately follow from 3(a), (b) by induction. To prove the third statement it suffices to show that each $\tilde{f}$ — cycle of order greater than one has a point in J and that each f-cycle contains a point smaller than or equal to 57. The second assertion is by 1(c) trivial and our foregoing considerations show that if there were an $\tilde{f}$ — cycle of order $n > 1$ having no point in J, then this whole cycle would be contained in $[p_6, p_7]$. But this contradicts the fact that $\tilde{f}$ is on $[p_6, p_7]$ a linear map with derivation smaller than -1. □

3. Existence and properties of the required function

We define the mapping f_0 by

$$f_0(x) = \begin{cases} 36 & \text{for } x \in [0, 33] \\ 69 - x & \text{for } x \in [33, 69] \end{cases}$$

Then f_0 satisfies the conditions 1, moreover $f_0^2 = f_0^4$ and therefore f_0 has cycles exactly of order one and two. Next we define by $f_{n+1} = \tilde{f}_n$ (in the sense of Construction 1) a sequence of continuous functions satisfying the conditions 1.

It follows by induction that for $n \geq 1$, $m > n$ and any $x \notin t^{n-1}([p_8, 57])$ we have $f_n(x) = f_m(x)$. Hence, we conclude $\|f_n - f_m\|_\infty \leq \lambda(s^{n-1}([0, 69])) = 69/3^{n-1}$, where λ denotes the Lebesgue measure. Now we put $f = \lim_{n \to \infty} f_n$. Since the f_n converge uniformly to f, f is a continuous map from I into itself and (by [1]) of type 2^∞. Indeed, for each $n \geq 1$ $t^n(p_F)$ belongs to an f— cycle of order 2^n, here p_F denotes the (common) unique fixed point of the f_n's, $n \geq 1$, which is contained in $[p_6, p_7]$. On the other hand any map f_n has no cycle of order greater than 2^n. Clearly f satisfies also the conditions 1. Because $f = \tilde{f}$, from the foregoing Lemma the following important properties of the map f follow:

4(a) If $x, f^n(x) \leq 57$, then $t(x)$, $t(f^n(x)) \leq 57$ and $f^{2n}(t(x)) = t(f^n(x))$.

4(b) If $x, f^n(x) \in J$, then $t^{-1}(f^n(x)) = f^{n/2}(t^{-1}(x))$

5. If $x, y \in I$, then $y = f(x)$ iff $s(y) = f(t(x))$.

Theorem 1. *There is a continuous mapping f from the interval I into itself of type 2^∞ and a point $x \in I$ such that the attractor $w_f(x)$ is an infinite set having isolated points. These isolated points do not belong to $\overline{\mathbf{Per}(f)}$, where $\mathbf{Per}(f)$ denotes the set of all periodic points of the function f.*

Proof. We choose f to be the function constructed above and x to be the point $b = 31$. Further denote $c = p_2$, $d = p_5$, $a_n = t^n(a)$ and similarly for b_n, c_n, d_n. Because $f(x) = f_2(x) = d + (x - d_1)$ for $x \in [d_1, t(p_6)]$, we obtain by induction from 4(a) the following:

6. $f^{2^n-1}(x) = d_{n-1} + (x - d_n)$ for $n \geq 1$ and $d_n \leq x \leq t^n(p_6)$. Similarly from $f(x) = f_1(x) = c_1 + (x - c)$ for $p_1 \leq x \leq c$.

7. $f^{2^n}(x) = c_{n-1} + (x - c_n)$ for $n \geq 0$ and $t^n(p_1) \leq x \leq c_n$ follows. Because $f^7(b) = f^5(a) = f^4(t(p_1)) = t(f^2(p_1)) = t(b) = b_1$, from the property 4(a) we get bys induction that

8. $b_{n-1} = f^{7 \cdot 2^n}(b_n)$ for $n \geq 0$.

Denote $S = \bigcup_{k=0}^{\infty} t^k([p_3, p_4])$, $\tilde{S} = \bigcup_{k=0}^{\infty} t^k([p_1, p_2] \cup [p_3, p_4] \cup [d, b] \cup \{p_7\}) \cup [57, 69]$ and $T = [p_1, c] \cup \{p_2\} \cup [d_1, t(p_6)]$. We obtain

9. $\bigcup_{k=1}^{2^n} f^k(t^n(T)) \subset S$ for $n \geq 1$.

Indeed, in case $n = 1$ the statement 9. is evident and now assume that it holds for some $n \geq 1$. For each point $x \in t^{n-1}(T)$ there is some $y \in t^n(T)$ satisfying $x = t(y)$. If $1 \leq k < 2^{n-1}$, then either $k = 2m$ for some integer m and in this case $f^k(x) = f^{2m}(t(y)) = t(f^m(y)) \in t(S) \subset S$ by assumption, or $k = 2.m + 1$ and then $f^k(x) = f(f^{2m}(x)) \in f(t(S) \cup t^{n-1}(T)) \subseteq [p_3, p_4]$; remark that $n + 1 \geq 2$. Furthermore, from $f(p_2) = b$ and $f(p_1) = p_7$ it follows by 4(a) that

10. $f^{2^n}(t^n(p_1)) = t^n(p_2)$, $f^{2^n}(t^n(p_7)) = t^n(b) = b_n$ if $n \geq 0$.

Finally, the statements 6., 7., 9., and 10. imply

11. $f^k(b) \subset \tilde{S}$, hence $f^k(b) \notin \bigcup_{i=0}^{\infty} t^i((c, p_3) \cup (p_4, d))$ for $k \geq 0$.

Note that $f^{2^{n+1}}(b_n) = a_n$ and $f^{2^n}(a) = t^{n+1}(p_1)$.

Iterating the properties 6., 8. and using the fact that $\lim_{n \to \infty} (b_n - d_n) = 0$ we get immediately $\{d_n, n \geq 0\} \subset w_f(b)$. Hence $w_f(b)$ contains also the points $f(d) = 66$, $f^2(d) = c$, and by statement 7. also all c_n, $n \geq 1$. Now take some $n \geq 1$, let $f^k(b) \in [t^n(p_1), c_n]$ and let i be the largest integer not greater than k with $f^i(b) = b_m$, $m \geq 0$. Then the statements 7., 9., and the equality $f^2(x) = c - (x - d)$ if $d \leq x \leq b$ imply $c_n - f^k(b) = b_m - d_m = (b - d)/3^m$. For $l = \min\{j > k, f^j(b) \in [t^n(p_1), c_n]\}$ we get similarly $c_n - f^l(b) = (c_n - f^k(b))/3$. By statement 11. c_n is therefore an isolated point of $w_f(b)$. Now denote $\mathbf{Orb} = \{d_n\}_{n=0}^{\infty} \cup \{66\} \cup \cup \{c_n\}_{n=0}^{\infty}$, remark that according to 6. and 7. all these points belong to one orbit of f. Since in the foregoing consideration n was chosen arbitrarily and since $f(w_f(b)) = w_f(b)$, it follows that each point of **Orb** is an isolated point of $w_f(b)$.

To finish the proof we show at first that $\mathbf{Per}(f) \subset [p_3, p_4] \cup \{p_F\} \cup [42, 57]$. Assume that k is the smallest positive integer such that there is an f-cycle $x_1, \ldots, x_{2^k}$ with $x_1 \in (p_4, p_6)$. From the proof of the Lemma we already know that there exists the largest i not greater than 2^k such that $x_i \in J = [38, 57]$. Now the statements 3a), b) imply that $f(t^{-1}(x_i)) > 57$, $f^2(x_i) = x_1$ and that $t^{-1}(x_i)$, $f(t^{-1}(x_i))$, and $t^{-1}(x_3)$ belong to an f-cycle of order 2^{k-1} which is not contained in $[p_3, p_4] \cup J$ again. But this contradicts the choice of k and implies that $\mathbf{Per}(f) \subset [p_3, p_4] \cup \{p_F\} \cup J$. The properties 4. yield now $[42, 57] \supset t(\mathbf{Per}(f)) = = \mathbf{Per}(f) \cap J$. This means $\mathbf{Per}(f) \subset \overline{\mathbf{Per}(f)} \subset [p_3, p_4] \cup \{p_F\} \cup [42, 57]$. Furthermore we obtain $t(\overline{\mathbf{Per}(f)}) = \overline{\mathbf{Per}(f)} \cap [42, 57]$ and since in general $f(\overline{\mathbf{Per}(f)}) = \overline{\mathbf{Per}(f)}$, from $c_1 \notin \overline{\mathbf{Per}(f)}$ we directly get the required statement $\mathbf{Orb} \cap \overline{\mathbf{Per}(f)} = \emptyset$.

Theorem 2. *For the attractor $w_f(b)$ considered in the proof of Theorem* 1 *more precisely, the following holds. Let $w_f(b) = P \cup D$ be the* Cantor—Bendixson *decomposition of $w_f(b)$, then the perfect kernel P is an affine image of the familiar* "middle third Cantor set" *satisfying* $\min P = 12$ *and* $\max P = 57$. *Moreover D is the set of all isolated points of $w_f(b)$ and forms a* "two-directional" *trajectory of f, that means $f(D) = D$ and for any $x, y \in D$ there is some $m \geq 0$ such that $f^m(x) = y$ or $f^m(y) = x$.*

Proof. First we show that $t(w_f(b) \cap [0, 57]) = w_f(b) \cap [38, 57]$ and $s(w_f(b) \cap [12, 57]) = w_f(b) \cap [12, 27]$. Hence, let $x \in w_f(b) \cap [0, 57]$. If $x = 57$, then $t(x) = x$ and in case of $x < 57$ the conclusion $t(x) \in w_f(b)$ follows immediately from 4a) and the fact that $t(b) = b_1 = f^7(b)$. Conversely, if $y \in w_f(b) \cap [38, 57]$, then 4a), and 11. give $t^{-1}(x) \in w_f(b)$. Next, if $y \in [12, 57] \cap \cap w_f(b)$, then clearly some $x \in w_f(b) \cap f^{-1}(y)$ exists, hence $x \leq 57$. From 5. we now obtain $s(y) = f(t(x)) \in w_f(b)$, as shown before. Finally, assume $y \in w_f(b) \cap$

$\cap [12, 27]$. Then $f(y) = y + 30 \in w_f(b) \cap [38, 57]$ and consequently $f(y) = t(x)$ for some $x \in w_f(b) \cap [0, 57]$. This implies $s(x) = t(x) - 30 = y$ and proves the last inclusion.

The proof of Theorem 1 moreover shows that $w_f(b) \cap [0, 12) = \{c\}$, $w_f(b) \cap (27, 33] = \{d\}$, $w_f(b) \cap [38, 42) = \{c_1\}$, and $w_f(b) \cap (57, 69] = \{66\}$. Now according to 11. $P \subset [12, 27] \cup [42, 57]$ and $P = s(P) \cup t(P)$. Under these assumptions the theory of selfsimilar sets, see [2], yields that P must be an affine image of the standard Cantor set with position as described in the assertion. Note that s, t satisfy the "*open set condition*". Recall that P was chosen to be the set of all *condensation* points of $w_f(b)$, but it is easy to see that also for the set $\tilde{P}$ of all nonisolated points of $w_f(b)$ $\tilde{P} = s(\tilde{P}) \cup t(\tilde{P})$ holds, hence we get $P = \tilde{P}$. We finish the proof by showing that $D = \tilde{D}$ with $\tilde{D} = \{f^i(d_{k+1}); k \geq 0, 0 \leq i \leq 2^k\} \cup \{d\} \cup \{66\} \cup \{f^i(c_k); k \geq 0, 0 \leq i \leq 2^k\}$. Using the properties 4., 5. one can conclude that $t(\tilde{D} \cap [12, 57]) \subset \tilde{D}$ and $s(\tilde{D} \cap [12, 57]) \subset \tilde{D}$. Clearly each inner component interval of $I \setminus P$ is an image of (27, 42) under some finite composition of the mappings s and t. Since (27, 42) contains two points of D, each inner component interval $\tilde{I}$ does too and therefore, by Theorem 1 in [5] $\tilde{I} \cap D = \tilde{I} \cap \tilde{D}$. The required identity $D = \tilde{D}$ now follows from the simple remark that $D \setminus [12, 57] = \tilde{D} \setminus [12, 57] = \{3, 66\}$.

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REFERENCES

[1] BLOCK, L.: Stability of periodic orbits in the theorem of Šarkovskii. Proc. Amer. Math. Soc. 82, 1981, 333—336.

[2] FALCONER, K. J.: Geometry of Fractal Sets. 1st ed. Cambridge University Press 1984.

[3] HSIN CHU-XIONG JINGCHENG: A counterexample in dýáynamical systems of [0, 1]. Proc. Amer. Math. Soc. 97, 1986, 361—366.

[4] KENŽEGULOV, CH. K.—ŠARKOVSKII, A. N.: On properties of the set of limit points of an iterated sequence of continuous functions (Russian). Volžsk. Mat. Sb. 3, 1965, 343 –348.

[5] ŠARKOVSKII, A. N.: Attracting sets containing no cycles (Russian). Ukrain. Mat. Žurn. 20, 1968, 136—142.

[6] ŠARKOVSKII, A. N.: On a theorem of G. D. Birkhoff (Russian). Dopov. Akad. Nauk USSR, 1967, No. 5, 429—432.

[7] SMÍTAL, J.: Chaotic functions with zero topological entropy. Trans. Amer. Math. Soc. 297, 1986, 269—282.

[8] VEREJKINA, M. B.—ŠARKOVSKII, A. N.: Recurrence in one-dimensional dynamical systems, in Approx. and Qualitative Methods of the Theory of Differential & Functional Equations (Russian). Instit. Math. AN USSR, Kiev 1983, pp. 35—46.

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Ústav aplikovanej matematiky
a výpočtovej techniky UK
Mlynská dolina
842 15 Bratislava