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ON COVERING SYSTEMS ON RINGS

ŠTEFAN PORUBSKÝ

In this paper we shall consider covering systems of residue classes on rings. Throughout this note it will be tacitly assumed that all rings under consideration are commutative with an identity.

Let $A_1, A_2, \ldots, A_k$ be proper (i.e. $(0)\neq A_i\neq R$) ideals in a ring R. A system of residue classes

$$a_i + A_i, \quad a_i \in R, \quad i = 1, \ldots, k \tag{1}$$

is called *covering* (on R) if the set theoretic union of these classes is R. Covering system (1) is called *exactly covering* if its classes are pairwise disjoint. Finally, covering system (1) is *irredundant* (or minimal) if no proper subsystem of its classes forms a covering system (on the original ring).

Although we shall examine only covering system on rings here, the notion of the covering system can be extended to a more general situation. Namely, considering only the additive group of a ring we obtain the following definition of a covering system on a group. If $G_1, G_2, \ldots, G_k$ are subgroups of a given group G (written multiplicatively), then the system

$$a_iG_i, \quad a_i \in G, \quad i = 1, \ldots, k \tag{2}$$

is called covering if

$$G = \bigcup_{i=1}^{k} a_iG_i,$$

and similarly exactly covering and irredundant. Some properties of covering systems of cosets on groups have been already considered in papers [3], [5], and [6]. H. B. Neumann proved these results (for (*i*) and (*ii*) see [6], for (*iii*) see [5], resp.):

Lemma. *Let* (2) *be a covering system on a group* G. *Then*

(*i*) *The index of (at least) one of the subgroups* $G_1, \ldots, G_k$ *does not exceed* k;

(*ii*) *Let* $G_1, \ldots, G_m$ *have infinite index. Then*

$$G = \bigcup_{i=m+1}^{k} a_i G_i ,$$

and consequently, if (2) *is irredundant, all the* G_i *have finite indices;*

(*iii*) *If* (2) *is irredundant, then the index*

$$\left[G\colon \bigcap_{i=1}^{k} G_i\right] \leqq c_1^{2^{k-1}}$$

with $c_1 = 1.3419$, *approximately, and for* $k \geqq 6$ *with* $c_1 = 1.3070$, *approximately.*

In this note as in [3], [5], and [6] we shall consider only covering systems consisting of a finite number of classes. However, H. B. Neumann notices in [6] (without giving any concrete example) that (*i*) becomes false if an infinite cardinal is substituted for k. If this is true, then the following extension of the second part of (*ii*) can be disproved under the assumption of the generalized continuum hypothesis: *If the cardinality of the set of cosets in an irredundant covering system is less than* $\aleph_s$, *then* $[G\colon G_i] < \aleph_s$ *for all* G_i. But perhaps it can be of some interest to find such a function $f(s)$ defined for ordinal numbers s for which:

If the cardinality of the set of all cosets in an irredundant covering system is less than $\aleph_t$ *for* $t \geqq s$, *then* $[G\colon G_i] < \aleph_k$ *for* $k \geqq f(s)$ *and all* G_i.

Our Lemma implies immediately $f(0) = 0$.

Another complex of questions arises with the estimation of the least upper bound Δ_k of the index $\left[G\colon \bigcap_{i=1}^{k} G_i\right]$, where G ranges over the groups that can be covered by an irredundant covering system consisting of k cosets. The present author proved in [8] that if G is restricted to the group of integers, then

$$\max \left[G\colon \bigcap_{i=1}^{k} G_i\right] = 2^{k-1},$$

where the maximum is taken over irredundant covering systems (on the set of integers) consisting of k cosets, moreover this bound is attained. It seems that Neumann's result (*iii*) is sharp only for small values of k (cf. the bottom lines of p. 237 in [5]). Maybe it can be sharpened for large values of k in the following way:

$$\Delta_k \leqq k^{k-1}.$$

In what follows we shall investigate only covering systems on rings, and we leave for the reader the obvious rephrasing of some subsequent results into a group-theoretic language.

For every $i = 1, \ldots, k$ fix such a system of the representatives of the residue-class ring R/A_i which contains the element a_i. Consider the ring structure on $R/A_1 \times R/A_2 \times \ldots \times R/A_k$ defined in the usual manner, that is with the addition and multiplication being defined coordinatewise. Further, let

$$h: R \to R/A_1 \times R/A_2 \times \ldots \times R/A_k$$

denote the mapping on R which associates x with the k-tuple of the representatives of those classes to which x belongs modulo corresponding A_i.

In the sequel $N_{i_1 \ldots i_r}$ will denote the intersection of all the ideals of (1) except for $A_{i_1}, \ldots, A_{i_r}$.

Theorem 1. *Let* (1) *be a covering system on a ring R and let* $t \in R$. *Let* $A_{i_1}, \ldots, A_{i_{m(t)}}$ *be the ideals of all the classes in* (1) *with* $t \in a_i + A_i$. *Then*

$$\bigcup_{j=1}^{m(t)} A_{i_j} \supseteq N_{i_1 \ldots i_{m(t)}}.$$

Proof. For the sake of simplicity suppose that $i_1 = 1, i_2 = 2, \ldots, i_{m(t)} = m(t)$. Let x be an element from $N_{1,2,\ldots,m(t)}$. Then the last $k - m(t)$ coordinates of $h(x)$ are certainly zeros. On the other hand, $h(t) = (a_1, a_2, \ldots, a_{m(t)}, z, \ldots, z)$, where "$z$" indicates that the entry is not a_i. Since $h(x) + h(t) = h(x+t)$ and since for every y at least one coordinate of $h(y)$ is a_i, then $x + t \in a_i + A_i$ for at least one $i = 1, \ldots, m(t)$. But this implies that x belongs to A_i for this i, and the theorem is proved.

Corollary 1. *In every irredundant covering system* (1) *we have* $A_i \supseteq N_i$ *for each* $i = 1, \ldots, k$.

The corollary does not hold for covering systems which are not irredundant, e.g. let R be the ring of rational integers and consider the covering system

$$0+(2), \quad 0+(3), \quad 1+(4), \quad 5+(6), \quad 3+(7), \quad 7+(12).$$

Corollary 2. *In an irredundant covering system* (1) *for every i there exists* $j \neq i$, j, $i = 1, \ldots, k$ *with*

$$A_i + A_j \neq R.$$

If A_i were comaximal with A_j for each $j \neq i$ and $j = 1, \ldots, k$, then A_i were also comaximal with N_i, which is impossible according to the previous corollary.

The Chinese remainder theorem shows that in exactly covering systems the statement of Corollary 2 holds even for every pair i, j of indices, but this evidently cannot hold for irredundant covering systems in general. The above stated example shows that Corollary 2 also fails in the case of covering systems which are not irredundant.

Corollary 3. *Let* (1) *be an irredundant covering system. If for some* A_i *there is unique* A_j *with* $j \neq i$ *and* $A_i + A_j \neq R$, *then* $A_i \supseteq A_j$.

We have $A_i \supseteq N_i \supseteq N_{i,j} \cdot A_j$. Since A_i is comaximal with A_s for every $i \neq s \neq j$, $s = 1, \ldots, k$, then A_i is comaximal with $N_{i,j}$, that is we have

$$A_i \supseteq N_{i,j} \cdot A_j \quad \text{and} \quad A_i + N_{i,j} = R$$

and therefore $A_i \supseteq A_j$ ([4], §2.8).

The last corollary is motivated by Schinzel's conjecture for covering systems [9] and confirms this conjecture under the assumptions of the corollary for covering systems on rings.

Theorem 2. *Let* (1) *be an irredundant covering system on a ring R, Let a P-primary ideal Q contain — upon relabelling if necessary — the ideals A_1, A_2, ..., A_s. Let the isolated P-primary component Q' of $\bigcap_{i=s+1}^{k} A_i$ be a proper overideal of Q. Then*

$$s \geqq 1 + \sum_{i=1}^{v-1} [|Q_i/Q_{i+1}| - 1],$$

where $Q' = Q_1 \supset Q_2 \supset \ldots \supset Q_v = Q$ is a maximal chain of distinct P-primary ideals between Q' and Q.

Proof. According to the Lemma there is only a finite number $P_1, P_2, \ldots, P_t$ of prime ideals which are over at least one of the ideals $A_1, \ldots, A_k$. These prime ideals are also of finite index and therefore they are maximal in R and simultaneously they are minimal prime overideals of A_i's. Then [2; 3.10]

$$\bigcap_{i=s+1}^{k} A_i = Q' \cap M,$$

where M is the intersection of the isolated P_i-primary components of $\bigcap_{i=s+1}^{k} A_i$ for $P_i \neq P$ and moreover Q' and M are comaximal.

Now, let w range over all the representatives of the non-zero classes in the residue-class rings

$$(Q_i \cap M)/(Q_{i+1} \cap M) \quad \text{for} \quad i = 1, 2, \ldots, v-1.$$

These w's are obviously distinct. For each such w there is an element x_w in R for which

$$h(x_w) = (\underbrace{w, \ldots, w}_{s \text{ times}}, \quad 0, \ldots, 0),$$

or more precisely, the ith coordinate of which is congruent to w modulo A_i for $i = 1, \ldots, s$. If $w \neq w'$, then $w - w' \notin A_i$ and consequently also $w_w - x_{w'} \notin A_i$ for every $i = 1, \ldots, s$ (and so x_w's are distinct). If $w - w'$ belongs to A_i, then $w - w'$ belongs to Q and $Q \cap M$, which is in contradiction with the fact that w's are representatives of non-zero classes in $(Q_i \cap M)/(Q_{i+1} \cap M)$.

Suppose that a_1 is chosen in such a way that a_1 belongs only to the class $a_1 + A_1$. Now it is easy to show that for each w element $a_1 + x_w$ can belong only to some of the first s classes in (1). Let $h(a_1) = (a_1, x_2, \ldots, x_k)$, where x_i is not a_i for $i = 2, \ldots, k$.

Since no x_w belongs to A_1, for every w at least one of the following relations holds

$$x_2+w\in a_2+A_2, \ldots, x_s+w\in a_s+A_s .$$

If

$$s-1\leqq\sum_{i=1}^{v-1}[|(Q_i\cap M)/(Q_{i+1}\cap M)|-1]$$

it follows by the pigeon-hole principle that there is one $i=2, 3, \ldots, s$ with

$$x_i+w\in a_i+A_i \quad \text{and} \quad x_i+w'\in a_i+A_i$$

but this is in contradiction with $w-w'\notin A_i$.

To finish the proof it remains to show that

$$|Q_i/Q_{i+1}|=|(Q_i\cap M)/(Q_{i+1}\cap M)| .$$

But this follows from the fact that the rings on the left and right hand side of this equality are isomorphic (Lemma 1.5 [7]). The Jordan—Hölder theorem implies that the statement of our theorem does not depend on the choice of the chain $Q_1, \ldots, Q_v$.

Corollary 1. *If a prime ideal P contains one ideal of an irredundant covering system on a ring R, then P contains at least $|R/P|$ ideals in* (1).

Corollary 2. *Let* (1) *be an irredundant covering system on R and let the P-primary ideals of R form a chain under the set inclusion. If a P-primary ideal $Q\neq P$ contains one ideal of* (1), *the Q contains at least $|Q'/Q|$ ideals of* (1), *where Q' is the minimal P-primary overideal of Q.*

There is a conspicuous similarity among the proof technique and result of Theorem 2 and the proof techniques and results from [3] and [7]. It would be of considerable interest to find the real inner connections between the ideas of these proofs.

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Matematický ústav SAV
Obrancov mieru 49
886 25 Bratislava

ПОКРЫВАЮЩИЕ СИСТЕМЫ НА КОЛЬЦАХ

Штефан Порубски

Резюме

Пусть R кольцо и $A_1, \ldots, A_k$ – идеалы этого кольца. Система

$$a_i + A_i, \quad a_i \in R, \quad i = 1, \ldots, k$$

называется покрывающей, если каждый элемент кольца R находится в одном из этих классов. В работе даются обобщения и улучшения некоторых результатов известных в случае кольца целых чисел.