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REPRESENTATIONS OF PALINDROMICALLY PRESENTED GROUPS ONTO FINITE SUBGROUPS OF $SO_3(\mathbb{R})$

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ABSTRACT. The purpose of this paper is to investigate representations of palindromically presented groups onto finite subgroups of $SO_3(\mathbb{R})$.

Let G be a group. Assume that G is generated by two elements which are conjugate in G. The equivalence classes of homomorphisms of G onto the alternating group A_5 are classified by R. Riley (see [10]). However, the method of finding such a homomorphism is just trial and error.

The aim of this note is to use the $SO_3(\mathbb{R})$-representation polynomial, introduced in [2], in order to give a useful algorithm to prove the existence or non-existence of homomorphisms of palindromically presented groups onto finite subgroups of $SO_3(\mathbb{R})$.

1. $SO_3(\mathbb{R})$-representations

Given two groups G and H, a representation of G into H is a homomorphism $\varrho\colon G \to H$. Two representations ϱ and ϱ' are equivalent (conjugate) if and only if they differ by an (inner) automorphism of H. Given a knot group G, we call a representation $\varrho\colon G \to H$ abelian if its image is abelian; then it must be cyclic. For a finite H, we define the order of a homomorphism $\varrho\colon G \to H$ to be the order of the element $\varrho(m) \in H$, where $m \in G$ is represented by a meridian.

We are interested in studying $SO_3(\mathbb{R})$-representations of palindromically presented groups. A group $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$ is called palindromically presented if there is a presentation

$$G = \langle S, T \mid L_S S = T L_S \rangle$$

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such that

$$L_S = S^{\varepsilon_1} T^{\varepsilon_2} \dots S^{\varepsilon_{\alpha-2}} T^{\varepsilon_{\alpha-1}} ,$$
$$\alpha \equiv 1 \bmod 2 , \quad \varepsilon_i \in \{\pm 1\} , \quad \varepsilon_i = \varepsilon_{\alpha-i} , \quad 1 \le i \le \alpha - 1 .$$

1.1. Remark. If there is an odd integer β, $-\alpha < \beta < \alpha$, relatively prime to α, such that $\varepsilon_i = (-1)^{\left[i\frac{\beta}{\alpha}\right]}$, then $G(\varepsilon_1, \dots, \varepsilon_{\alpha-1}) =: G(\alpha, \beta)$ is the group of the two-bridge knot $\mathfrak{b}(\alpha, \beta)$. (For every real x, $[x]$ is the greatest integer n such that $n \le x$.)

1.2. Remark. Knots and links with two bridges are classified by H. Schubert (see [11]).

1.3. THEOREM. (Schubert) *$\mathfrak{b}(\alpha, \beta)$ and $\mathfrak{b}(\alpha', \beta')$ are equivalent as oriented knots (or links) if and only if*

$$\alpha = \alpha' \qquad \text{and} \qquad \beta^{\pm 1} \equiv \beta' \bmod 2\alpha .$$

For more information about 2-bridge knots see also [3; Chapter 12].

The three-dimensional sphere S^3 can be viewed as the set of the unit quaternions

$$(P, \varphi) = \cos\Big(\frac{1}{2}\varphi\Big) + \sin\Big(\frac{1}{2}\varphi\Big) P ,$$

where $0 \le \varphi \le 2\pi$ and P is a pure unit quaternion satisfying $P^2 = -1$; note that the set of such P can naturally be identified with the two-sphere S^2. There is a 2-fold covering $\delta \colon S^3 \to \mathrm{SO}_3(\mathbb{R}) = \mathbb{R}\mathrm{P}^3$, $(P, \varphi) \mapsto \delta(P, \varphi)$, which is a group homomorphism with $\mathrm{Kernel}(\delta) = \{\pm 1\}$. $\delta(P, \varphi)$ is a rotation through φ about the axis P.

1.4. Remark. It is usual to identify the unit quaternions with the group $\mathrm{SU}_2(\mathbb{C})$ of special unitary matrices, $\mathrm{SU}_2(\mathbb{C}) = \left\{ \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix} \right) \,\middle|\, a\bar{a} + b\bar{b} = 1 \right\}$.

Let $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$ be a palindromically presented group and $\varrho \colon G \to \mathrm{SO}_3(\mathbb{R})$ a non-abelian representation, i.e.

$$\varrho : \begin{cases} S \mapsto \delta(P, \varphi) \\ T \mapsto \delta(Q, \varphi) \end{cases} \qquad (P \neq Q).$$

The equivalence class of ϱ is determined by the parameters $\tau = \langle P, Q \rangle = \cos\psi$, $\psi = \sphericalangle(P, Q)$, and $y = \cot^2 \dfrac{\varphi}{2}$. Here $\langle P, Q \rangle$ denotes the scalar product in $\mathbb{R}^3$ (for details see [2]). There is a restriction of the parameters, and we denote by $D \subseteq \mathbb{R}^2$ the domain

$$D = \big\{ (\tau, y) \mid y \ge 0 , \ -1 < \tau < 1 \big\} .$$

In order to analyse the representations of a palindromically presented group $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$, we consider the sequence $(\mu_1, \dots, \mu_n)$, $n = \dfrac{\alpha - 1}{2}$, $\mu_i = \varepsilon_{n-i+1} \cdot \varepsilon_{n-i+2}$, $1 \le i \le n$.

1.5. THEOREM. *Given a palindromically presented group* $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$*, there exists a polynomial* $z_n(\tau, y) \in \mathbb{Z}[\tau, y]$*,* $\deg z_n = n$*, which only depends on the sequence* $(\mu_1, \dots, \mu_n)$ *such that a pair* $(\tau_0, y_0) \in D$ *determines an equivalence class of* $\mathrm{SO}_3(\mathbb{R})$*-representations if and only if* $z_n(\tau_0, y_0) = 0$*.*

Proof. See [2]. □

The polynomial $z_n(\tau, y)$ which only depends on the sequence (μ_i) is called the $\mathrm{SO}_3(\mathbb{R})$-representation polynomial. There is an inductive procedure to calculate this polynomial.

1.6. LEMMA. *For* $n \geq 3$ *we have:*

$$z_n(\tau, y) = \big(\mu_n(y-1)-2\tau\big)\big(z_{n-1}(\tau, y)+\mu_n(y+1)z_{n-2}(\tau, y)\big)-(y+1)^3\mu_n z_{n-3}(\tau, y)$$

starting with

$$z_0(\tau, y) = 1\,, \quad z_1(\tau, y) = y - 2\tau - 1\,,$$

and

$$z_2(\tau, y) = y^2 + y\big(1 - 2\mu_2 + (1 + 2\mu_2)(-2\tau - 1)\big) + 4\tau^2 + 2\tau - 1\,.$$

Proof. See [2]. □

The finite subgroups of $\mathrm{SO}_3(\mathbb{R})$ are well known (see [4] or [8]); we summarize these groups in the following table:

group	order of the group	symbol
cyclic	$m,\ m \geq 1$	$\mathbb{Z}_m$
dihedral	$2m,\ m \geq 1$	D_m
tetrahedral	12	Θ
octahedral	24	Ω
icosahedral	60	$\mathfrak{I}$

Non-abelian conjugacy classes of representations of knot groups onto the dihedral group are studied by E. Klassen (see [7]), the equivalence classes can be found in [3; 14.8]. As a corollary, we obtain:

1.7. COROLLARY. *Let* $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$ *be a palindromically presented group. There are exactly* $n = \frac{\alpha - 1}{2}$ *conjugacy classes of non-abelian representations* $G \to D_\alpha$*, but only one equivalence class of homomorphisms* $G \to D_\alpha$*. In fact, every non-abelian representation* $G \to \mathrm{SO}_3(\mathbb{R})$ *of order 2 is a dihedral representation.*

2. Tetrahedral representations

Each orientation preserving symmetry of the tetrahedron is given by an even permutation of its vertices. Therefore we have

$$\Theta \cong \mathrm{A}_4 \cong \mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2).$$

It follows immediately that Θ is metabelian. But every non-abelian homomorphism of a knot group onto a finite metabelian group factorizes through $\beta_n \colon G \to \mathbb{Z}_n \ltimes H_1(\hat{C}_n)$ (n is the order of the homomorphism, and the n-fold cyclic branched covering of the knot is denoted by $\hat{C}_n$, see [3; 14.3]). Using this fact, it is possible to derive a criterion for the existence of tetrahedral representations in terms of the Alexander polynomial $\Delta(t)$, which was done by Hartley.

2.1. THEOREM. (Hartley) *Let G be a knot group. There exists a surjective representation $G \twoheadrightarrow \mathrm{A}_4$ if and only if $2 \mid \Delta(\omega)\Delta(\omega^2)$, where ω is a primitive 3rd root of the unity.*

Proof. See [5; Example 1.13]. □

2.2. Remark. Every non-surjective homomorphism $G \to A_4$ is abelian. On the other hand, there is a surjective homomorphism $G \to A_4$ if and only if $\Delta(\omega) \equiv 0 \bmod 2$.

In fact, if $G = G(\alpha, \beta)$ is a 2-bridge knot group, we have:

2.3. THEOREM. (Murasugi) *Let $\Delta(t)$ be the Alexander polynomial of a 2-bridge knot $\mathfrak{b}(\alpha, \beta)$. Then there exists a number $r \in \mathbb{N}$ such that*

$$1 + t + \cdots + t^{r-1} + t^r \equiv \Delta(t) \bmod 2.$$

Proof. See [9] or [1]. □

2.4. Remark. $\lambda := r + 1$ is the linking number of the knot $\mathfrak{b}(\alpha, \beta)$ with the axis of the period two symmetry (see [9]).

2.5. COROLLARY. *Let $G = G(\alpha, \beta)$ be a 2-bridge knot group. There exists a surjective homomorphism $G \twoheadrightarrow \mathrm{A}_4$ if and only if $\lambda \equiv 0 \bmod 3$.*

Proof. By Theorem 2.3, we have $\Delta(\omega) \equiv 1 + \omega + \cdots + \omega^{r-1} + \omega^r \bmod 2$, and therefore $\Delta(\omega) \equiv 0 \bmod 2$ if and only if $r \equiv 2 \bmod 3 \iff \lambda \equiv 0 \bmod 3$. □

Using the σ-normal form of the knot $\mathfrak{b}(\alpha, \beta)$ (see [1]), it is possible to calculate r. For that reason, let $\mathfrak{b}(\alpha, \beta) = [a_1, b_1, \dots, b_k, a_{k+1}]$ be a σ-normal form of $\mathfrak{b}(\alpha, \beta)$. By $(b_{j_i})_{0 \le i \le l}$ we denote the odd coefficients b_ν in the σ-normal form.

$$A(j_i, j_{i+1}) := \sum_{j_i < j \le j_{i+1}} a_j, \qquad \text{where } j_0 := 0 \quad \text{and} \quad j_{l+1} := k + 1.$$

We obtain: $\lambda = r + 1 = \left| \sum_{i=0}^{l} (-1)^i A(j_i, j_{i+1}) \right|$ (see [1]).

2.6. Example. The σ-normal form of $\mathfrak{b}(19,5)$ is given by $[3,1,-1,-1,-1]$:

$$A(j_0,j_1)=3\,,\quad A(j_1,j_2)=-1\,,\quad A(j_2,j_3)=-1 \implies \lambda=3+1-1=3\,.$$

Consequently, there exists a surjective homomorphism $G(19,5)\twoheadrightarrow A_4$.

Tetrahedral representations of palindromically presented groups.

Let $G=G(\varepsilon_1,\dots,\varepsilon_{\alpha-1})$ be a palindromically presented group. There are at most two equivalence classes of representations, represented by $\theta_1,\theta_2\colon G\twoheadrightarrow \mathrm{A}_4$:

$$\theta_1:\begin{cases} S\mapsto(123)\\ T\mapsto(124)\end{cases},\qquad \theta_2:\begin{cases} S\mapsto(123)\\ T\mapsto(142)\end{cases}. \tag{1}$$

The fixed axis of these symmetries is given by the following points of $S^2=\mathbb{E}^3\cap S^3$ (see Figure 1):

$$(123)\leftrightarrow\frac{i+j+k}{\sqrt3}\,,\qquad (124)\leftrightarrow\frac{i+j-k}{\sqrt3}\,,\qquad (142)\leftrightarrow\frac{-i-j+k}{\sqrt3}\,.$$

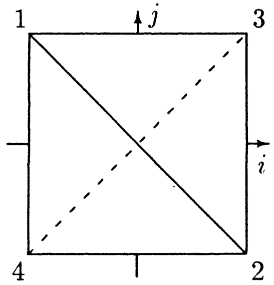

Figure 1. Tetrahedron in $\mathbb{E}^3$.

Interpreting θ_i as representations into $\mathrm{SO}_3(\mathbb{R})$ converts (1) into:

$$\theta_1:\begin{cases} S\mapsto\delta\left(\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)\frac{i+j+k}{\sqrt3}\right)\\ T\mapsto\delta\left(\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)\frac{i+j-k}{\sqrt3}\right)\end{cases},$$

$$\theta_2:\begin{cases} S\mapsto\delta\left(\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)\frac{i+j+k}{\sqrt3}\right)\\ T\mapsto\delta\left(\cos\left(\frac{\pi}{3}\right)+\sin\left(\frac{\pi}{3}\right)\frac{-i-j+k}{\sqrt3}\right)\end{cases}.$$

2.7. Remark. θ_1 and θ_2 are represented in the (τ, y)-plane by the points $y = \dfrac{1}{3}$ and $\tau_1 = \dfrac{1}{3}$, resp. $\tau_2 = -\dfrac{1}{3}$.

By Theorem 1.5, $(\tau, y) \in D$ represents a $SO_3(\mathbb{R})$-representation of $G(\varepsilon_1, \dots$ $\dots, \varepsilon_{\alpha-1})$ if and only if $z_n(\tau, y) = 0$. Defining $z_n^{\pm} := z_n\big(\pm\frac{1}{3}, \frac{1}{3}\big)$, an application of Lemma 1.6 leads to

$$z_0^{\pm} = 1\,, \quad z_1^{\pm} = -\frac{2}{3}(1 \pm 1)\,, \quad z_2^{\pm} = \mp\frac{4}{9}\big(-1 + \mu_2 \pm (1 + 3\mu_2)\big)\,,$$

and

$$z_n^{\pm} = \mp\frac{2}{3}(1 \pm \mu_n)\Big(z_{n-1}^{\pm} + \mu_n \frac{4}{3} z_{n-2}^{\pm}\Big) - \Big(\frac{4}{3}\Big)^3 \mu_n z_{n-3}^{\pm}\,,$$

where $\mu_i = \varepsilon_{n-i-+1}\varepsilon_{n-i+2}$, $i = 1, \dots, n$. In order to make all formulas as simple as possible, we introduce $t_n^{\pm} := \Big(\dfrac{3}{2}\Big)^n z_n^{\pm}$ and obtain:

$$t_0^{\pm} = 1\,, \quad t_1^{\pm} = -(1 \pm 1)\,, \quad t_2^{\pm} = \pm\big(1 - \mu_2 \mp (1 + 3\mu_2)\big)\,, \tag{2}$$

and

$$t_n^{\pm} = \mp(1 \pm \mu_n)(t_{n-1}^{\pm} + 2\mu_n t_{n-2}^{\pm}) - 8\mu_n t_{n-3}^{\pm}\,. \tag{3}$$

2.8. Lemma. *Let $n \geq 3$. Then we have:*

$$t_n^{\pm}(\mu_1, \dots, \mu_n) = \begin{cases} -8\mu_n t_{n-3}^{\pm}(\mu_1, \dots, \mu_{n-3}) & \text{if } \mu_n = \mp 1\,, \\ 2\mu_n t_{n-1}^{\pm}(\mu_1, \dots, \mu_{n-2}, -\mu_{n-1}) & \text{if } \mu_n = \pm 1\,. \end{cases}$$

Proof. If $\mu_n = \mp 1$, the lemma is a direct consequence of (3). For that reason, we assume that $\mu_n = \pm 1$. Hence

$$t_n^{\pm} = \mp 2(t_{n-1}^{\pm} \pm 2t_{n-2}^{\pm}) \mp 8t_{n-3}^{\pm}\,.$$

By (3), we earn:

$$\begin{aligned} t_n^{\pm} &= \mp 2\big(\mp(1 \pm \mu_{n-1})(t_{n-2}^{\pm} + 2\mu_{n-1}t_{n-3}^{\pm}) - 8\mu_{n-1}t_{n-4}^{\pm} \pm 2t_{n-2}^{\pm}\big) \mp 8t_{n-3}^{\pm} \\ &= \big(2(1 \pm \mu_{n-1}) - 4\big)t_{n-2}^{\pm} + \big(4\mu_{n-1}(1 \pm \mu_{n-1}) \mp 8\big)t_{n-3}^{\pm} \pm 16\mu_{n-1}t_{n-4}^{\pm} \\ &= -2(1 \mp \mu_{n-1})t_{n-2}^{\pm} + 4\mu_{n-1}(1 \mp \mu_{n-1})t_{n-3}^{\pm} \pm 16\mu_{n-1}t_{n-4}^{\pm}\,. \end{aligned}$$

In the same way, we obtain:

$$\begin{aligned} &2\mu_n t_{n-1}^{\pm}(\mu_1, \dots, \mu_{n-2}, -\mu_{n-1}) \\ &= \pm\, 2\big(\mp(1 \mp \mu_{n-1})(t_{n-2}^{\pm} - 2\mu_{n-1}t_{n-3}^{\pm}) + 8\mu_{n-1}t_{n-4}^{\pm}\big) \\ &= -\, 2(1 \mp \mu_{n-1})t_{n-2}^{\pm} + 4\mu_{n-1}(1 \mp \mu_{n-1})t_{n-3}^{\pm} \pm 16\mu_{n-1}t_{n-4}^{\pm}\,. \end{aligned}$$

□

2.9. THEOREM. *Given a palindromically presented group* $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$, $\alpha \equiv 1 \bmod 2$, *then there is at most one equivalence class of representations of* G *onto* Θ *represented by* θ_2. *Moreover,* $\theta_2 \colon G \twoheadrightarrow \mathrm{A}_4$ *defines a representation if and only if* $t_n^-(\mu_1, \dots, \mu_n) = 0$, *where* $n = \dfrac{\alpha - 1}{2}$ *and* $\mu_i = \varepsilon_{n-i+1} \cdot \varepsilon_{n-i+2}$, $i = 1, \dots, n$.

Proof. We have to show that $t_n^+ \neq 0$. Applying Lemma 2.8, we obtain $t_n^+ = C t_{n^*}^+$ for $0 \le n^* \le 2$, where $C \neq 0$. Formulas (2) lead to $t_{n^*}^+ \neq 0$ if $0 \le n^* \le 2$, and the corollary is proved. □

2.10. Remark. It is possible to define palindromically presented groups $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$ even if $\alpha \equiv 0 \bmod 2$ (see [2]). There may exist representations conjugate to θ_1, although we assume that the generators S and T of G are mapped onto elements which are conjugate (see also Remark 4.13).

2.11. Example. $G(4,1) = \langle S, T \mid L_S T = T L_S \rangle$, where $L_S = STS$. A homomorphism is defined by $S \mapsto (123)$ and $T \mapsto (124)$ because $L_S \mapsto (123)(124)(123) = (142)$.

Given the sequence $(\mu_1, \dots, \mu_n)$ associated to the group $G(\varepsilon_1, \dots, \varepsilon_n)$, $\alpha \equiv 1 \bmod 2$, Lemma 2.8 leads to an algorithm to decide whether or not there exists a surjective homomorphism onto A_4.

2.12. Example.

$$\begin{aligned}&G(19,5) = G(1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1)\\ &\Longrightarrow\ (\mu_i) = (1,1,-1,1,1,1,-1,1,1)\,.\end{aligned}$$

Applying Lemma 2.8 we come up with:

$$\begin{aligned} t_9^-(1,1,-1,1,1,1,-1,1,1) &= -8 t_6^-(1,1,-1,1,1,1) \\ &= 64 t_3^-(1,1,-1) \\ &= -128 t_2^-(1,-1) \\ &= 0\,. \end{aligned}$$

Therefore, a surjective homomorphism of $G(19,5)$ onto A_4 is given by $S \mapsto (123)$ and $T \mapsto (142)$.

2.13. COROLLARY. *Let* $\alpha \equiv 1 \bmod 2$, *and let* $G(\alpha, \beta)$ *be the group of the knot* $\mathfrak{b}(\alpha, \beta)$.

$G(\alpha, 1)$ *has a non-abelian tetrahedral representation* $\iff \alpha \equiv 0 \bmod 3$;
$G(\alpha, 3)$ *has a non-abelian tetrahedral representation* $\iff \alpha \equiv \pm 5 \bmod 18$;
$G(\alpha, \alpha-2)$ *has a non-abelian tetrahedral representation* $\iff \alpha \equiv \pm 3 \bmod 8$.

Proof. The following sequences (μ_i) are associated to the knot $\mathfrak{b}(\alpha,\beta)$, $\beta \in \{1,3,\alpha-2\}$:

$$\mathfrak{b}(\alpha,1) \implies (\mu_i) = (\underbrace{1,\dots,1}_{n}), \qquad \text{where} \quad n = \frac{\alpha-1}{2}\,;$$

$$\mathfrak{b}(\alpha,3) \implies (\mu_i) = (\underbrace{1,\dots,1}_{k},-1,\underbrace{1,\dots,1}_{n-k-1}), \qquad \text{where} \quad n = \frac{\alpha-1}{2}$$

$$\text{and} \quad k = \left[\frac{\left[\frac{\alpha}{3}\right]+1}{2}\right];$$

$$\mathfrak{b}(\alpha,\alpha-2) \implies (\mu_i) = (1,\underbrace{-1,\dots,-1}_{n-1}), \qquad \text{where} \quad n = \frac{\alpha-1}{2}\,.$$

There is a non-abelian tetrahedral representation $G(\alpha,\beta) \twoheadrightarrow \mathrm{A}_4$ if and only if $t_n^-(\mu_1,\dots,\mu_n) = 0$.

$\mathfrak{b}(\alpha,1)$: Now Lemma 2.8 yields: $t_n^-(1,\dots,1) = -8t_{n-3}^-(1,\dots,1)$.

Using formula (2), we get:

$$t_n^-(1,\dots,1) = 0 \iff n \equiv 1 \bmod 3 \iff \alpha \equiv 0 \bmod 3\,.$$

$\mathfrak{b}(\alpha,3)$: First, by Lemma 2.8, we have:

$$t_{k+1}^-(1,\dots,1,-1) = (-2)^{k-2}t_2^-(1,-1) = 0\,.$$

If $l \equiv 0 \bmod 3$, we gain:

$$t_{k+l+1}^-(\underbrace{1,\dots,1}_{k},-1,\underbrace{1,\dots,1}_{l}) = (-8)^{l/3}t_{k+1}^-(1,\dots,1,-1) = 0\,.$$

On the other hand, if $l \not\equiv 0 \bmod 3$, it follows that

$$t_{k+l+1}^-(\underbrace{1,\dots,1}_{k},-1,\underbrace{1,\dots,1}_{l}) = 0 \iff k+l+1 \equiv 1 \bmod 3\,.$$

If $n \equiv 1 \bmod 3$, we have $\alpha \equiv 0 \bmod 3$, but this contradicts $\alpha \equiv \pm 1 \bmod 6$; remember $\gcd(\alpha,\beta) = 1$ and $\alpha \equiv 1 \bmod 2$. As a result, there is a non-abelian representation $G(\alpha,3) \twoheadrightarrow \mathrm{A}$ if and only if $l = n-k-1 \equiv 0 \bmod 3$.

Let $\alpha \equiv \pm 1 \bmod 6$. Then, by definition, $k = \dfrac{\alpha \mp 1}{6}$, and therefore $l = n-k-1 = \dfrac{2\alpha - 9 \pm 1}{6}$. There is a non-abelian tetrahedral representation of $G(\alpha,3)$ if and only if

$$n-k-1 \equiv 0 \bmod 3 \implies 2\alpha \equiv \pm 8 \bmod 18\,.$$

But $\alpha \equiv \pm 1 \bmod 6$ and $2\alpha \equiv \pm 8 \bmod 18 \implies \alpha \equiv \mp 5 \bmod 18$.

$\mathfrak{b}(\alpha, \alpha-2)$: An application of Lemma 1.6 procures the formulas:

$$\begin{aligned} t_n^-(1,-1,\dots,-1) &= -2t_{n-1}^-(1,-1,\dots,-1,1) \\ &= 16t_{n-4}^-(1,-1,\dots,-1). \end{aligned}$$

Now (2) gives:

$$t_n^-(1,-1,\dots,-1) = 0 \iff n \equiv 1,2 \bmod 4 \iff \alpha \equiv \pm 3 \bmod 8.$$

This proves the corollary. □

2.14. Remark. Without giving a proof, we remark that there is a surjective representation $G(\alpha, 5) \twoheadrightarrow \mathrm{A}_4$ if and only if $\alpha \equiv \pm 3 \bmod 30$ or $\alpha \equiv \pm 11 \bmod 30$.

3. Octahedral representations

Reviewing the fact that the octahedron and the cube are dual, it can be easily seen that each orientation preserving symmetry of the octahedron is an orientation preserving symmetry of the cube and vice versa. But each orientation preserving symmetry of the cube is given by the permutation of the four diagonals of the cube, therefore Ω is isomorphic to the permutation group $\mathcal{S}_4$ (see [4] or [8]).

3.1. THEOREM. *Let G be a knot group. There is a surjective homomorphism $G \twoheadrightarrow \mathcal{S}_4$ if and only if $\Delta(-1) \equiv 0 \bmod 3$, where $\Delta(t)$ denotes the Alexander polynomial of the knot.*

Proof. Each homomorphism $\lambda\colon G \twoheadrightarrow \mathcal{S}_4$ is the lift of a homomorphism $G \twoheadrightarrow \mathcal{S}_3$ (see [6]). By [15; 1.11] there exists a surjective homomorphism $G \twoheadrightarrow \mathcal{S}_3$ if and only if $\Delta(-1) \equiv 0 \bmod 3$. □

3.2. Remark. For every special symmetric group $G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$, $\alpha \equiv 1 \bmod 2$, there is at most one conjugacy class of octahedral representations. This class is associated to the point $\tau = 0$ and $y = 1$ in the (τ, y)-plane, and it exists if and only if $\alpha \equiv 0 \bmod 3$.

4. Icosahedral representations

The group $\mathfrak{I}$ of orientation preserving symmetries of the icosahedron is isomorphic to the alternating group A_5 (see [4] or [8]). Let G be a knot group. It is not possible to apply the theory of metabelian representations to prove the existence of surjective representations $G \twoheadrightarrow \mathrm{A}_5$ because the alternating group A_5 is perfect.

Let $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$ be a given palindromically presented group, $\alpha \equiv 1 \bmod 2$. There are at most two equivalence classes of surjective homomorphisms

$\varrho\colon G \twoheadrightarrow A_5$ of order 5, and one class of order 3; this was proved by R. Riley (see [10; Section 2]). However, there are always pairs of classes of conjugate representations $G \twoheadrightarrow A_5$. There are exactly 6 classes of non-abelian conjugate homomorphisms.

4.1. Example.

$$\varrho : \begin{cases} S \mapsto (123) \\ T \mapsto (145) \end{cases}, \qquad \varrho' : \begin{cases} S \mapsto (123) \\ T \mapsto (154) \end{cases}.$$

By the very definition, ϱ and ϱ' are equivalent, but not conjugate, as representations onto A_5.

In order to determine the points of the (τ, y)-plane which are associated to non-abelian surjective icosahedral representations, we consider an icosahedron in $\mathbb{E}^3$ (see Figure 2).

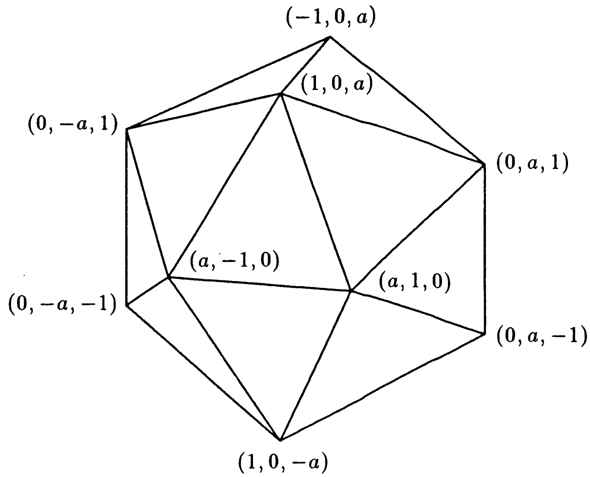

Figure 2. Icosahedron in $\mathbb{E}^3$

All 12 vertices of the icosahedron are obtained by a cyclic permutation of the coordinates of the points $(\pm 1, 0, \pm a)$, $a = \dfrac{1+\sqrt{5}}{2}$ (see [4]).

A homomorphism $G \to A_5$ of order 2 cannot be surjective because every non-abelian subgroup of $SO_3(\mathbb{R})$ generated by two element of order 2 is a dihedral group.

The axes of the symmetries of order 5 are determined by the 12 vertices of the icosahedron. Let ψ be the angle spanned by two of those axes. All possible values of $\tau = \cos\psi$ are $\tau = \pm 1$ and $\tau = \pm \dfrac{a}{1+a^2} = \pm \dfrac{1}{\sqrt{5}}$. The associated homomorphism is abelian if and only if $\tau = \pm 1$. In order to describe homomorphisms of order 5, the possible values of y are $y = \cot^2(\pi/5)$ and $y = \cot^2(2\pi/5)$.

The axes of order 3 symmetries pass through the 20 centers of the triangles of the icosahedron. $\tau = \pm 1$, $\tau = \pm\frac{1}{3}$, and $\tau = \pm\frac{\sqrt{5}}{3}$ are the possible values of τ. We obtain again an abelian representation if and only if $\tau = \pm 1$, and a tetrahedral representation if and only if $\tau = \pm\frac{1}{3}$ ($A_4 \subset A_5$). For every representation of order 3 we have $y = \frac{1}{3}$.

Altogether, we have proved the following lemma:

4.2. Lemma. *There are at most six conjugacy classes of representations* $G \twoheadrightarrow \mathfrak{I}$.

We denote representatives of these classes by $\iota_i \colon G \twoheadrightarrow \mathfrak{I}$, $i = 1, \dots, 6$. We fix the (τ, y)-parameter associated to the ι_i as in the following diagram:

ι_1	$\left(\frac{1}{\sqrt{5}}, 1 + \frac{2}{\sqrt{5}}\right)$	ι_2	$\left(-\frac{1}{\sqrt{5}}, 1 - \frac{2}{\sqrt{5}}\right)$	ι_3	$\left(\frac{1}{\sqrt{5}}, 1 - \frac{2}{\sqrt{5}}\right)$
ι_4	$\left(-\frac{1}{\sqrt{5}}, 1 + \frac{2}{\sqrt{5}}\right)$	ι_5	$\left(\frac{\sqrt{5}}{3}, \frac{1}{3}\right)$	ι_6	$\left(-\frac{\sqrt{5}}{3}, \frac{1}{3}\right)$

4.3. Remark. All the coordinates correlated to ι_i are elements of $\mathbb{Q}(\sqrt{5})$. The pairs of coordinates fixed by ι_{2i-1} and ι_{2i} are exchanged by the Galois automorphism $\sqrt{5} \mapsto -\sqrt{5}$, $i = 1, 2, 3$.

4.4. Remark. All points in the (τ, y)-plane which are related to non-abelian representations onto finite subgroups of $SO_3(\mathbb{R})$ are contained in the following list:

point	group
$\left(\cos\left(\frac{2\pi}{2n+1}\nu\right), 0\right)$, $1 \le \nu \le n$	D_{2n+1}
$\left(\pm\frac{1}{3}, \frac{1}{3}\right)$	Θ
$(0, 1)$	Ω
$\left(\pm\frac{1}{\sqrt{5}}, 1 \pm \frac{2}{\sqrt{5}}\right)$, $\left(\mp\frac{1}{\sqrt{5}}, 1 \pm \frac{2}{\sqrt{5}}\right)$, $\left(\pm\frac{1}{\sqrt{5}}, \frac{1}{3}\right)$	$\mathfrak{I}$

The points of the (τ, y)-plane associated to dihedral representations are elements of the line $y = 0$. It is easy to see that the remainder points are obtained as intersection points of the lines

$$y = \frac{1}{3}, \quad y = \pm 2\tau + 1 \quad \text{and} \quad y = \pm(3 \pm \sqrt{5})\tau + (2 \pm \sqrt{5})$$

(see Figure 3).

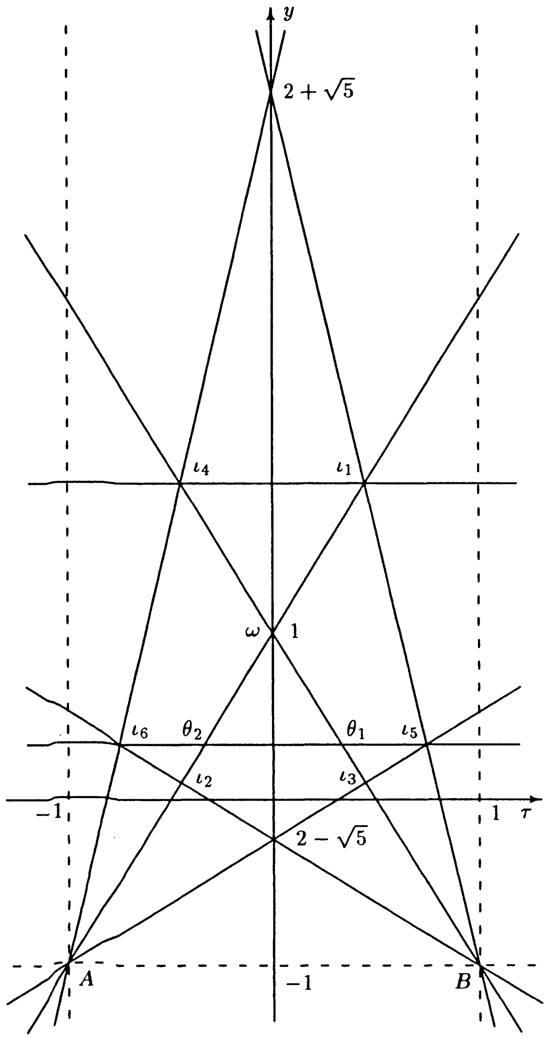

Figure 3.
Points in the (τ, y)-plane associated to representations with finite image.

Given a group $G = G(\varepsilon_1, \ldots, \varepsilon_{\alpha-1})$, $\alpha \equiv 1 \bmod 2$, there is a representation $\iota_i\colon G \twoheadrightarrow \mathrm{A}_5$ if and only if $z_n(\tau_0, y_0) = 0$, where (τ_0, y_0) are the coordinates correlated to ι_i. With the intention to prove the existence of icosahedral representations, we apply Lemma 1.6. Let (τ_0, y_0) be a pair of coordinates which is associated to an icosahedral representation. We have $z_{n^*}(\tau_0, y_0) \in \mathbb{Q}(\sqrt{5})$ if $0 \le n^* \le 2$, and, by Remark 4.3, we obtain $z_n(\tau_0, y_0) \in \mathbb{Q}(\sqrt{5})$ because $z_n(\tau, y) \in \mathbb{Z}[\tau, y]$. Therefore, it is sufficient to consider the points $\left(\pm\frac{1}{\sqrt{5}}, 1+\frac{2}{\sqrt{5}}\right)$ and $\left(\frac{\sqrt{5}}{3}, \frac{1}{3}\right)$.

Icosahedral representations of order 5.

We fix $f_n^{\pm} := z_n\left(\pm\frac{1}{\sqrt{5}}, 1 + \frac{2}{\sqrt{5}}\right)$ and $\eta := 2\left(1 + \frac{1}{\sqrt{5}}\right)$. Lemma 1.6 yields the following formulas:

$$f_0^+ = 1\,, \qquad f_1^+ = 0\,, \qquad f_2^+(1,\mu_2) = \begin{cases} -\eta^2 & \text{if } \mu_2 = 1\,, \\ \frac{\sqrt{5}}{3}\eta^3 & \text{if } \mu_2 = -1\,, \end{cases}$$

$$f_3^+(1,\mu_2,\mu_3) = \begin{cases} -\eta^3 & \text{if } \mu_3 = 1\,, \\ \frac{\sqrt{5}}{4}\eta^4 & \text{if } \mu_3 = -1\,,\ \mu_2 = 1\,, \\ 0 & \text{if } \mu_3 = -1\,,\ \mu_2 = -1\,, \end{cases} \tag{4}$$

and

$$f_0^- = 1\,, \qquad f_1^- = \frac{4}{\sqrt{5}}\,, \qquad f_2^-(1,\mu_2) = \begin{cases} 0 & \text{if } \mu_2 = 1\,, \\ \eta^2 & \text{if } \mu_2 = -1\,, \end{cases}$$

$$f_3^-(1,\mu_2,\mu_3) = \begin{cases} \eta^3 & \text{if } \mu_3 = -1\,, \\ -\frac{4}{\sqrt{5}}\eta^2 & \text{if } \mu_3 = 1\,,\ \mu_2 = 1\,, \\ 0 & \text{if } \mu_3 = 1\,,\ \mu_2 = -1\,, \end{cases} \tag{5}$$

and

$$f_n^{\pm} = \frac{2}{\sqrt{5}}(\mu_n \mp 1)(f_{n-1}^{\pm} + \mu_n \eta f_{n-2}^{\pm}) - \mu_n \eta^3 f_{n-3}^{\pm}\,. \tag{6}$$

4.5. Theorem. *Let $G = G(\varepsilon_1, \ldots, \varepsilon_{\alpha-1})$ be a palindromically presented group and $(\mu_1, \ldots, \mu_n)$ the associated (μ_i)-sequence. There are at most two equivalence classes of surjective homomorphisms of order 5 of G onto A_5 represented by:*

$$\varrho_+ : \begin{cases} S \mapsto (12345) \\ T \mapsto (13254) \end{cases}, \qquad \textit{resp.} \qquad \varrho_- : \begin{cases} S \mapsto (12345) \\ T \mapsto (14523) \end{cases}.$$

Additionally, $\varrho_{\pm}\colon G \twoheadrightarrow \mathrm{A}_5$ defines a surjective homomorphism of order 5 if and only if $f_n^{\pm}(\mu_1, \ldots, \mu_n) = 0$.

Proof. The first part of the theorem is proved by R. Riley (see [10]). The remainder is a result of the discussion above. □

4.6. LEMMA. *Let $n \geq 4$. Then we have:*

$$f_n^{\pm}(\mu_1, \ldots, \mu_n) = \begin{cases} -\mu_n \eta^3 f_{n-3}^{\pm}(\mu_1, \ldots, \mu_{n-3}) & \text{if } \mu_n = \pm 1, \\ -\eta^2 f_{n-2}^{\pm}(\mu_1, \ldots, \mu_{n-3}, -\mu_{n-2}) & \text{if } \mu_n = \mp 1, \ \mu_{n-1} = \mp 1, \\ -\mu_n \eta f_{n-1}^{\pm}(\mu_1, \ldots, \mu_{n-3}, -\mu_{n-2}, -\mu_{n-1}) & \text{if } \mu_n = \mp 1, \ \mu_{n-1} = \pm 1. \end{cases}$$

Proof. If $\mu_n = \pm 1$, the lemma is a direct conclusion of (6). Therefore, we can assume $\mu_n = \mp 1$.

Applying (6) twice we obtain:

$$f_n^{\pm}(\mu_1, \ldots, \mu_n) = \mp 4\Big(\frac{2}{5}(\mu_{n-1} \mp 1) \mp \frac{1}{\sqrt{5}}\eta\Big) f_{n-2}^{\pm} \mp \Big(\frac{8}{5}(1 \mp \mu_{n-1})\eta - \eta^3\Big) f_{n-3}^{\pm} \frac{4}{\sqrt{5}} \mu_{n-1} \eta^3 f_{n-4}^{\pm}.$$

The equation $\frac{16}{5} + \frac{4}{\sqrt{5}}\eta = \eta^2$ ensures that

$$f_n^{\pm}(\mu_1, \ldots, \mu_n) = \eta^2 f_{n-2}^{\pm} \pm \frac{4}{\sqrt{5}}\eta^2 f_{n-3}^{\pm} - \frac{4}{\sqrt{5}}\eta^3 f_{n-4}^{\pm} \qquad \text{if} \quad \mu_{n-1} = \mp 1,$$

and

$$f_n^{\pm}(\mu_1, \ldots, \mu_n) = \eta \frac{4}{\sqrt{5}} f_{n-2}^{\pm} \pm \eta^3 f_{n-3}^{\pm} + \frac{4}{\sqrt{5}}\eta^3 f_{n-4}^{\pm} \qquad \text{if} \quad \mu_{n-1} = \pm 1.$$

An analogous application of (6) results in:

$$f_{n-2}^{\pm}(\mu_1, \ldots, \mu_{n-3}, -\mu_{n-2}) = -f_{n-2}^{\pm} \mp \frac{4}{\sqrt{5}} f_{n-3}^{\pm} + \frac{4}{\sqrt{5}}\eta f_{n-4}^{\pm} \qquad \text{if} \quad \mu_{n-1} = \mp 1,$$

and

$$f_{n-1}^{\pm}(\mu_1, \ldots, \mu_{n-3}, -\mu_{n-2}, -\mu_{n-1}) = \pm \frac{4}{\sqrt{5}} f_{n-2}^{\pm} + \eta^2 f_{n-3}^{\pm} \pm \frac{4}{\sqrt{5}}\eta^2 f_{n-4}^{\pm} \qquad \text{if} \quad \mu_{n-1} = \pm 1.$$

Combining these equations yields the formulas

$$f_n^{\pm}(\mu_1, \ldots, \mu_n) = -\eta^2 f_{n-2}(\mu_1, \ldots, \mu_{n-3}, -\mu_{n-2}) \qquad \text{if} \quad \mu_n = \mu_{n-1} = \mp 1,$$

and

$$f_n^{\pm}(\mu_1, \ldots, \mu_n) = -\mu_n \eta f_{n-1}(\mu_1, \ldots, \mu_{n-3}, -\mu_{n-2}, -\mu_{n-1}) \qquad \text{if} \quad \mu_n = -\mu_{n-1} = \mp 1.$$

This proves the desired recursion formulas. □

4.7. Example. The sequence $(\mu_i) = (1,-1,-1,-1,1,-1,-1,-1)$ is determined by the knot $\mathfrak{b}(17,13)$. We have:

$$\begin{aligned} f_8^+(1,-1,-1,-1,1,-1,-1,-1) &= -\eta^2 f_6^+(1,-1,-1,-1,1,1) \\ &= \eta^5 f_3^+(1,-1,-1) = 0, \\ f_8^-(1,-1,-1,-1,1,-1,-1,-1) &= \eta^3 f_5^-(1,-1,-1,-1,1) \\ &= \eta^4 f_4^-(1,-1,1,1) = -\eta^6 f_4^-(1,1) = 0. \end{aligned}$$

We proved that there are two non equivalent surjective representations $G(17,13) \twoheadrightarrow \mathrm{A}_5$.

4.8. COROLLARY. *Let $\alpha \equiv 1 \bmod 2$ and let $G(\alpha,\beta)$ be the group of the knot $\mathfrak{b}(\alpha,\beta)$.*

$\varrho_+ : G(\alpha,1) \twoheadrightarrow \mathrm{A}_5$ *defines a representation if and only if* $\alpha \equiv 0 \bmod 3$.

$\varrho_- : G(\alpha,1) \twoheadrightarrow \mathrm{A}_5$ *defines a representation if and only if* $\alpha \equiv 0 \bmod 5$.

There is no icosahedral representation of $G(\alpha,3)$ which is equivalent to ϱ_+.

$\varrho_- : G(\alpha,3) \twoheadrightarrow \mathrm{A}$ *defines a representation if and only if* $\alpha \equiv \pm 7 \bmod 15$.

$\varrho_+ : G(\alpha,\alpha-2) \twoheadrightarrow \mathrm{A}_5$ *defines a representation if and only if* $\alpha \equiv \pm 2 \bmod 5$.

There is no icosahedral representation of $G(\alpha,\alpha-2)$ which is equivalent to ϱ_-.

Proof. The sequences (μ_i) associated to the knots can be found in the proof of Corollary 2.13. Fix $n = \dfrac{\alpha-1}{2}$ and $k = \left[\dfrac{[\frac{\alpha}{3}]+1}{2}\right]$. There is a representation $\varrho_+ : G(\alpha,\beta) \twoheadrightarrow \mathrm{A}_5$ (resp. $\varrho_- : G(\alpha,\beta) \twoheadrightarrow \mathrm{A}_5$) if and only if $f_n^+(\mu_1,\dots,\mu_n) = 0$ (resp. $f_n^-(\mu_1,\dots,\mu_n) = 0$).

$\mathfrak{b}(\alpha,1)$: By Lemma 4.6, we obtain $f_n^+(1,\dots,1) = -\eta^3 f_{n-3}^+(1,\dots,1)$, and, using (5), one obtains

$$\begin{aligned} f_n^+(1,\dots,1) = 0 &\iff n \equiv 1 \bmod 3 \iff \alpha \equiv 0 \bmod 3, \\ f_n^-(1,\dots,1) &= -\eta^2 f_{n-2}^-(1,\dots,1,-1) = -\eta^5 f_{n-5}^-(1,\dots,1). \end{aligned}$$

Again (5) gives

$$f_n^-(1,\dots,1) = 0 \iff n \equiv 2 \bmod 5 \iff \alpha \equiv 0 \bmod 5.$$

$\mathfrak{b}(\alpha,3)$: In the first step we are interested in $f_{k+l+1}^+(\underbrace{1,\dots,1}_{k},-1,\underbrace{1,\dots,1}_{l})$.

If $l \not\equiv 0 \bmod 3$, we get

$$f_{k+l+1}^+(\underbrace{1,\dots,1}_{k},-1,\underbrace{1,\dots,1}_{l}) = 0 \iff k+l+1 \equiv 1 \bmod 3.$$

The assumption $n \equiv 1 \bmod 3$ leads to $\alpha \equiv 0 \bmod 3$, which is impossible because $\gcd(\alpha,3) = 1$. Therefore, there is no representation equivalent to ϱ_+ if $n-k-1 \not\equiv 0 \bmod 3$.

In the second step, we will use again Lemma 4.6 and (5).

If $k \geq 1$, we obtain:

$$f^{+}_{k+1}(1,\dots,1,-1) = 0 \iff k \equiv 0 \bmod 3\,. \tag{7}$$

Let $l \equiv 0 \bmod 3$. Then (7) gives

$$f^{+}_{k+l+1}(\underbrace{1,\dots,1}_{k},-1,\underbrace{1,\dots,1}_{l}) = 0 \iff k \equiv 0 \bmod 3\,.$$

By definition, we have $\alpha \equiv \pm 1 \bmod 6$, which gives $k = \dfrac{\alpha \mp 1}{6}$ and $l = \dfrac{2\alpha - 9 \pm 1}{6}$. Accordingly, we come up with two congruence equations:

$$\begin{aligned} k \equiv 0 \bmod 3 &\implies \alpha \equiv \pm 1 \bmod 18\,,\\ l \equiv 0 \bmod 3 &\implies \alpha \equiv \pm 4 \bmod 18\,. \end{aligned}$$

It is evident that there is no solution. This proves the non-existence of a representation $\varrho_{+}\colon G(\alpha,3) \twoheadrightarrow \mathrm{A}_5$.

Next we consider the equation

$$f^{-}_{k+l+1}(\underbrace{1,\dots,1}_{k},-1,\underbrace{1,\dots,1}_{l}) = 0. \tag{8}$$

Applying Lemma 4.6 and (5) it is possible to derive circumstances under which there exists a solution of (8). The results are summarized in the following table:

l	k
$l \equiv 0 \bmod 5$	$k \equiv 4 \bmod 5$
$l \equiv 1 \bmod 5$	k arbitrary
$l \equiv 2 \bmod 5$	$k \equiv 1 \bmod 5$
$l \equiv 3 \bmod 5$	$k \equiv 3 \bmod 5$
$l \equiv 4 \bmod 5$	$k \equiv 2 \bmod 5$

where $k = \dfrac{\alpha \mp 1}{6}$, $l = \dfrac{2\alpha - 9 \pm 1}{6}$ and $\alpha \equiv \pm 1 \bmod 6$. There exists an α which solves these equations if and only if $l \equiv 1 \bmod 5$.

$$l \equiv 1 \bmod 5 \implies \alpha \equiv \pm 7 \bmod 15\,.$$

$\mathfrak{b}(\alpha, \alpha - 2)$: Now Lemma 4.6 gives:

$$\begin{aligned} f_n^+(1, -1, \dots, -1) &= -\eta^2 f_{n-2}^+(1, -1, \dots, -1, 1) \\ &= -\eta^5 f_{n-5}^+(1, -1, \dots, -1). \end{aligned}$$

From (4), we get:

$$f_n^+(1, -1, \dots, -1) = 0 \iff n \equiv 1, 3 \bmod 5 \iff \alpha \equiv \pm 2 \bmod 5.$$

An analogous application of Lemma 4.6 and (5) results in $f_n^-(1, -1, \dots, -1) \neq 0$.

This finishes the proof of the corollary. □

Icosahedral representations of order 3.

We use the same methods as in the last Subsection in order to study icosahedral representations of palindromically presented groups of order 3. Let $G = G(\varepsilon_1, \dots, \varepsilon_{\alpha-1})$ be a palindromically presented group, $\alpha \equiv 1 \bmod 2$. There is a homomorphism $G \twoheadrightarrow \mathrm{A}_5$ of order 3 if and only if $z_n\left(\frac{1}{\sqrt{5}}, \frac{1}{3}\right) = 0$. In order to keep formulas as simple as possible, we introduce $d_n := \left(\frac{3}{2}\right)^n z_n\left(\frac{1}{\sqrt{5}}, \frac{1}{3}\right)$. The use of Lemma 1.6 gives:

$$\begin{gathered} d_0 = 1, \qquad d_1 = -\left(1 + \sqrt{5}\right), \qquad d_2(1, \mu_2) = \begin{cases} 0 & \text{if } \mu_2 = 1, \\ \left(1 + \sqrt{5}\right)^2 & \text{if } \mu_2 = -1, \end{cases} \\ d_3(1, \mu_2, \mu_3) = 0 \iff \mu_2 = 1, \quad \mu_3 = -1, \\ d_4(1, \mu_2, \mu_3, \mu_4) = 0 \iff \mu_2 = \mu_3 = -1, \quad \mu_4 = 1, \end{gathered} \tag{9}$$

and

$$d_n(\mu_1, \dots, \mu_n) = -\left(\mu_n + \sqrt{5}\right) d_{n-1} - 2\mu_n\left(\mu_n + \sqrt{5}\right) d_{n-2} - 8\mu_n d_{n-3}. \tag{10}$$

4.9. Lemma. *Let $n \geq 5$. Then we have:*

$$\begin{aligned} &d_n(\mu_1, \dots, \mu_n) \\ &= \begin{cases} -\mu_n 2^5 d_{n-5}(\mu_1, \dots, \mu_{n-5}) & \text{if } \mu_n = \mu_{n-1} = \mu_{n-2}, \\ -\mu_n 2^3 d_{n-3}(\mu_1, \dots, \mu_{n-4}, -\mu_{n-3}) & \text{if } \mu_n = \mu_{n-1} = -\mu_{n-2}, \\ -\mu_n 2 d_{n-1}(\mu_1, \dots, \mu_{n-3}, -\mu_{n-2}, \mu_{n-1}) & \text{if } \mu_n = -\mu_{n-1}. \end{cases} \end{aligned}$$

Proof. We apply the formula (10) three times and come up with:

$$\begin{aligned} d_n(\mu_1, \dots, \mu_{n-3}, \mu, \mu, \mu) &= -\left(\mu + \sqrt{5}\right) d_{n-1} - 2\mu\left(\mu + \sqrt{5}\right) d_{n-2} - 8\mu d_{n-3} \\ &= 4 d_{n-2} + 4\left(\mu + \sqrt{5}\right) d_{n-3} + 8\mu\left(\mu + \sqrt{5}\right) d_{n-4} \\ &= -32\mu d_{n-5}. \end{aligned}$$

An analogous calculation leads to:

$$\begin{aligned} d_n(\mu_1,\dots,\mu_{n-3},-\mu,\mu,\mu) &= 4d_{n-2}+4(\mu+\sqrt{5})d_{n-3}+8\mu(\mu+\sqrt{5})d_{n-4} \\ &= 8\mu(d_{n-3}+2\sqrt{5}d_{n-4}+4d_{n-5}) \\ &= -8\mu d_{n-3}(\mu_1,\dots,\mu_{n-4},-\mu_{n-3}), \end{aligned}$$

and

$$\begin{aligned} d_n(\mu_1,\dots,\mu_{n-2},-\mu,\mu) &= -(\mu+\sqrt{5})d_{n-1}-2\mu(\mu+\sqrt{5})d_{n-2}-8\mu d_{n-3} \\ &= -2\mu((-\mu+\sqrt{5})d_{n-2}+8d_{n-3}+4(\mu+\sqrt{5})d_{n-4}) \\ &= -2\mu d_{n-1}(\mu_1,\dots,\mu_{n-3},-\mu_{n-2},-\mu). \end{aligned}$$

□

4.10. THEOREM. *Let $G = G(\varepsilon_1,\dots,\varepsilon_{\alpha-1})$ be a palindromically presented group and $(\mu_1,\dots,\mu_n)$ the associated (μ_i)-sequence. There is at most one equivalence class of surjective homomorphisms of order 3 of G onto A_5 given by:*

$$\varrho : \begin{cases} S \mapsto (123) \\ T \mapsto (145) \end{cases}.$$

In fact, $\varrho\colon G \twoheadrightarrow A_5$ defines a surjective homomorphism of order 3 if and only if $d_n(\mu_1,\dots,\mu_n) = 0$.

Proof. See proof of Theorem 4.5. □

Lemma 4.6 and Lemma 4.9 yield useful methods to find icosahedral representations of palindromically presented groups.

4.11. Example. Let $6_3 = \mathfrak{b}(13,5)$. The (μ_i)-sequence $(1,-1,1,1,-1,1)$ is associated to this knot.

$$\begin{aligned} f_6^+(1,-1,1,1,-1,1) &= -\eta^3 f_3^+(1,-1,1) \neq 0, \\ f_6^-(1,-1,1,1,-1,1) &= -\eta f_5^+(1,-1,1,-1,1) \\ &= \eta^2 f_4^+(1,-1,-1,1) = -\eta^3 f_3^+(1,1,1) \neq 0, \\ d_6(1,-1,1,1,-1,1) &= -2d_5(1,-1,1,-1,-1) = -8d_2(1,1) = 0. \end{aligned}$$

We obtain that $G(13,5)$ does not possess a surjective icosahedral representation of order 5. But there are two conjugation classes (one equivalence class) of representations of order 3.

4.12. Corollary. *Let $\alpha \equiv 1 \bmod 2$, and let $G(\alpha,\beta)$ be the group of the knot $\mathfrak{b}(\alpha,\beta)$.*

There is a surjective representation $G(\alpha,1) \twoheadrightarrow \mathrm{A}_5$ if and only if $\alpha \equiv 0 \bmod 5$.
There is no surjective icosahedral representation of $G(\alpha,3)$ of order 3.
There is no surjective icosahedral representation of $G(\alpha,\alpha-2)$ of order 3.

Proof. The corollary is proved in the same way as Corollary 4.8 by use of Lemma 4.9 and the initial conditions (9). □

4.13. Remark. One may also consider palindromically presented groups $G = G(\varepsilon_1,\dots,\varepsilon_{\alpha-1})$ in case of α even (including links with two-bridges).

$$G = G(\varepsilon_1,\dots,\varepsilon_{\alpha-1}) = \langle S,T \mid L_S T = T L_S\rangle, \quad L_S = S^{\varepsilon_1}T^{\varepsilon_2}\dots T^{\varepsilon_{\alpha-2}}S^{\varepsilon_{\alpha-1}},$$

where $\varepsilon_i \in \{\pm 1\}$ and $\varepsilon_i = \varepsilon_{\alpha-i}$.

Let $G = G(\varepsilon_1,\dots,\varepsilon_{\alpha-1})$ (α even). We would like to study non-abelian representations $\varrho\colon G \to \mathrm{SO}_3(\mathbb{R})$ which factorize through $\mathrm{SU}_2(\mathbb{C})$ such that $\varrho(S)$ and $\varrho(T)$ are conjugate in $\mathrm{SO}_3(\mathbb{R})$, i.e.

$$\varrho : \begin{cases} S \mapsto \delta(P,\varphi) \\ T \mapsto \delta(Q,\varphi) \end{cases} \qquad (P \neq Q).$$

The equivalence class of such a representation is also determined by $\tau = \langle P,Q\rangle$ and $y = \cot^2 \frac{\varphi}{2}$. Moreover, Lemma 1.6 remains valid with slight changes:

Put $n = \frac{\alpha-2}{2}$, $\mu_i = \varepsilon_{n-i+1}\varepsilon_{n-i+2}$, $1 \le i \le n$ (see [2]).

The polynomial $z_n(\tau,y)$ is determined by the numbers ε_{n+1}, μ_i, $1 \le i \le n$. The recursion is the same as in Lemma 1.6, but the beginning changes to:

$$z_0 = \varepsilon_1, \quad z_1 = 2\varepsilon_2(\mu_1 y - \tau), \quad z_2 = \varepsilon_3\big(\mu_1(y-1) - 2\tau\big)\big(2\mu_2 y - 2\tau + \mu_1(y+1)\big).$$

Therefore, Lemma 2.8 is still true; but we have to change formulas (2) to:

$$t_0^{\pm} = \varepsilon_1, \quad t_1^{\pm} = \varepsilon_2(\mu_1 \mp 1), \quad t_2^{\pm} = -\varepsilon_3(\mu_1 \pm 1)(2\mu_1 + \mu_2 \mp 1).$$

The same holds for Lemma 4.6 (resp. Lemma 4.9) if one changes formulas (4) and (5) in an analogous way.

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