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CLASSIFICATION OF σ-ideals

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In the paper, σ-ideals of complete separable perfect metric spaces are considered. For any σ-ideal $\mathscr{I}$, let $S(\mathscr{B}, \mathscr{I})$ denote the σ-field generated by Borel sets and sets from $\mathscr{I}$. Ordinal numbers $RT(\mathscr{I})$, $RZ(\mathscr{I})$, characteristic of $\mathscr{I}$, which describe some special properties of $S(\mathscr{B}, \mathscr{I})$-measurable sets and functions, are investigated. Examples concerning meager sets and sets of measure zero are discussed. Connections with Mauldin's results on generalized Baire systems are observed.

Throughout the paper, we shall assume that X is a complete separable perfect metric space. Let $\mathscr{B}$ denote the family of all Borel subsets of X. We shall also consider Borel classes F_α, G_α, $\alpha < \omega_1$, (comp. [2], pp. 251—252). Here ω_1 denotes the first uncountable ordinal number; ω will denote the first infinite ordinal number. The closure of a set $A \subseteq X$ will be written as $\bar{A}$.

A family $\mathscr{I}$ of subsets of X will be called a σ-ideal if and only if it fulfils the conditions:

(i) if $A \in \mathscr{I}$ and $B \subseteq A$, then $B \in \mathscr{I}$;

(ii) if $A_n \in \mathscr{I}$ for all $n < \omega$, then $\bigcup_{n<\omega} A_n \in \mathscr{I}$;

(iii) if $A \in \mathscr{I}$, then the interior of A is empty;

(iv) if $x \in X$, then $\{x\} \in \mathscr{I}$.

Assume that $\mathscr{I}$ is a σ-ideal. let $S(\mathscr{B}, \mathscr{I})$ denote the σ-field generated by all sets from $\mathscr{B} \cup \mathscr{I}$. It is easily checked that

$$S(\mathscr{B}, \mathscr{I}) = \{B \triangle A : B \in \mathscr{I}, A \in \mathscr{I}\}$$

where $B \triangle A$ denotes the symmetric difference of the sets B, A.

Define $RT(\mathscr{I})$ as the first of ordinal numbers $\alpha \leqq \omega_1$ such that

$$S(\mathscr{B}, \mathscr{I}) = \{B \triangle A : B \in F_\alpha, A \in \mathscr{I}\}$$

(here $F_{\omega_1} = \mathscr{B}$). Observe that it will not matter if we replace above F_α by G_α. We may interpret $RT(\mathscr{I})$ as follows. Let $\mathscr{B}/\mathscr{I}$ denote the Boolean algebra of Borel sets modulo $\mathcal{I}$, i.e. the set of all equivalence classes of the relation

$$A \sim B \quad \text{if and only if} \quad A \triangle B \in \mathcal{I}$$

defined for all Borel sets. Then $RT(\mathcal{I})$ means the first of numbers α such that equivalence classes generated by sets from F_α give the whole $\mathcal{B}/\mathcal{I}$.

In the sequel, $\mathcal{K}$, $\mathcal{L}$ will denote, respectively, the σ-ideal of all meager subsets of X and the σ-ideal of all subsets of $\mathbf{R}$ (the real line) of the Lebesgue measure zero. It is an easy exercise to show that $RT(\mathcal{K}) = 0$, $RT(\mathcal{L}) = 1$ (comp. [4], [8]). In [7] Miller constructed, for each $\alpha < \omega_1$, a σ-ideal $\mathcal{I}$ of subsets of the Cantor set 2^ω, such that $RT(\mathcal{I}) = \alpha$ (for related results, see [8]).

Proposition 1. *If $\mathcal{I}$, $\mathcal{J}$ are σ-ideals such that $\mathcal{I} \subseteq \mathcal{J}$, then $RT(\mathcal{J}) \leqq RT(\mathcal{I})$.*

Proof. Let $RT(\mathcal{I}) = \alpha$ and $E \in S(\mathcal{B}, \mathcal{J})$. Then $E = B \triangle A$ where $B \in \mathcal{B}$, $A \in \mathcal{J}$. Obviously, $B \in S(\mathcal{B}, \mathcal{I})$, and so $B = C \triangle D$ where $C \in F_\alpha$, $D \in \mathcal{I}$. Thus $E = C \triangle (A \triangle D)$ and $C \in F_\alpha$, $(A \triangle D) \in \mathcal{J}$. Hence $RT(\mathcal{J}) \leqq \alpha$.

We shall say that the functions f, g: $X \to \mathbf{R}$ are $\mathcal{I}$-equivalent if and only if $\{x : f(x) \neq g(x)\} \in \mathcal{I}$.

Let χ_A denote the characteristic function of a set A and let $f|A$ be the restriction of a function f to A.

Theorem 1. *Let f: $X \to R$. The following conditions are equivalent:*

(1) *f is $S(\mathcal{B}, \mathcal{I})$-measurable;*

(2) *there is a $\mathcal{B}$-measurable function g: $X \to \mathbf{R}$ such that f, g are $\mathcal{I}$-equivalent;*

(3) *there is a set $A \in \mathcal{I}$ such that the function $f | X \setminus A$ is $\mathcal{B}$-measurable.*

Proof. (1)$\Rightarrow$(2). Assume first that $f = \chi_A$, $A \in S(\mathcal{B}, \mathcal{I})$. Let $A = B \triangle C$ where $B \in \mathcal{B}$, $C \in \mathcal{I}$. Put $g = \chi_B$. Then g is $\mathcal{B}$-measurable and we have

$$\{x : f(x) \neq g(x)\} = A \triangle B = C \in \mathcal{I}.$$

Thus f, g are $\mathcal{I}$-equivalent.

Next, assume that f is a simple function

$$f = \sum_{i=1}^{n} a_i \chi_{A_i}; \quad a_i \in \mathbf{R}, \quad A_i \in S(\mathcal{B}, \mathcal{I}), \quad i \leqq n.$$

Let g_i be $\mathcal{B}$-measurable functions such that

$$\{x : \chi_{A_i}(x) \neq g_i(x)\} \in \mathcal{I}, \qquad i \leqq n.$$

Put $g = \sum_{i=1}^{n} a_i g_i$. Then g is $\mathcal{B}$-measurable and we have

$$\{x : f(x) \neq g(x)\} \subseteq \bigcup_{i=1}^{n} \{x : \chi_{A_i}(x) \neq g_i(x)\} \in \mathcal{I}.$$

Hence f, g are $\mathcal{I}$-equivalent.

Now, let f be an arbitrary $S(\mathcal{B}, \mathcal{I})$-measurable function. Then there exists a sequence $\{f_n\}_{n<\omega}$ of $S(\mathcal{B}, \mathcal{I})$-measurable simple functions which converges pointwise to f. For any $n < \omega$, let g_n be a $\mathcal{B}$-measurable function such that f_n, g_n are $\mathcal{I}$-equivalent.

Let

$$(*)\qquad g(x) = \begin{cases} \limsup_{n\to\infty} g_n(x) & \text{if } \limsup_{n\to\infty} g_n(x) < +\infty \\ 0 & \text{otherwise.} \end{cases}$$

Then g is $\mathcal{B}$-measurable and we have

$$\{x : f(x) \neq g(x)\} \subseteq \bigcup_{n<\omega} \{x : f_n(x) \neq g_n(x)\} \in \mathcal{I}.$$

Thus f, g are $\mathcal{I}$-equivalent.

(2) $\Rightarrow$ (3). Let $A = \{x : f(x) \neq g(x)\}$. By the assumption we have $A \in \mathcal{I}$ and since g is $\mathcal{B}$-measurable, $f|X\setminus A$ is $\mathcal{B}$-measurable.

(3) $\Rightarrow$ (1). Denote $h = f|X\setminus A$. Let G be an arbitrary open subset of $\mathbf{R}$. Since h is $\mathcal{B}$-measurable, there is a set $B \in \mathcal{B}$ such that $h^{-1}(G) = B\setminus A$. So we have

$$f^{-1}(G) = (B\setminus A) \cup C \quad \text{where} \quad C = (f|A)^{-1}(G).$$

Let $D = (C\setminus B) \cup (A \cap B\setminus C)$. Since $A \in \mathcal{I}$ and $D \subseteq A$, therefore $D \in \mathcal{I}$. It is easily checked that $f^{-1}(G) = B \triangle D$. Thus f is $S(\mathcal{B}, \mathcal{I})$-measurable.

The proof is completed.

Let $M(\mathcal{I})$ denote the family of all $S(\mathcal{B}, \mathcal{I})$-measurable functions $f\colon X \to \mathbf{R}$.

For $\alpha < \omega_1$, let $M_\alpha(\mathcal{I})$ denote the family of all functions $f\colon X \to \mathbf{R}$ such that there is a function $g\colon X \to \mathbf{R}$, $\mathcal{B}$-measurable of class α (see [2], p. 280), and f, g are $\mathcal{I}$-equivalent. Furthermore, put $M_{\omega_1}(\mathcal{I}) = M(\mathcal{I})$.

Next, define $RZ(\mathcal{I})$ as the first of ordinal numbers $\alpha \leqq \omega_1$ such that $M_\alpha(\mathcal{I}) = M(\mathcal{I})$ (note that a similar classification for topological measure spaces was proposed by Zink in [10], [11]). We may interpret $RZ(\mathcal{I})$ as follows. Consider the lattice of all equivalence classes of the relation

$$f \sim g \text{ if and only if } f,\ g \text{ are } \mathcal{I}\text{-equivalent}$$

defined for all $\mathcal{B}$-measurable functions. Then $RZ(\mathcal{I})$ signifies the first of numbers α such that the above lattice consists of equivalence classes generated by all $\mathcal{B}$-measurable functions of the class α.

Proposition 2. $RZ(\mathcal{I}) > 0$.

Proof (comp. [11], th. 6). Suppose that $RZ(\mathcal{I}) = 0$. This implies that the closure of every open set is again open, i.e. X is extremally disconnected. Indeed, let U be an open set. By the supposition, there exists a continuous function $f\colon X \to \mathbf{R}$ such that $\chi_{\bar{U}}$, f are $\mathcal{I}$-equivalent. Then the set $\{x : f(x) \notin \{0, 1\}\}$ is open and belongs to $\mathscr{I}$, thus, in virtue of (iii), it is empty. Consequently, f is the characteristic function of some set V which must be open since f is continuous. Moreover, $U \triangle V \in \mathcal{O}$. Since $V \setminus \bar{U}$ is open and belongs to $\mathscr{I}$, therefore it is empty. Similarly, $U \setminus V$ is empty since it is an open subset of $\bar{U} \setminus V \in \mathscr{I}$. Hence we conclude that $U \subseteq V \subseteq \bar{U}$ and, consequently, $\bar{U} = V$. We have obtained a contradiction since a metric space X cannot simultaneously be perfect and extremally disconnected. Indeed, suppose that X has these two properties. Consider any point $x_0 \in X$. Choose a sequence $\{x_n\}_{n<\omega}$ of distinct points converging to x_0 and sequence $\{U_n\}_{n<\omega}$ of open pairwise disjoint sets such that for each $n < \omega$ we have $x_n \in U_n$ and U_n is contained in the ball with the centre x_0 and the radius $1/n$. Then the set $\overline{\bigcup_{n<\omega} U_{2n}}$ is open and easily seen to be equal to $\{x_0\} \cup \bigcup_{n<\omega} \overline{U_{2n}}$. Thus, it must contain almost all points x_n, which is impossible.

Proposition 3. *If $\mathscr{I}$, $\mathscr{J}$ are σ-ideals such that $\mathscr{I} \subseteq \mathscr{J}$, then $RZ(\mathscr{J}) \leqq RZ(\mathscr{I})$.*

The proof is analogous to that of Proposition 1.

We shall now study the relationships between $RT(\mathscr{I})$ and $RZ(\mathscr{I})$.

Theorem 2. $RT(\mathscr{I}) \leqq RZ(\mathscr{I}) \leqq RT(\mathscr{I}) + 3$.

Proof. Let $RT(\mathscr{I}) = \alpha$, $RZ(\mathscr{I}) = \beta$. First, we shall show that $\alpha \leqq \beta$. It is enough to consider the case $\beta < \omega_1$. Assume, for example, that β is even. Let $f \in S(\mathscr{B}, \mathscr{I})$. Then $A = B \triangle C$ where $B \in \mathscr{B}$, $C \in \mathscr{I}$. Since $RZ(\mathscr{I}) = \beta$, there is a function $g \in M_\beta(\mathscr{I})$ such that the set $D = \{x: \chi_B(x) \neq f(x)\} \in \mathscr{I}$. Let $E = g^{-1}(\{1\})$. We have $A = E \triangle (E \triangle A)$, $E \in F_\beta$, $E \triangle A \in \mathscr{I}$ (because $E \triangle A = (E \triangle B) \triangle C$, $E \triangle B \subseteq D \in \mathscr{I}$, $C \in \mathscr{I}$). Hence $\alpha \leqq \beta$. If α is odd, the proof is analogous. We shall now show that $\beta \leqq \alpha + 3$. Consider the non-trivial case $\alpha < \omega_1$ only. Let $f \in M(\mathscr{I})$. If $f = \chi_A$ where $A \in S(\mathscr{B}, \mathscr{I})$, then $A = B \triangle C$, $B \in F_\alpha$, $C \in \mathscr{I}$ by the definition of α. Put $g = \chi_B$; then g is $\mathscr{B}$-measurable of class $\alpha + 1$ (comp. [2], p. 281) and f, g are $\mathscr{I}$-equivalent. Thus, $f \in M_{\alpha+1}(\mathscr{I})$. The same happens when f is a simple function. In the general case, choose a sequence $\{f_n\}_{n<\omega}$ of $S(\mathscr{B}, \mathscr{I})$-measurable functions converging to f. For any $n < \omega$, let g_n be a $\mathscr{B}$-measurable function of the class $\alpha + 1$, such that f_n, g_n are $\mathscr{I}$-equivalent. Let g be defined by formula (*) given in the proof of Theorem 1. Define $E = \{x: \limsup_{n \to \infty} g_n(x) = +\infty\}$. It is easy to verify that E belongs to $F_{\alpha+2}$ or $G_{\alpha+2}$. Moreover, $g \mid X \setminus E$ is $\mathscr{B}$-measurable of class $\alpha + 3$ since it can be obtained by starting from functions of the clas $\alpha + 1$ and using twice the operation of pointwise convergence (comp. [2], p. 284). This implies that g is of class $\alpha + 3$, too. Since f, g are $\mathscr{I}$-equivalent, we conclude that $f \in M_{\alpha+3}(\mathscr{I})$. Thus $\beta \leqq \alpha + 3$.

Corollary 0. *The conditions $RT(\mathcal{J}) = \omega_1$, $RZ(\mathcal{J}) = \omega_1$ are equivalent.*

Below, we shall prove that, for some σ-ideals, we can obtain more precise estimations of $RZ(\mathcal{J})$ than the second inequality in Theorem 2. We shall need the following proposition which generalizes the well-known characterization of functions possessing the Baire property (see[2], p. 306). This result was already published in the cases $\alpha = 0$ ([3], p. 408) and $\alpha = 1$ ([4], 8 (ii)). In the general case, the proof is analogous and will be omitted.

Proposition 4. *Let $\alpha < \omega_1$ and assume that*

$$S(\mathcal{B}, \mathcal{J}) = \{B \triangle A \colon B \in F_\alpha,\ A \in \mathcal{J}\}.$$

A function $f\colon X \to \mathbf{R}$ is $S(\mathcal{B}, \mathcal{J})$-measurable if and only if there is a set $A \in \mathcal{J}$ such that the restriction $f|X \setminus A$ is $\mathcal{B}$-measurable of class α.

Theorem 3. *Let $\alpha < \omega_1$. Assume that each set $A \in \mathcal{J}$ is included in a Borel set $B \in \mathcal{J}$ of the additive class α. Then $RZ(\mathcal{J}) \leqq \max(\alpha, RT(\mathcal{J}))$.*

Proof. Let $\beta = \max(\alpha, RT(\mathcal{J}))$. It is enought to show that $M(\mathcal{J}) \subseteq M_\beta(\mathcal{J})$. Let $f \in M(\mathcal{J})$. Since $RT(\mathcal{J}) \leqq \beta$, therefore $S(\mathcal{B}, \mathcal{J}) = \{B \triangle A \colon B \in F_\beta,\ A \in \mathcal{J}\}$. In virtue of Theorem 1, there is a set $A \in \mathcal{J}$ such that the function $f|X \setminus A$ is $\mathcal{B}$-measurable of class β. It follows from the assumption that there is a Borel set $B \in \mathcal{J}$ of the additive class β, such that $A \subseteq B$. The function $f|X \setminus B$ is $\mathcal{B}$-measurable of class β and the set $X \setminus B$ is of the multiplicative classs β; thus (see [2], p. 341), there exists an extension of $f|X \setminus B$ to a function $g\colon X \to \mathbf{R}$ which is $\mathcal{B}$-measurable of class β. Since $\{x\colon f(x) \neq g(x)\} \subseteq B \in \mathcal{J}$, therefore f, g are $\mathcal{J}$-equivalent. Hence $f \in M_\beta(\mathcal{J})$.

Example 1. It follows from Theorem 3 that $RZ(\mathcal{K}) \leqq 1$. Thus, by Proposition 2, we have $RZ(\mathcal{K}) = 1$.

Example 2. Theorem 3 easily implies that $RZ(\mathcal{L}) \leqq 2$, which gives the well-known property that each Lebesgue measurable function is equal almost everywhere to a function of the Baire class 2. Consider a Borel set $E \subseteq \mathbf{R}$ which is of positive measure on every interval and whose complement has the same property. Then $\chi_E \in M(\mathcal{L}) \setminus M_1(\mathcal{L})$. Thus $RZ(\mathcal{L}) = 2$.

Example 3. Consider the σ-ideal $\mathcal{K} \cap \mathcal{L}$ of all meager subsets o $\mathbf{R}$ of measure zero. Observe that

$$(**) \qquad S(\mathcal{B}, \mathcal{K} \cap \mathcal{L}) = S(\mathcal{B}, \mathcal{K}) \cap S(\mathcal{B}, \mathcal{L}).$$

Indeed, the inclusion "$\subseteq$" is obvious. In order to prove the converse inclusion, assume that $A \in S(\mathcal{B}, \mathcal{K}) \cap S(\mathcal{B}, \mathcal{L})$. Then A has the Baire property, hence it can be expressed in the form $A = B \cup C$ where B is of type G_δ and $C \in \mathcal{K}$. We may assume that B, C are disjoint. Since $S(\mathscr{B}, \mathscr{K}) \cap S(\mathscr{B}, \mathscr{L})$ is a σ-field and A, B belong to it $C = A \setminus B$ also belongs to this field. Then C is Lebesgue measurable, so it can be expressed in the form $C = D \cup E$ where D is of type F_σ and $E \in \mathscr{L}$. We may assume that D, E are disjoint. Thus we have $A = (B \cup D) \triangle E$, $B \cup D \in \mathscr{B}$, $E \in \mathscr{K} \cap \mathscr{L}$. In consequence, the inclusion "$\supseteq$" in (**) holds. Moreover, $B \cup D$ is of type $F_{\sigma\delta}$, thus it turns out that $RT(\mathscr{K} \cap \mathscr{L}) \leqq 2$. The inequality $RT(\mathscr{K} \cap \mathscr{L}) \geqq 2$ will be proved when we find a set $E \in \mathscr{B}$ such that $D \triangle E \notin \mathscr{K} \cap \mathscr{L}$ for each set D of type F_σ. Let A, $B \in \mathscr{B}$ be disjoint sets such that $A \cup B = (0, 1)$, $A \in \mathscr{K}$, $B \in \mathscr{L}$ (see [9]). Put $E = (A + 1) \cup B$ where $A + 1 = \{a + 1 \colon a \in A\}$. Of course, $E \in \mathscr{B}$. Let D be an arbitrary set of type F_σ. If $D \cap (0, 2) \in \mathscr{L}$, then $(A + 1) \setminus D \notin \mathscr{L}$, and so, $D \triangle E \notin \mathscr{K} \cap \mathscr{L}$. Next, consider the case $D \cap (0, 2) \notin \mathscr{L}$. If $D \cap (0, 1) \notin \mathscr{L}$, then $D \setminus B \notin \mathscr{L}$, and so, $D \triangle E \notin \mathscr{K} \cap \mathscr{L}$. If $D \cap (0, 1) \in \mathscr{L}$, then $D \cap (0, 1)$, being a set of type F_σ and of measure zero is meager. Thus $B \setminus D \notin \mathscr{K}$ and, consequently, $D \triangle E \notin \mathscr{K} \cap \mathscr{L}$. Hence, we have proved that $RT(\mathscr{K} \cap \mathscr{L}) = 2$. Observe now that Theorems 2, 3 easily imply $RZ(\mathscr{K} \cap \mathscr{L}) = 2$.

Problem. Do there exist σ-ideals $\mathscr{I}$, $\mathscr{J}$ such that $RZ(\mathscr{I}) = RT(\mathscr{I}) + 2$, $RZ(\mathscr{J}) = RT(\mathscr{J}) + 3$?

We shall now show that Mauldin's results concerning generalized Baire systems have some connections with $RZ(\mathscr{I})$, $RT(\mathscr{I})$.

Let $C_{\mathscr{I}}$ denote the family of all functions $f\colon X \to \mathbf{R}$ whose sets of points of discontinuity belong to $\mathscr{I}$. Define the Baire system $\Phi_\alpha(\mathscr{I})$, $\alpha \leqq \omega_1$, as follows (see [5]): let $\Phi_0(\mathscr{I}) = C_{\mathscr{I}}$ and, for each α, $0 < \alpha \leqq \omega_1$, let $\Phi_\alpha(\mathscr{I})$ be the family of all pointwise limits of sequences with terms from $\bigcup_{\nu < \alpha} \Phi_\nu(\mathscr{I})$. The first of ordinal numbers α such that $\Phi_{\omega_1}(\mathscr{I}) = \Phi_\alpha(\mathscr{J})$ will be called the Baire order of $\mathscr{I}$. Observe that the Baire order is always positive since if A denotes any countable dense subset of X, then we have $\chi_A \in \Phi_1(\mathscr{I}) \setminus \Phi_0(\mathscr{I})$.

Let Φ_α, $\alpha \leqq \omega_1$, denote the usual Baire system (defined analogously as above by taking Φ_0 equal to the family of all continuous functions).

Denote by $\mathscr{I}_1$ the σ-ideal of all sets from $\mathscr{I}$ which have supersets from $\mathscr{I}$, of type F_σ.

Theorem I ([5]). *Let* $0 < \alpha < \omega_1$. *Then* $f \in \Phi_\alpha(\mathscr{I})$ *if and only if there exists* $g \in \Phi_\alpha$ *such that* f, g *are* $\mathscr{I}_1$*-equivalent.*

Theorem II ([6]). *The Baire order of* $\mathscr{L}$ *is* ω_1.

Proposition 5. *Let* $0 < \alpha \leqq \omega_1$. *Then*

$$\Phi_\alpha(\mathscr{I}) = \begin{cases} M_\alpha(\mathscr{I}_1) & \text{if is finite or equal to } \omega_1 \\ M_{\alpha+1}(\mathscr{I}_1) & \text{otherwise.} \end{cases}$$

Proof. For $\alpha < \omega_1$, the assertion follows from Theorem I and the known fact that g belongs to Φ_α if and only if it is $\mathscr{B}$-measurable of class α or $\alpha + 1$ when α is finite or infinite, respectively (see [2], p. 299). Next, observe that

$$\Phi_{\omega_1}(\mathscr{I}) = \bigcup_{\nu<\omega_1} \Phi_\nu(\mathscr{I}) = \bigcup_{\nu<\omega_1} M_\nu(\mathscr{I}_1) = M_{\omega_1}(\mathscr{I}_1),$$

hence the assertion holds for $\alpha = \omega_1$, as well.

Remark. Since, for any function, its set of points of discontinuity is of type F_σ, we always have $C_{\mathscr{I}} = C_{\mathscr{I}_1}$. Thus the Baire systems and orders of $\mathscr{I}$, $\mathscr{I}_1$ are identical.

Corollary 1. *If $\mathscr{I} \subseteq \mathscr{L}_1$, then $RZ(\mathscr{I}) = RT(\mathscr{I}) = \omega_1$.*

Proof. By Theorem II and Proposition 5, we have $RZ(\mathscr{I}_1) = \omega_1$. Thus, by Proposition 3, $RZ(\mathscr{I}) = \omega_1$, and Corollary 0 gives $RT(\mathscr{I}) = \omega_1$.

Since $RZ(\mathscr{K}) = 1$ (comp. Example 1) and $\mathscr{K} = \mathscr{K}_1$, therefore Proposition 5 implies the known result:

Corollary 2 ([1]). *The Baire order of $\mathscr{K}$ is* 1.

Remarks. (a) From Theorem 2 and Miller's result ([7]) stating that there is a σ-ideal $\mathscr{I}$ of subsets of 2^ω with arbitrary $RT(\mathscr{I})$, we conclude that there are σ-ideals $\mathscr{I}$ with arbitrarily high $RZ(\mathscr{I})$. The question may be posed whether, for each $\alpha < \omega_1$, there exists a σ-ideal $\mathscr{I}$ such that we have exactly $RZ(\mathscr{I}) = \alpha$. Theorems 2, 3 suggest a method for seeking such σ-ideal. Miller's construction is expected to be useful as well.

(b) Mauldin in [6] asked whether, for each $\alpha < \omega_1$, there exists a σ-ideal with the Baire order α. One may associate this problem with that posed in (a) and try to solve it in the affirmative by using Proposition 5.

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КЛАССИФИКАЦИЯ σ-ИДЕАЛОВ

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Резюме

Пусть $\mathscr{I}$ — σ-идеал множеств в полном, сепарабельном, совершенном метрическом пространстве. Пусть $S(\mathscr{B}, \mathscr{I})$ — σ-поле, порожденное борелевскими множествами и множествами из $\mathscr{I}$. В статье исследуются ординалы $RT(\mathscr{I})$, $RZ(\mathscr{I})$, описывающие специальные свойства $S(\mathscr{B}, \mathscr{I})$-измеримых множеств и функций. Приведены примеры и указама связ с обобщенными системами Бэра.