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DISTRIBUTION OF IRREGULAR PAIRS

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(*Communicated by Gejza Wimmer*)

ABSTRACT. All irregular primes up to 12 million were computed in [BUHLER, J.—CRANDALL, R.—ERNVALL, R.—METSÄNKYLÄ, T. SHOKROLLAHI, M. A.: *Irregular primes and cyclotomic invariants to* 12 *million*, J. Symbolic Comput. **31** (2001), 89–96]. K. Woolridge tried to examin the fractions $\frac{2a}{p}$ in his Ph.D. thesis in 1975, but only a few irregular primes were known in that time. In this paper new data lead to the compelling statistical view to the fractions $\frac{2a}{p}$ and their distribution.

Let us denote $B_n = \frac{N_n}{D_n}$ the Bernoulli numbers in lowest terms. A pair of integers $(p, 2a)$ is said to be an *irregular pair* if a is an integer satisfying $0 < 2a < p-1$, and p is a prime (so-called *irregular prime*) dividing B_{2a} (resp. N_{2a} in $\mathbb{Z}$).

It is a very time consuming problem for computers to decide whether a given prime is irregular or not. "Smallest" irregular primes together with their characteristics (including all possible a's with the above-mentioned property) are tabulated (historically) thanks to e.g. Kummer (the idea of regular and irregular primes originates from him), Vandiver, Lehmers, Nicol, Selfridge, Kobelev, Johnson, Wagstaff, Tauner, Buhler, Crandall, Sompolski, Ernvall, Metsänkylä, Shokrollahi, etc.

Let us consider irregular pairs $(p, 2a)$ and fractions $\frac{2a}{p}$. Clearly, $0 < \frac{2a}{p} < 1$. One may expect that because Bernoulli numbers and their numerators grow rapidly fast, then "more" such pairs will be of the form where $2a$ is greater than for example $\frac{p}{2}$. But this is probably not true. Taking some upper bound for primes (at present $< 12M$), one can compute how many irregular pairs is of form where $\frac{2a}{p}$ is smaller or greater than $\frac{1}{2}$. Let us denote Y_1 the number of the formers, resp. Y_2 the latters. On Picture 1 we can see the trend of the ratio

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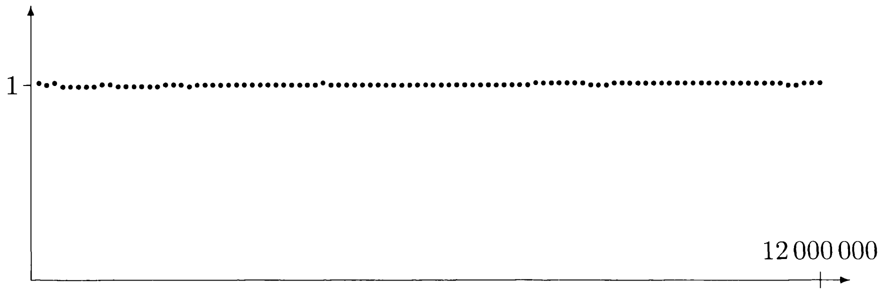
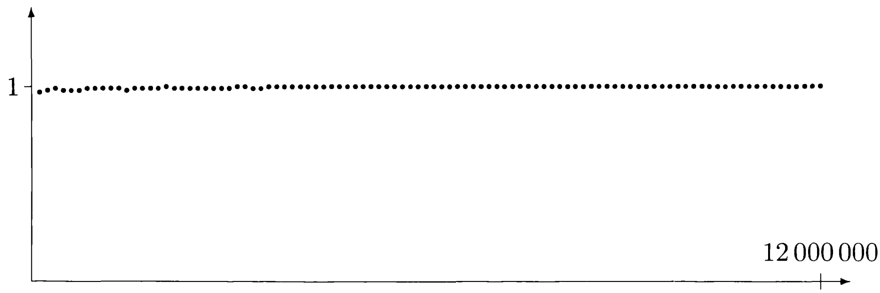
$Y_1 : Y_2$ when changing the limit for maximal irregular prime which is used. The scale of both axes is linear and we can see that the ratio $Y_1 : Y_2$ is very close to 1 even if we count irregular pairs with $p < 120\,000$ (the first point).

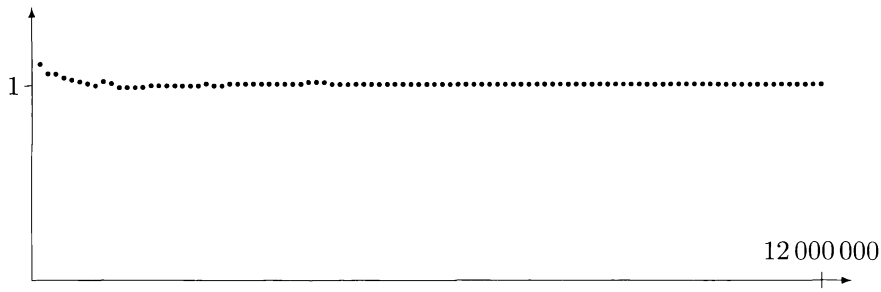


PICTURE 1. Ratio $Y_1 : Y_2$.

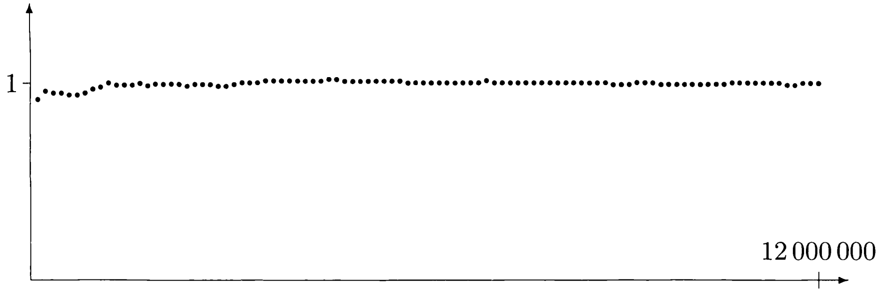
Situation is the same even if we use only irregular primes with constant index of irregularity (index of irregularity of the prime p is the number of different irregular pairs including the given prime p). We can see graphs for index 1, 2 and 3:



PICTURE 2. Ratio $Y_1 : Y_2$ for index of irregularity 1.



PICTURE 3. Ratio $Y_1 : Y_2$ for index of irregularity 2.

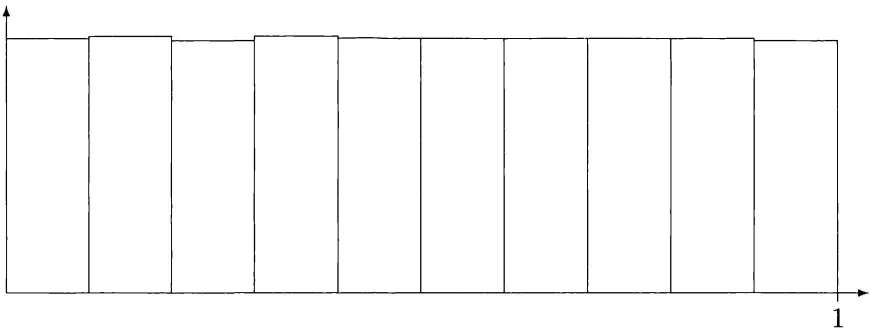


PICTURE 4. Ratio $Y_1 : Y_2$ for index of irregularity 3.

This analysis is of course only a special case of more general one. Let us denote K some integer greater than 1, and $I_i = \left(\frac{i-1}{K}, \frac{i}{K}\right)$ the subinterval of the real interval $\langle 0,1\rangle$ (for $i = 1, 2, \dots, K$). To avoid conflicts at the edge-points, let us consider for example K in the form $2^\alpha 5^\beta$ — it means, K is prime to all tested prime numbers. Then it is not hard to show that $\frac{2a}{p} \neq \frac{i}{K}$ for all "possible" i's and p's. Furthermore,

$$Y_i = \#\left\{(p, 2a) : \ \tfrac{2a}{p} \in I_i\right\}, \qquad y_i = \frac{Y_i}{\sum_{i=1}^{K} Y_i}.$$

We can take the values Y_i's (or y_i's) as depending ones on the index i the index of subinterval I_i, in which the fractions $\frac{2a}{p}$ lie. For example, we can see the situation for $K = 10$ — the histogram of occurrences of fractions $\frac{2a}{p}$ in intervals I_i (again, the scaling factor of both axes is linear). The horizontal axis displays the interval I_i, and the vertical one displays the value Y_i.



PICTURE 5. Histogram for $K = 10$.

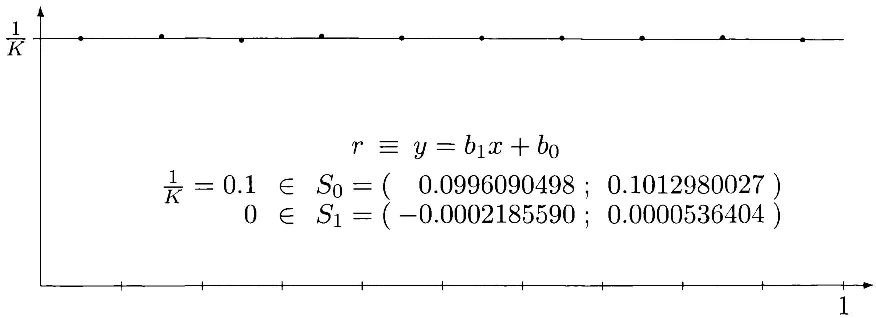
It is important to test the significance of the hypothesis that this distribution is uniform. We will use the standard statistical methods for the construction of the regression line ($r \equiv y = b_1 x + b_0$) for obtained (relative) sizes ([3; 194 196]).

Our hypothesis says that $r \equiv y = 0 \cdot x + \frac{1}{K}$. We will construct 95% confidence intervals $\mathcal{S}_0 = (b_0-\omega_0, b_0+\omega_0)$, $\mathcal{S}_1 = (b_1-\omega_1, b_1+\omega_1)$ for parameters b_0, b_1, respectively. The facts that this regression line possesses the zero slope and the intercept equal to $\frac{1}{K}$ are not rejected at significance level 0.05.

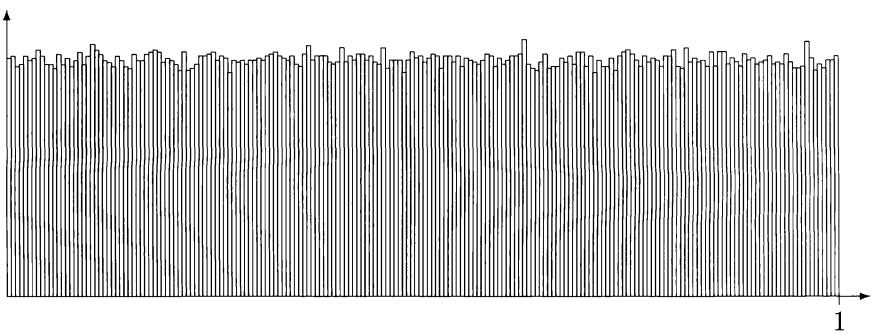
We can demostrate this fact on Pictures 6 and 8 (situations for 10 and 200 subintevals). Similar results will be obtained (again) if we will consider only irregular pairs with primes with given index of irregularity. However, Pictures 5 8 represent the situations for all irregular pairs $(p, 2a)$ where $p < 12\,000\,000$.

We reject the hypothesis about the uniform distribution of studied data using Pearson χ^2 goodness-of-fit test at significance level 0.05. For $K = 200$, we have $\chi^2 = 248.236 > 232.912 = \chi^2_{199}(0.95)$.

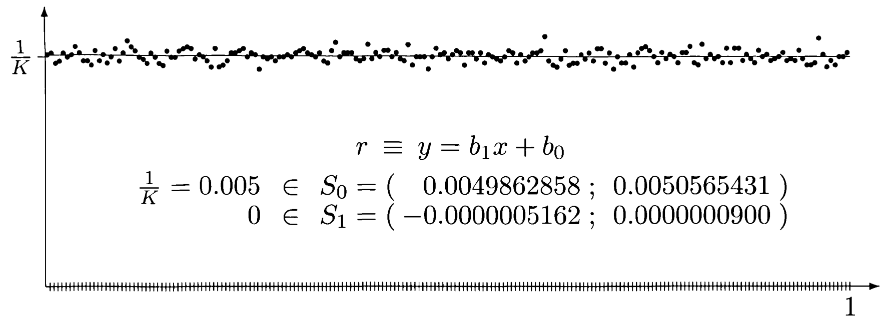
Pearson χ^2 goodness-of-fit test seems to be strong although objective tool for testing. As we can see on Picture 9, uniform distribution is predominantly not rejected, and if it is rejected (i.e. $\chi^2 > \chi^2_{K-1}(0.95)$), then the fraction $\frac{\chi^2}{\chi^2_{K-1}(0.95)}$ is not much greater than 1. Picture 9 shows the ratio $\frac{\chi^2}{\chi^2_{K-1}(0.95)}$ for all irregular pairs with primes $< 12M$, and the number of intervals $K = 5, 10, 15, \ldots, 1000$.



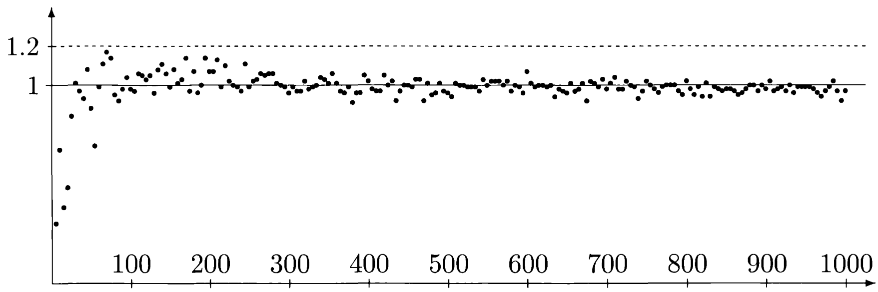
PICTURE 6. Regression line for $K = 10$.



PICTURE 7. Histogram for $K = 200$.



PICTURE 8. Regression line for $K = 200$.



PICTURE 9. Ratio $\frac{\chi^2}{\chi^2_{K-1}(0.95)}$ for $K = 5, 10, \dots, 1000$.

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