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ON THE STRUCTURE OF THE SPACE OF CONTINUOUS MAPS WITH ZERO TOPOLOGICAL ENTROPY

ROMAN HRIC

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ABSTRACT. Every continuous map of a compact interval into itself has every trajectory approximable by cycles, or is ε-chaotic for some $\varepsilon > 0$. For some functions (called stable) the first property is stable in the sense that perturbed functions can be only little chaotic whenever the perturbation is small. In the paper, we give the topological structure (with respect to stable functions) of the space of continuous maps of an interval with zero topological entropy.

Introduction

Let $(C(I,I),\varrho)$ be the space of continuous functions $I \to I$ with the uniform metric ϱ, where I is a compact real interval. For $f \in C(I,I)$, f^n denotes the nth iterate of f; the orbit of an interval $J \subseteq I$ is the set $\{f^n(J);\ n = 0,1,2,\dots\}$, sometimes we consider the orbit as $\bigcup\limits_{n=0}^{\infty} f^n(J)$; $\omega_f(x)$ denotes the ω-limit set of the trajectory $\{f^n(x)\}_{n=0}^{\infty}$ of x. A periodic point of f is any $x \in I$ such that $f^n(x) = x$ for some $n \in \mathbb{N}$, the smallest such n is called the period of x; analogously for an interval $J \subseteq I$. The set of all periodic points of f is denoted by $\operatorname{Per}(f)$.

From the viewpoint of applicability in mathematical modelling, remarkable attention is given to asymptotical properties of trajectories, especially to regular asymptotical behavior like asymptotical periodicity. Unfortunately, an arbitrarily small perturbation of the map can destroy the last-named property. In connection with this, the notion of stability introduced in [4] (see Def. below)

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is significant. By [3], [5] (cf. also [4]), every $f \in C(I,I)$ has just one of the following two properties:

A. f is non-chaotic, every trajectory of f is approximable by cycles, i.e.,

$$\forall x\in I\ \forall \varepsilon>0\ \exists p\in \operatorname{Per}(f):\quad \limsup_{n\to\infty} |f^n(x)-f^n(p)| < \varepsilon. \tag{1}$$

B. f is ε-chaotic, i.e.,
$\exists \varepsilon>0\ \exists S\neq\emptyset$ a perfect set $\forall x,y\in S,\ x\neq y\ \forall p\in \operatorname{Per}(f)$:

$$\limsup_{n\to\infty}|f^n(x)-f^n(y)| > \varepsilon,$$
$$\liminf_{n\to\infty} |f^n(x)-f^n(y)| = 0,$$
$$\limsup_{n\to\infty}|f^n(x)-f^n(p)| > \varepsilon.$$

Note that ε-chaos is weaker than positive topological entropy.

There is a natural question about the size (in the topological sense) of the set of stable functions as a subset of the space of functions with zero topological entropy (or, equivalently, functions with no cycle of period $\neq 2^n$). Now we can recall the definition of a stable function and present the main result with some consequences.

DEFINITION. ([4]) *A* non-chaotic map $f \in C(I,I)$ is called *stable* if for any $\varepsilon > 0$, any $g \in C(I,I)$ sufficiently near to f (relative to the uniform metric) has every trajectory ε-approximable by cycles (i.e., for any x there is a $p \in \operatorname{Per}(g)$ such that (1) is true for f replaced with g).

THEOREM. *The set of all functions f in $C(I,I)$ with the following two properties*

(i) $\operatorname{Per}(f)$ *is nowhere dense,*
(ii) *all trajectories of f are asymptotically periodic (i.e., for any $x \in I$ there is a point $p \in \operatorname{Per}(f)$ such that $\lim_{n\to\infty} |f^n(x)-f^n(p)| = 0$)*

is dense in the set $Z(I,I) \subseteq C(I,I)$ of maps with zero topological entropy.

COROLLARY 1. *The set $S(I,I) \subseteq C(I,I)$ of stable functions is dense in $Z(I,I)$.*

P r o o f . This simply follows from Theorem by Theorem 2 (below). □

Now let ε be an arbitrary positive number. We put $S_\varepsilon(I,I) = \{f \in Z(I,I);\ \exists \delta>0\ \forall g\in C(I,I):$ if $\varrho(f,g) < \delta$, then g has all trajectories ε-approximable by cycles$\}$. From Corollary 1, we have:

COROLLARY 2. *$S_\varepsilon(I,I)$ is open and dense in $Z(I,I)$ for any $\varepsilon > 0$.*

Note that $Z(I,I)$ is complete, and therefore, a second category space.

COROLLARY 3. *$S(I,I)$ is a residual G_δ subset of $Z(I,I)$.*

Now we define:

$$F(I,I) = \{f \in C(I,I)\,;\ f \text{ has cycles exactly of the periods } 1,2,\dots,2^n \text{ for some } n\}\,,$$

$$N(I,I) = \{f \in C(I,I)\,;\ f \text{ is non chaotic}\}\,.$$

COROLLARY 4. *$N(I,I) \setminus F(I,I)$ is residual in $Z(I,I)$; $F(I,I)$ and $Z(I,I) \setminus F(I,I)$ are first category sets in $Z(I,I)$.*

Proof. By Theorem 1 (below), we obtain that $F(I,I)$ is a first category set in $Z(I,I)$. Corollary 3 finishes the proof if we realize that $S(I,I) \subseteq N(I,I)$. □

Preliminaries

THEOREM 1. ([1]) *Let $f \in C(I,I)$ have a point of period n. Then there is a neighborhood U of f inside $C(I,I)$ such that for all $g \in U$ and every k to the right of n in the Šarkovskij ordering g has a point of period k.*

THEOREM 2. ([4]) *A non-chaotic map $f \in C(I,I)$ is stable if and only if the following two conditions are satisfied:*

- (i) *$\mathrm{Per}(f)$ is nowhere dense (or, equivalently, $\mathrm{Per}(f)$ is a first category set, or, equivalently, $\mathrm{Per}(f)$ contains no interval);*
- (ii) *for any infinite ω-limit set $\omega_f(x)$ and any positive integer n there is a system $I(n) = \{I(n,i)\,;\ 1 \le i \le 2^n\}$ of periodic intervals such that $\omega_f(x) = \bigcap_{n=1}^{\infty} \left(\bigcup I(n)\right)$.*

Note that there is a non-chaotic map for which condition (ii) is not valid, so it is not equivalent to non-chaos.

LEMMA. *Let $f \in Z(I,I)$ have at least one infinite ω-limit set. Then for any $n = 0,1,2,\dots$ there exists a system $\mathcal{S}$ of orbits of compact periodic intervals of period $m = 2^n$ such that*

- (i) *the interiors of orbits (as subsets of I) in $\mathcal{S}$ are pairwise disjoint,*
- (ii) *each orbit in $\mathcal{S}$ consists of intervals with pairwise disjoint interiors,*
- (iii) *$\mathcal{S}$ covers the union of all infinite ω-limit sets of f.*

Proof. Let $\omega_f(x)$ be infinite. Then there is an orbit $J(n) = \{J(n,1),\dots,J(n,m)\}$ of pairwise disjoint compact periodic intervals of period m covering $\omega_f(x)$ (see [4]). Now consider another infinite $\omega_f(y)$ and an orbit $K(n) = \{K(n,1),\dots,K(n,m)\}$ of compact periodic intervals of period m covering

$\omega_f(y)$. Assume $\left(\bigcup J(n)\right) \cap \left(\bigcup K(n)\right) \neq \emptyset$. We suppose $J(n,1)$ and $K(n,1)$ to be the left-hand end-interval of $J(n)$ and $K(n)$ respectively. We easily have $J(n,1) \cap K(n,1) \neq \emptyset$. Now we will show that $\{J(n,1) \cup K(n,1), \dots, J(n,m) \cup K(n,m)\}$ is an orbit of pairwise disjoint compact periodic intervals of the period m.

Suppose we have $\left(J(n,i) \cup K(n,i)\right) \cap \left(J(n,j) \cup K(n,j)\right) \neq \emptyset$ for some $i \neq j$. Then clearly $K(n,i) \cap J(n,j) \neq \emptyset$ or $J(n,i) \cap K(n,j) \neq \emptyset$. We can assume without loss of generality that $K(n,i) \cap J(n,j) \neq \emptyset$, and $K(n,1)$ contains the right-hand end-point of $J(n,1) \cap K(n,1)$. Then $\emptyset \neq f^{m-j+1}\left(K(n,i) \cap J(n,j)\right) \subseteq J(n,1)$, and, at the same time, $f^{m-j+1}\left(K(n,i) \cap J(n,j)\right) \subseteq K(n,k)$ for suitable $k \equiv i + m - j + 1 \pmod{m}$, and hence $k \neq 1$. But $J(n,1) \cap K(n,k) = \emptyset$, and this is a contradiction.

For each infinite ω-limit set of f consider an orbit of pairwise disjoint compact periodic intervals of period m covering this set. These orbits create a system on which, for a given equivalence, we can make a new system $\mathscr{S}$ taking for each equivalence class the orbit given by the closure of the union of all left-hand end-intervals in the class. It is easy to see that there is an equivalence for which $\mathscr{S}$ satisfies the statement of Lemma. □

Proof of the main result

We give the proof in two steps. The first one is a construction of a function h near to f without infinite ω-limit sets. The second one is then "making" $\mathrm{Per}(h)$ to be nowhere dense. Since the first step is in detail a little unclear, we give a short indication of it.

By Lemma, for f we take a system $\mathscr{S}$ with m sufficiently large. Then every $J(n)$ in $\mathscr{S}$ has a sufficiently small interval. We take its pre-image in $J(n)$ and change f to h on it, h being constant on the middle of this interval and linearly completed on its margins. On the other intervals of $J(n)$, we change f to h so that we "shrink down" a little every rectangular $J(n,i) \times f\left(J(n,i)\right)$ with a part of the graph of f to its centre and on the margins of $J(n,i)$ we complete h linearly. All this we can do in such a way that every point in the interior of $\bigcup J(n)$ will be attracted to the unique cycle inside of $\bigcup J(n)$.

P r o o f. Let $f \in Z(I,I)$, and ε be an arbitrary positive number. f is uniformly continuous on I, i.e.,

$$\exists \delta{>}0 \ \ \forall x,y{\in}I: \quad |x-y| < \delta \implies |f(x)-f(y)| < \frac{\varepsilon}{4}\,. \tag{2}$$

Now we can choose $t \in (0,1)$ so that

$$\frac{1-t}{2t} \operatorname{diam} I < \delta \,, \tag{3}$$

$$(1-t) \cdot \operatorname{diam} I < \frac{\varepsilon}{4} \,, \tag{4}$$

and $m = 2^n$ so that

$$\frac{1}{m} \operatorname{diam} I < \frac{\varepsilon}{2} \,. \tag{5}$$

Now we are going to construct a function h with all points asymptotically periodic satisfying $\varrho(f,h) < \frac{\varepsilon}{2}$.

If f already has all points asymptotically periodic, then we define $h = f$ on I. Otherwise we will construct h in the following way. For f, let $\mathscr{S}$ be a system from Lemma, and m be as in (5).

We put

$$h(x) = f(x) \qquad \text{for} \quad x \in I \setminus \bigcup_{J(n) \in \mathscr{S}} \big(\operatorname{Int} J(n,1) \cup \cdots \cup \operatorname{Int} J(n,m)\big) \,.$$

Now let $\{J(n,1), \dots, J(n,m)\}$ be an arbitrary orbit from $\mathscr{S}$. Denote

$$J(n,i) = [a_i, b_i] \qquad \text{for} \quad i = 1, \dots, m$$

and take

$$c_i = \frac{(1+t)a_i + (1-t)b_i}{2} \qquad \text{and} \qquad d_i = \frac{(1-t)a_i + (1+t)b_i}{2} \,.$$

We have $a_i < c_i < d_i < b_i$. Since $J(n,1), \dots, J(n,m)$ have pairwise disjoint interiors, we can choose one of them, $J(n,j)$, with $\operatorname{diam} f\big(J(n,j)\big) \le \frac{\operatorname{diam} I}{m} < \frac{\varepsilon}{2}$. Now we will finish the definition of h:

$$h(x) = \begin{cases} \dfrac{a_{j+1} + b_{j+1} - 2f(a_j)}{(1-t)(b_j - a_j)}(x - a_j) + f(a_j) & \text{for } x \in (a_j, c_j)\,, \\[2ex] \dfrac{a_{j+1} + b_{j+1}}{2} & \text{for } x \in [c_j, d_j]\,, \\[2ex] \dfrac{a_{j+1} + b_{j+1} - 2f(b_j)}{(1-t)(b_j - a_j)}(b_j - x) + f(b_j) & \text{for } x \in (d_j, b_j)\,, \end{cases}$$

and for each $i = 1, \dots, m$, $i \neq j$

$$h(x) = \begin{cases} \dfrac{(a_i + b_i - 2x)f(a_i) + (a_{i+1} + b_{i+1})(x - a_i)}{(b_i - a_i)} & \text{for } x \in (a_i, c_i)\,, \\[2ex] \dfrac{(1-t)(a_{i+1} + b_{i+1})}{2} + tf\left(\dfrac{(t-1)(a_i + b_i) + 2x}{2t}\right) & \text{for } x \in [c_i, d_i]\,, \\[2ex] \dfrac{(2x - a_i - b_i)f(b_i) + (a_{i+1} + b_{i+1})(b_i - x)}{(b_i - a_i)} & \text{for } x \in (d_i, b_i)\,, \end{cases}$$

where we take $a_{m+1} = a_1$, $b_{m+1} = b_1$, and, in the following, also $c_{m+1} = c_1$, $d_{m+1} = d_1$, $J(n, m+1) = J(n, 1)$.

One can easily see that $h \in C(I, I)$.

$\omega_h(x)$ is a cycle for each $x \in I$: we have $h\big(J(n,i)\big) \subseteq J(n, i+1)$ for $i = 1, \dots, m$. So, if h has an infinite ω-limit set, then it is contained in some $\bigcup J(n) \in \mathscr{S}$. $h\big([c_i, d_i]\big) \subseteq [c_{i+1}, d_{i+1}]$, h is linear on $[a_i, c_i]$ and $[d_i, b_i]$, and $h^m\big([c_i, d_i]\big)$ is a one-point set. Using this we easily obtain that every point in $\operatorname{Int} J(n,i)$ is attracted to the unique cycle of period m in $\bigcup_{i=1}^{m} \operatorname{Int} J(n)$. The case of end-points is then trivial.

$\varrho(f, h) < \frac{\varepsilon}{2}$ because of the following: $f(x) \neq h(x)$ is possible only on some $J(n) \in \mathscr{S}$. Since $h\big(J(n,i)\big) \subseteq J(n, i+1)$ and $f(J(n,i)) \subseteq J(n, i+1)$, we could have $|f(x) - h(x)| \geq \frac{\varepsilon}{2}$ only if $\operatorname{diam} J(n, i+1) \geq \frac{\varepsilon}{2}$. Let $J(n,i)$ be such an interval. We will distinguish two cases:

(i) $x \in [c_i, d_i]$. Since, by (3), $\left| \dfrac{(t-1)(a_i + b_i) + 2x}{2t} - x \right| < \delta$, we have

$$|h(x) - f(x)|$$
$$= \left| \frac{(1-t)(a_{i+1} + b_{i+1})}{2} - (1-t)f(x) + tf\left(\frac{(t-1)(a_i + b_i) + 2x}{2t} \right) - tf(x) \right| < \frac{\varepsilon}{2}$$

by (2) and (4).

(ii) $x \in [a_i, c_i)$, and analogously, $x \in (d_i, b_i]$. $|h(x) - f(x)| \leq |h(x) - h(a_i)| + |h(a_i) - f(a_i)| + |f(a_i) - f(x)| < \frac{\varepsilon}{2}$ by (2) and (4), using $h(a_i) = f(a_i)$ and $|c_i - a_i| < \delta$.

Now, if $\operatorname{Per}(h)$ is nowhere dense, h has all properties required in Theorem. Therefore we suppose $\operatorname{Per}(h)$ is not nowhere dense. We are going to construct a function F with all points asymptotically periodic and moreover $\operatorname{Per}(F)$ nowhere dense satisfying $\varrho(f, F) < \varepsilon$.

Since h is non-chaotic, $\operatorname{Per}(h)$ must contain an interval (see [4; Prop. 4.1]). All points of such an interval, except one point at the most, must have the same period. Let us consider all such intervals. The union of them can be covered by pairwise disjoint orbits of a system of periodic compact intervals $\mathcal{K} = \big\{K(i);\ i \in L\big\}$ for a suitable set L. We define F as follows:

$$F = h \qquad \text{on} \quad I \setminus \bigcup_{i \in L} K(i).$$

If $[u, v] \in \mathcal{K}$, then

$$F(x) = \begin{cases} h\Big(u + \dfrac{x-u}{t}\Big) & \text{for } x \in \big[u,\ (1-t)u + tv\big], \\ h(v) & \text{for } x \in \big((1-t)u + tv,\ v\big]. \end{cases}$$

Clearly, $F \in C(I,I)$. Using (2) and $\varrho(f,h) < \frac{\varepsilon}{2}$ we have that, if $x \in [u,\,(1-t)u+tv]$, then $|F(x) - f(x)| \le \left|h\left(u + \frac{x-u}{t}\right) - f\left(u + \frac{x-u}{t}\right)\right| + \left|f\left(u + \frac{x-u}{t}\right) - f(x)\right| < \varepsilon$ since $\left|u + \frac{x-u}{t} - x\right| < 2\delta$, and also, if $x \in \big((1-t)u+tv,\, v\big]$, then $|F(x) - f(x)| \le |h(v) - f(v)| + |f(v) - f(x)| < \varepsilon$ since $|v - x| < 2\delta$. Thus we have obtained $\varrho(f,F) < \varepsilon$.

$\operatorname{Per}(F)$ is nowhere dense: if $K = [u,v] \in \mathcal{K}$, then there is the smallest $k \in \mathbb{N}$ such that $F^k(K) = h^k(K) = K$. We have two possibilities:

(i) $F^k(u) = u$ and $F^k(x) = v$ for $x \in \big[(1-t)u + tv,\, v\big]$. Then v is a globally attracting fixed point of F^k on $(u,v]$.

(ii) $F^k(u) = v$ and $F^k(x) = u$ for $x \in \big[(1-t)u + tv,\, v\big]$. Then F^k has a repulsing fixed point $p \in \big(u,\,(1-t)u + tv\big)$ and a cycle $\{u,v\}$ of the period 2 which is globally attracting on $K \setminus \{p\}$.

From the construction of function F, we obviously have every $\omega_F(x)$ finite for each $x \in I$. That completes the proof. □

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