Mathematica Slovaca

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Mathematica Slovaca, Vol. 37 (1987), No. 1, 89--110

Persistent URL: http://dml.cz/dmlcz/129055

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REGULAR SYNTHESIS FOR THE LINEAR-CONVEX OPTIMAL CONTROL PROBLEM WITH CONVEX CONTROL CONSTRAINTS

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1. Introduction

Sufficient conditions for the existence of a regular optimal control synthesis for an abstract optimal control problem have been given in [2]. It has been proved in [3] that these conditions are satisfied for the linear-quadratic optimal control problem with linear control constraints. This paper constitutes an extension towards linear-convex optimal control problems with convex control constraints. This extension includes the linear-quadratic optimal control problem [3] as well.

In the course of the work on this paper it turned out that the theory of [2] cannot be directly applied to the linear-quadratic problem as claimed in [3]. The reason is that extremal trajectories from the interior of the reachable set may enter the boundary of the latter. In order to straighten out this gap we present a modification of the concept of regular synthesis to a set which need not be open. It is proved that the modified definition of a regular synthesis remains the sufficient condition of optimality. Finally, it is proved that the linear-convex problem admits a regular synthesis in the modified sense.

2. The linear-convex optimal control problem with convex control contraints

We consider the optimal control problem given by the linear system

$$\dot{x} = Ax + Bu \tag{1}$$

where A, B are $n \times n$ and $n \times m$ matrices, respectively, and the $n \times (nm)$ matrix $(B, AB, A^2B, \dots, A^{n-1}B)$ has rank n. The cost functional of the problem is prescribed by

$$J(u) = \int_0^T f^0(x, u)\,\mathrm{d}t. \tag{2}$$

Here $f^\circ(x, u)$ is a given real convex analytic function on $R^n \times R^m$ such that $\frac{\partial^2 f^\circ(x, u)}{\partial u^2} > 0$ for all $(x, u) \in R^n \times U$.

The control domain U is assumed to be of the form

$$U = \{u \in R^m / g^i(u) \leqslant 0,\ i \in P\}, \qquad P = \{1, \ldots, p\},$$

where

a) $g^i: R^m \to R$, $i \in P$, are analytic functions,

b) $\frac{\partial^2 g^i(u)}{\partial u^2} \geqslant 0$ for all $u \in R^m$, $i \in P$,

c) among the inequalities defining U there are no redundant ones, i.e., for every $i \in P$ there exists a $u \in R^m$ such that $g^i(u) > 0$ and $g^j(u) \leq 0$ for $j \in P - \{i\}$,

d) U is bounded and U contains the origin in its interior,

e) if for $u \in U$ one has $g^i(u) = 0$ for $i \in P_1 \subset P$, then the vectors $\frac{\partial g^i(u)}{\partial u}$, $i \in P_1$, are linearly independent.

By an admissible control we understand any measurable function u: $[0, T] \to U$.

Given a point $(y, T) \in R^{n+1}$ and a control u on $[0, T]$ by $x(t; T, y, u)$ we denote the solution of (1) with $u = u(t)$ such that $x(0; T, y, u) = y$. We say that the control u steers the system (1) from y to 0 on $[0, T]$ if $x(T; T, y, u) = 0$. The control u on $[0, T)$ is called optimal (for given $(y, T) \in R^{n+1}$) if it minimizes J among all controls steering the system (1) from y to 0 on $[0, T]$.

The optimal control problem just formulated for given $(y, T) \in R^{n+1}$ we will denote LK(y, T).

Let us denote that for our purpose it is sufficient if the assumptions a), b) are valid on some neighbourhood of U.

Adding the equation

$$\dot{x}^{n+1} = -1 \tag{1'}$$

to (1) we reformulate the fixed time problem LK(y, T) as a free time problem which we denote by LK$'(y, T)$. Here the admissible controls are defined on the intervals of type $[0, T]$, $T > 0$, and the responses $\tilde{x}(t) = (x(t), x^{n+1}(t))$ satisfy the conditions $(x(0), x^{n+1}(0)) = (y, -T)$ and $(x(T), x^{n+1}(T)) = (0, 0) \in R^{n+1}$.

Let G be an open subset of R^{n+1}. We will say that the system of problems LK(y, T), $(y, -T) \in G$ admits a regular synthesis in G if the system of problems LK$'(y, T)$, $(y, -T) \in G$ admits a regular synthesis in the sense of [2].

Let $K(T)$ be a set of all points y in R^n from which the system (1) can be steered to the origin by admissible controls $u(t)$ on $[0, T]$, $T > 0$. Let us denote

$$K_1 = \bigcup_{T>0} (K(T)\, x\{-T\}).$$

Our goal is to prove that the system of problems LK(y, T), $(y, -T) \in \operatorname{int} K_1$ admits a regular synthesis in int K_1.

3. Some basic properties of LK problems

We recall some properties of the LK problems which will be used later. The properties LK1—LK5 are simple corollaries of some general properties of optimal control problems for which [5] is a good reference. The property LK6 uses more a special form of the given LK problem.

LK1. *For every $(y, -T) \in K_1$ there exists an optimal control for the LK(y, T) problem.*

LK2. *Let $\bar{u}(t)$ be an optimal control with response $\bar{x}(t)$ for the LK(y, T) problem. Then there exists a non-zero solution $\psi(t) = (\psi^o, \eta(t))$, $\psi^o \leqslant 0$, of the adjoint system*

$$(3) \qquad \begin{aligned} \dot{\psi}^o &= 0 \\ \dot{\eta} &= \left(\frac{\partial f^o(\bar{x}, \bar{u})}{\partial x}\right)^* \psi^o - A^* \eta \end{aligned}$$

such that $\bar{u}(t)$, $\bar{x}(t)$ are extremal (with respect to $\psi(t)$), i.e.,

$$(4) \qquad M(\bar{x}(t), \psi(t), \bar{u}(t)) = \max_{u \in U} M(\bar{x}(t), \psi(t), u) \text{ a.e. on } [0, T].$$

Here

$$(5) \qquad M(x, \psi, u) = \psi^o f^o(x, u) + \eta^* Bu$$

(asterisk standing for transpose).

A $(u(t), x(t), \psi(t))$ is called an extremal triple for the LK(y, T) problem provided it satisfies (3), (4) and $\psi(t) \not\equiv 0$, $\psi^o \leqslant 0$.

Let L$\tilde{\text{K}}$(y, T) be the optimal control problem given by the linear process

$$(6) \qquad \dot{x} = -Ax - Bu$$

where A, B, the performance index, the control domain and the admissible controls are such as in LK(y, T). The initial state is prescribed by $x(0) = 0$ and the target state by $x(T) = y$. Let $\tilde{K}(T)$ denote the set of all points of R^n to which the system (6) can be steered from 0 by admissible controls on $[0, T]$.

LK3. a) $K(T) = \tilde{K}(T)$. b) A $(u(t), x(t), \psi(t))$ *is an extremal triple for the* $LK(y, T)$ *problem if and only if* $(\tilde{u}(t), \tilde{x}(t), \tilde{\psi}(t))$, *where* $\tilde{u}(t) = u(T - t)$, $\tilde{x}(t) = x(T - t)$, $\tilde{\psi}(t) = (\tilde{\psi}^o, \tilde{\eta}(t)) = (\psi^o, -\eta(T - t)) = \psi(t)$, *is an extremal triple for the* $L\tilde{K}(y, T)$ *problem.*

LK4. *The set* $K(T)$ *is compact, convex and varies continuously with* T, *on* $T > 0$.

LK5. Int K_1 *is a non-empty connected set,* $0 \in \bar{K}_1$.

Remark. Due to LK5 it makes sense to consider the existence of regular synthesis for the LK problem on int K_1.

LK6. a) *Let* $(y, -T) \in \operatorname{int} K_1$, *let* $(\bar{u}(t), \bar{x}(t)), (\bar{\psi}^o, \bar{\eta}(t))$ *be an extremal triple for the* $LK(y, T)$ *problem. Then,* $\psi^o \neq 0$.

b) Let $(\bar{u}(t), \bar{x}(t), (\bar{\psi}^o, \bar{\eta}(t))$ *and* $(\tilde{u}(t), \tilde{x}(t), (\tilde{\psi}^o, \tilde{\eta}(t)))$ *be extremal triples for the* $LK(y, T)$ *problem such that* $\bar{\psi}^o \neq 0$, $\tilde{\psi}^o \neq 0$. *Then* $\bar{u}(t) = \tilde{u}(t)$ *almost everywhere on* $[0, T]$.

Proof. According to LK3 it is sufficient to show that LK6 is valid for the $L\tilde{K}(y, T)$ problem. Also we assume that $\bar{u}(t), \tilde{u}(t)$ are extremal controls for the $L\tilde{K}(y, T)$ problem. We extend the system (6) to the system

$$\dot{x}^o = f^o(x, u)$$
$$\dot{x} = -Ax - Bu. \tag{7}$$

Let $u(t)$, $t \in [0, T]$, be an admissible control. Consider the solution $z(t) = (x^o(t), x(t))$ of the system (7) satisfying the condition $z(0) = 0$. Then $x^o(T) = J(u, T)$. Let us denote $\hat{K}(T)$ the set of all endpoints $(x^o, x) = (x^o(T), x(T))$ of the solution $z(t)$ of (7) satisfying the initial condition $z(0) = 0$ for all admissible controls on $[0, T]$. Note that the natural projection of the set $\hat{K}(T) \subset R \mathrm{x} R^n$ on the x- space is the set $K(T)$.

Let $(\bar{u}(t), \bar{x}(t), \bar{\psi}(t))$, $\bar{\psi}^o \neq 0$, be an extremal triple for the LK problem. Then $\bar{z}(t) = (\bar{x}^o(t), \bar{x}(t))$ is a solution of (7) satisfying $\bar{z}(0) = 0$, $\bar{z}(T) = (\bar{x}^o(T), \bar{x}(T)) = (J(\bar{u}, T), y)$. We can prove that $\bar{z}(T)$ is a boundary point of $\hat{K}(T)$ and $(\bar{\psi}^o, \bar{\eta}(T))$ is an exterior normal to $\hat{K}(T)$ at $\bar{z}(T)$, i.e., for an arbitrary admissible control $u(t)$ such that the corresponding solution of (7) $z(t)$ satisfying $z(0) = 0$ there holds

$$\bar{\psi}(T)^* \bar{z}(T) - \bar{\psi}(T)^* z(T) \geqslant 0. \tag{8}$$

First we derive two expressions which will be needed for the proof of (8).

Since $(\bar{x}(t), \bar{u}(t), \bar{\psi}(t))$ is an extremal triple for $L\tilde{K}(y, T)$, it satisfies

$$-\bar{\psi}^o f^o(\bar{x}, \bar{u}) + \bar{\eta}^* B\bar{u} = \min_{u \in U} [-\bar{\psi}^o f^o(\bar{x}, u) + \bar{\eta}^* Bu]. \tag{9}$$

Formula (9) defines a problem of convex programming and, therefore, it can be written

$$(10) \qquad \left(-\bar{\psi}^{\circ}\frac{\partial f^{\circ}(\bar{x},\bar{u})}{\partial u}+\bar{\eta}^{*}B\right)(u-\bar{u})\geqslant 0$$

for all $u \in U$ almost everywhere on $[0, T]$.

Since the function $f^{\circ}(x, u)$ is assumed to be convex and since $\psi^{\circ} \leqslant 0$ there holds

$$(11) \quad \psi^{\circ}f^{\circ}(\bar{x},\bar{u})-\psi^{\circ}f^{\circ}(x,u)\geqslant -\psi^{\circ}\frac{\partial f^{\circ}(\bar{x},\bar{u})}{\partial x}(x-\bar{x})-\psi^{\circ}\frac{\partial f^{\circ}(\bar{x},\bar{u})}{\partial u}(u-\bar{u})$$

for all $x, \bar{x} \in R^n$, $u, \bar{u} \in U$.

Using the condition $\bar{z}(0) = z(0) = 0$ and conditions (7), (10), (11) we obtain

(11′)

$$\bar{\psi}(T)^{*}\bar{z}(T)-\bar{\psi}(T)^{*}z(T)=\int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\psi}(t)^{*}\bar{z}(t))\,\mathrm{d}t-\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\psi}(t)^{*}z(t))\right)\mathrm{d}t=$$

$$=\int_0^T \bar{\psi}^{\circ}f^{\circ}(\bar{x},\bar{u})-\bar{\psi}^{\circ}f^{\circ}(x,u)+\bar{\psi}^{\circ}\frac{\partial f^{\circ}(\bar{x},\bar{u})}{\partial x}(x-\bar{x})+\bar{\eta}^{*}B(u-\bar{u}))\,\mathrm{d}t\geqslant$$

$$\geqslant\int_0^T\left(-\bar{\psi}^{\circ}\frac{\partial f^{\circ}(\bar{x},\bar{u})}{\partial u}(u-\bar{u})+\bar{\eta}^{*}B(u-\bar{u})\right)\mathrm{d}t\geqslant 0.$$

Thus, the inequality (8) is proved. From this inequality it follows that $\bar{z}(T) = (\bar{x}^{\circ}, \bar{x}(T))$ lies on the boundary of $\hat{K}(T)$ and that $(\bar{\psi}^{\circ}, \bar{\eta}(\mathrm{T}))$ is an exterior normal to $\hat{K}(T)$. If $\bar{\psi}^{\circ} = 0$, then $\bar{\eta}(T)$ would be an exterior normal to $K(T)$ at $\bar{x}(T)$ and thus $\bar{x}(T) = y$ would be a boundary point of $K(T)$. It is a contradiction to our assumption. Therefore $\bar{\psi}^{\circ} \neq 0$ and the statement a) of LK6 is proved.

Now let $(\bar{u}(t), \bar{x}(t), \bar{\psi}(t))$, $\bar{\psi}^{\circ} \neq 0$ and $(\tilde{u}(t), \tilde{x}(t), \tilde{\psi}(t))$, $\tilde{\psi}^{\circ} \neq 0$ be extremal triples of the LK(y, T) problem. Let there exist a non-trivial interval $I \subset [0, T]$ such that $\bar{u}(t) \neq \tilde{u}(t)$ almost everywhere on I. Using the calculations (11)′ for $\bar{x}(t)$, $\tilde{x}(t)$, $\bar{\psi}(t)$ and for $\tilde{x}(t)$, $\bar{x}(t)$, $\tilde{\psi}(t)$ and taking into account the strict convexity of $f^{\circ}(x, u)$ in u for fixed x we obtain $\bar{x}^{\circ}(T) < \tilde{x}^{\circ}(T)$ as well as $\tilde{x}^{\circ}(T) < \bar{x}^{\circ}(T)$ and therefore $\bar{u}(t) = \tilde{u}(t)$ almost everywhere on $[0, T]$.

4. The existence domain of a regular synthesis

The concept of a regular synthesis in the sense of [2] and the corresponding existence theorem are formulated for an open set $G \subset R^n$. In our case, because of fixed time, it would be necessary to choose in same way a suitable subset of R^{n+1}. As in the linear-quadratic problem from [3] it would be natural to take as this set G the open set int K_1. This choice would also have the advantage that according to LK6 one would take $\psi^0 = -1$ and conclude easily the unicity property required in assumptions of the existence theorem.

To take int K_1 for G one would have to prove that the extremal responses steering the points from int K_1 to 0 are staying in int K_1, i.e., if $(y, -T) \in \text{int } K_1$ and $x(t)$ is the corresponding extremal response of the LK(y, T) problem, then $(x(t), -t) \in \text{int } K_1$ on $[0, T)$.

We demonstrate by a simple example that this property fails to be valid in general. The example satisfies the assumptions of the linear-quadratic problem from [3] as well and hence shows that in [3] the above claim was erroneous.

Example. Consider the optimal control problem given by a system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{12}$$

a control domain $U = [-1, 1]$ and a performance index

$$J(u) = \int_0^T u^2 \, \mathrm{d}t. \tag{13}$$

The corresponding adjoint system is

$$\begin{aligned} \dot{\psi}_1 &= \; = 0 \\ \dot{\psi}_2 &= -\psi_1 \end{aligned}$$

and $\psi_1(\mathrm{t}) \equiv a$, $\psi_2(t) = -at + b$, $a, b \in R$, is its solution.

If $u(t)$ has to be an extremal solution of our problem, it must maximize the function $\psi_1 u + \psi_0 u^2$. Since we consider responses steering the points from int K_1 to 0 only, according to LK6 we can take $\psi^0 = -1$ and then the maximum condition is given by

$$u\psi_2 - u^2 = \max_{u \in U} (u\psi_2 - u^2). \tag{14}$$

Solving the condition (14) we obtain that $(u(t), x(t), \psi(t))$ is an extremal triple if and only if $u(t)$ has the following values only:

$$u = \begin{cases} 1 & \text{if } \psi_2 \geqslant 2 \\ \psi_2/2 & \text{if } -2 \leqslant \psi_2 \leqslant 2 \\ -1 & \text{if } \psi_2 \leqslant -2 \end{cases}$$

Now let us consider the time optimal control problem given by (12). Then according to [5] the control $u(t)$ and response $x(t)$, $t \in [0, T]$ are time optimal if and only if $x(0) \in \partial K(T)$ and according to [1] a time extremal control has only values 1 and -1.

Evidently, if $u(t)$ is extremal for the primary linear-quadratic problem given by (12), (13), $x(t)$ is its response on $u(t)$ and $u(t) = 1$ on $[t_1, T]$, $t_1 > 0$, then $(x(t), -t) \in \partial K_1$ on $[t_1, T]$. We shall demonstrate that such a control can be constructed for $x(0) \in \operatorname{int} K_1$.

Let $\psi_1 < 0$, $\psi_2(0) \in (-2, 2)$. Then $\psi_2 t) = -\psi_1 + \psi_2(0)$ is increasing and $\psi_1(t_1) = 2$ if and only if $t_1 = (\psi_2(0) - 2)/\psi_1$, Let $T > (\psi_2(0) - 2)/\psi_1$. Then the control

$$u(t) = \begin{cases} (-\psi_1 t + \psi_2(0)) & \text{for } t \in [0, (\psi_2(0) - 2)/\psi_1] \\ 1 & \text{for } t \in [\psi_2(0) - 2/\psi_1, T] \end{cases}$$

uniquely determines a $y \in K(T)$ such that a $u(t)$ steers y to 0 on $[0, T]$. A simple computation proved that if $y = (y_1, y_2)$, then

$$y_1 = \left(\frac{\psi_2(0) - 2}{\psi_1}\right)^3 \left(\frac{\psi_1 + 3}{6}\right) + \left(\frac{\psi_2(0) - 2}{\psi_1}\right)^2 \left(\frac{\psi_2(0) + 4T}{4}\right) + \frac{T^2}{2}\,\frac{\psi_2(0) - 2}{\psi_1}$$

$$y_2 = \left(\frac{\psi_2(0) - 2}{\psi_1}\right)^3 \left(\frac{\psi_1 + 2}{4}\right) + \left(\frac{\psi_2(0) - 2}{\psi_1}\right) \left(\frac{\psi_2(0)}{2} + T\right) + \frac{T^2}{2}.$$

The control $u(t)$ is evidently extremal for the LK(y, T) problem and from the fact that it is not time optimal there follows $(y, T) \in \operatorname{int} K_1$. Then there exists such a point $(y, T) \in \operatorname{int} K_1$ for which the extremal response is from the boundary of K_1 on a non trivial interval.

By this example it was demonstrated that for the linear-convex problem (and for the linear-quadratic problem as well) it is not possible to consider the existence of a regular synthesis on int K_1 in the sense of [2]. The reason is that in general the requirement about extremal responses to stay in int K_1 is not satisfied.

In the next part we shall make a slight modification of the regular synthesis concept to include the cases when the extremal responses reach the boundary of G. We shall prove the optimality of controls generated by the modified regular synthesis. The formulation of the existence theorem from [2] will be modified as well.

5. The modification of the regular synthesis concept

As the regular synthesis concept is rather space consuming and some points from its definition from [2] remain without change for the modified definition we shall not give the full regular synthesis concept for the set G not necessarily open.

Definition 1. *Let $G \subset R^n$,* int G *connected, $x \in \bar{G}$. By a* regular synthesis *in G of the optimal control problem from* [2] *given by the equation*

$$\dot{x} = f(x, u) \tag{15}$$

the performance index

$$J(u, T) = \int_0^T f^0(x, u)\,\mathrm{d}t \tag{16}$$

with initial points from int G *and with target point $\hat{x}$ we shall understand a 6-tuple $(\mathscr{S}, \mathscr{S}_1, \mathscr{S}_2, \Pi, \Sigma, v)$ where the symbols $\mathscr{S}, \mathscr{S}_1, \mathscr{S}_2, \Pi, \Sigma$ and v have the same significance as in the definition of* [2] *and the conditions A, B, C from* [2] *are satisfied on the whole G, the conditions D in the interior of G and in addition the following condition is satisfied*

E. *If $\tilde{u}(t)$, $t \in [0, T]$ is an admissible control steering the system* (15) *from $x \in$* int G *to $\hat{x}$, $\tilde{x}(t)$ is the response of $\tilde{u}(t)$, then there exists a sequence $x_n \to x$ and $\delta_n \to 0$, $x_n, x \in G$, $\delta_n \in R$, such that $x_n(t) = x(t, x_n, \tilde{u}(t)) \in$* int G *on* $[0, T - \delta_n)$.

We shall show this modification of a regular synthesis concept to preserve the optimality of controls generated by the regular synthesis.

Theorem 1. *Let $G \supset R^n$,* int G *connected, $\hat{x} \in \bar{G}$. Let $(\mathscr{S}, \mathscr{S}_1, \mathscr{S}_2, \Pi, \Sigma, v)$ be a regular synthesis in G of the control problems* (15), (16) *with initial points $x \in$* int G *and a target point $\hat{x}$. Then for every $x \in$* int G *the control $u_x(t)$ generated by the closed-loop control v (equation* (10) *from* [2]) *is optimal for the initial state x.*

This theorem can be concluded from the next in the same way as theorem A2 from A1 in [2] for G open.

Theorem 2. *Let $G \subset R^n$,* int G *connected. Let M be a closed stratified subset of G of dimension $< n$. Let $(G -$* int $G) \subset M$. *Let the following assumptions be satisfied:*

a) *$\hat{x} \in \bar{G}$, the function $V: G \cup \{\hat{x}\} \to R^1$ is continuous in* int $G \cup \{\hat{x}\}$ *and continuously differentiable in $G - M$, $V(\hat{x}) = 0$.*
b) *For every $x \in$* int G *there exists the control $u_x(t)$, $t \in [0, T(x)]$, steering the system* (15) *from x to $\hat{x}$ such that its response $x(t) \in G$ on $[0, T(x))$ and $J(u_x, T(x)) = V(x)$.*

Let the assumption E from Definition 1 be satisfied as well. If the condition

$$f^0(x, u) + \frac{\partial V}{\partial x}(x) f(x, u) \geqslant 0$$

holds in $G - M$, then u_x is an optimal control for every initial point $x \in \operatorname{int} G$.

Proof. Let $x_0 \in \operatorname{int} G$, $\varepsilon > 0$. From the continuity of $V(x)$ in x_0 and in $\hat{x}$ it follows that there are neighbourhoods W_0 of x_0 and W_1 of $\hat{x}$ such that $|V(x) - V(x_0)| < \varepsilon$ for $x \in W_0$ and $|V(x) - V(\hat{x})| < \varepsilon$ for $x \in W_1 \cap G$. Let $u(t)$, $t \in [0, T]$ be an arbitrary control steering the system (15) from x to $\hat{x}$, let $x(t)$ be its response. Then there exists such a $\delta > 0$ that $x(T - \delta) \in W_1 \cap G$ and

$$\text{(17)} \qquad \int_{T-\delta}^{T} f^0(x(t), u(t))\, \mathrm{d}t > -\varepsilon$$

Futher, there exists a neighbourhood $W_0' \subset W_0$ of x_0 such that if $y(t)$ is the response of $u(t)$ starting at an arbitrary $y_0 \in W_0'$ and satisfies $y(t) \in \operatorname{int} G$ on $[0, T - \delta]$, then $y(T - \delta) \in W_1 \cap G$ and

$$\text{(18)} \qquad -\int_0^{T-\delta} f^0(y(t), u(t))\, \mathrm{d}t + \int_0^{T-\delta} f^0(x(t), u(t))\, \mathrm{d}t > -\varepsilon$$

By the assumption E there exists a n such that $x_n \in W_0'$, $\delta_n < \delta$ and $x(t, x_n, u) \in$ $\in \operatorname{int} G$ for $t \in [0, T - \delta_n]$. Let W_0'' be such a neighbourhood of x_n that $W_0'' \subset W_0'$. According to [Lemma 4, 2] applied to int G there exists an $y_0 \in W_0''$ such that the response $y(t)$ of $u(t)$ starting at y_0 meets M at at most finitely many points and furthemore $y(t) \in \operatorname{int} G$ on $[0, T - \delta]$ and $y(T - \delta) \in W_1 \cap G$. Then using [2, Lemma 2] applied to $y(t)$ and int G one has

$$\text{(19)} \qquad -V(y(T - \delta)) + V(y_0) \leqslant \int_0^{T-\delta} f^0(y(t), u(t))\, \mathrm{d}t.$$

Because of $y(T - \delta) \in W_1 \cap G$ one has

$$\text{(20)} \qquad -V(y_0) + V(x_0) < \varepsilon$$

$$\text{(21)} \qquad -V(\hat{x}) + V(y(T - \delta)) < \varepsilon.$$

Adding (17)—(21) we obtain

$$-V(\hat{x}) + V(x_0) - 2\varepsilon \leqslant \int_0^T f^0(x(t), u(t))\, \mathrm{d}t + 2\varepsilon.$$

Since $\varepsilon > 0$ may be taken arbitrarily small we have

$$-V(\hat{x}) + V(x_0) \leqslant \int_0^T f^0(x(t), u(t))\,dt$$

which due to assumption b) proves the optimality of the control u_{x_0}.

Now we adapt the assumptions of the existence theorem from [2] to be sufficient conditions of the regular synthesis existence in the sense of the modified Definition 1. Once again the full assumptions of the theorem will not be presented but rather the differences will be pointed out.

The modification of the existence theorem. *Let $G \subset R^n$, int G connected and let the conditions 1—7 of existence theorem [2] be satisfied on G with the following exceptions:*

— *Assumption 2: The sets N_i cover $\bar{G} \times R_0^{n+1}$, $R_0^{n+1} = \{\psi \in R^{n+1} / \psi^0 < 0\}$ and N_i are closed subsets of $R^n \times R_0^{n+1}$; the functions w_i satisfy the maximum condition*

$$(22') \qquad H(x, \psi, w_i(x, \psi)) = \max_{u \in U} H(x, \psi, u) \text{ for } (x, \psi) \in N_i$$

with $\psi^0 \neq 0$;

— *Assumption 3: It suffices to require the uniqueness of extremal controls satisfying $\psi^0 \neq 0$ and the existence for $x \in$ int G only;*

— *Assumption 5: It suffices to require continuity of $J(x, u_x)$ on* int *G.*

Let the assumption E from Definition 1 hold. Then there exists a regular synthesis of the optimal control problems (15), (16) for $x \in$ int G in the sense of Definition 1.

Remark. As the basic space in which the cells are inductively constructed backwards the open set $G \times (R^{n+1} - \{0\})$ (resp. $G \times S^n$ where $S^n = \{\psi \in R^{n+1} / |\psi| = 1\}$) is taken in the proof of [2]. In the case of the modified theorem one can take as this basic space $R^n \times (R_0^{n+1} - \{0\})$ or $G \times S_0^n$, where $S_0^n = \{\psi \in R^{n+1} / |\psi| = 1, \psi^0 \neq 0\}$. In this way the extremal trajectories reaching the boundary of G which cannot be extended as extremals into int G are excluded. Therefore the proof of the existence theorem [2] is valid for our case as well.

At the end of this paper it will be proved that our linear-convex problem satisfies the assumptions of the modified existence theorem and then there exists a regular synthesis of the LK(y, T), $(y, -T) \in$ int K_1, problem in the sense of Definition 1. To this aim some lemmas and theorems will be needed.

6. The solution of the maximum condition

First, we formulate the control u from the condition (4) as a function of x and ψ and prove its continuity.

Lemma 1. *For any* $(x, \psi) \in R^n \times R_0^{n+1}$ *there exists a unique solution* $w = w(x, \psi)$ *of the condition*

$$M(x, \psi, w) = \max_{u \in U} M(x, \psi, u) \tag{22}$$

where the function M *is defined by* (5). *The function* w *is continuous.*

Proof. The existence of the (22) condition solution for given $(x, \psi) \in \in R^n \times R_0^{n+1}$ follows from the continuity of $M(x, \psi, u)$ and from the compactness of U. The uniqueness follows from the strict concavity of M and convexity of U. The proof of the continuity proceeds as in [3].

Now, we shall construct a partition of U and a corresponding partition of $R^n \times R_0^{n+1}$ to prove the analycity of $w(x, \psi)$ on every number of this partition.

Let $I \subset P$, where P is the index set defined earlier. We shall write $g^I \leqslant 0$ if $g^i \leqslant 0$ for all $i \in I$.

For every index set I such that $I \subset P$ and $|I| \leqslant m$ ($|I|$ denote cardinality of I) we shall denote

$$U_I = \{u \in R^m / g^I(u) = 0,\ g^{P-I}(u) < 0\}.$$

The index set I will be called admissible if $U_I \neq \emptyset$. Since there are no redundant constraints the family of the sets U_I, I admissible, is a partition of U. For $I = \emptyset$, U_I is the interior of U. Since g^i, $i \in I$, are analytic, every U_I has a finite number of connected components.

Lemma 2. *The family of the sets* U_I, *I admissible, is an analytic stratification of* U.

Proof. It is clear that every U_I, I admissible, is an analytic submanifold of R^m of codimension $|I|$. Let U_I, U_J be such that $U_I \neq U_J$ and $U_I \cap \bar{U}_J \neq \emptyset$. From the definition of U_I it follows that

$$\bar{U}_J = \bigcup_{J' \supseteq J} U_{J'}. \tag{23}$$

Let u be an arbitrary point of $U_I \cap \bar{U}_J$. Then, because of $u \in U_I$ there holds $g^I(u) = 0$, $g^{P-I}(u) < 0$ and since $u \in \bar{U}_J$ there exists a $J' \supset J$ such that $g^{J'}(u) = 0$, $g^{P-J'}(u) < 0$. Hence $I = J'$ and $U_I \subset \bar{U}_J$. Since $I = J' \supset J$, $I \neq J$, there holds $|I| > |J|$ and $\dim U_I = m - |I| < m - |J| = \dim U_J$ and therefore the partition of u into the sets U_I, I admissible, is an analytic stratification of U.

Denote

$$W_I = \{(x, \psi) \in R^n \times R_0^{n+1} / w(x, \psi) \in U_I\}.$$

It is clear that the family of the sets W_I, I admissible, is a partition of $R^n \times R_0^{n+1}$.

Lemma 3. *For every I admissible the set W_I is subanalytic.*

Proof. The subanalycity of W_I follows from the fact that the point (x, ψ) belongs to the set W_I if and only if

$$(x, \psi) \in (R^n \times R_0^{n+1}) \wedge (\exists w \in U_I \Rightarrow (\forall u \in U \Rightarrow M(x, \psi, u) \leqslant M(x, \psi, w))$$

where R^n, R_0^{n+1} are analytic submanifolds, U is compact semianalytic, $U_I \subset U$ is semianalytic and $M(x, \psi, u)$ is analytic [7].

Lemma 4. *Let $I \supset P$ be admissible, then*

$$\bar{W}_I \subset \bigcup_{I' \supseteq I} W_{I'}.$$

Proof. If we take into account the equality (23), we obtain

$$w^{-1}(\bar{U}_I) = w^{-1}\left(\bigcup_{I' \supseteq I} U_{I'}\right) = \bigcup_{I' \supseteq I} w^{-1}(U_{I'}) = \bigcup_{I' \supseteq I} W_{I'}.$$

Since w is continuous the set $w^{-1}(\bar{U}_I)$ is closed and therefore the conclusion of Lemma 4 holds.

7. The properties of the function w(x, ψ)

In this section we shall prove that the function $w(x, \psi)$ is analytic on every W_I, I admissible, and that it can be extended as an analytic function to a neighbourhood of W_I. To this aim we shall take into account that the Kuhn--Tucker theorem is a necessary and sufficient condition of the existence of the maximum of the function $M(x, \psi, u)$ on U for every $(x, \psi) \in R^n \times R_0^{n+1}$ in our case. The next lemma is a characterization of the set W_I, I admissible.

Lemma 5. *Let $(x, \psi) \in R^n \times R_0^{n+1}$, I admissible. Then $(x, \psi) \in W_I$ if and only if there exists a $u \in U_I$ and an $a^i \geqslant 0$, $i \in I$, such that*

$$K(x, \psi, u) + \sum_{i \in I} a^i \frac{\partial g^i(u)}{\partial u} = 0 \tag{24}$$

where

$$K(x, \psi, u) = -B^*\psi - \psi^0 \frac{\partial f^0(x, u)}{\partial u}. \tag{25}$$

Here $\dfrac{\partial g^i(u)}{\partial u}$, $\dfrac{\partial f^0(x, u)}{\partial u}$ *are understood as column vectors. Thus* $u = w(x, \psi)$.

Proof. Let $(x, \psi) \in W_I$. Take $u = w(x, \psi)$. According to the definition of W_I there holds $u \in U_I$ and therefore $g^I(u) = 0$ and $g^{P-I}(u) \leqslant 0$. From the Kuhn--Tucker theorem the existence of an $a^i \geqslant 0$, $i \in P$, follows such that

$$K(x, \psi, u) + \sum_{i \in P} a^i \frac{\partial g^i(u)}{\partial u} = 0 \tag{26}$$

$$a^i g^i(u) = 0 \text{ for every } i \in P \tag{27}$$

$$g^i(u) \leqslant 0 \text{ for every } i \in P \tag{28}$$

$$a^i \geqslant 0 \text{ for every } i \in P. \tag{29}$$

Because of $g^{P-I}(u) < 0$ we have $a^{P-1} = 0$ and so the equation (24) holds.

Let now exist to every $(x, \psi) \in R^n \times R_0^{n+1}$ such $u \in U_I$ and such $a^i \geqslant 0$, $i \in I$, that the condition (24) holds. Since $u \in U_I$ we have $g^I(u) = 0$, $g^{P-I}(u) < 0$. Take $a^{P-I} = 0$. Note that the conditions (26)—(29) are satisfied and since they are in our case sufficient conditions for u to be a solution of (23) we have $u = w(x, \psi)$. Since $w(x, \psi) \in U_I$ we have $(x, \psi) \in W_I$.

Remark. If $(x, \psi) \in W_I$, I admissible, then $u \in U_I$ from the last theorem is uniquely defined because of $u = w(x, \psi)$. The vectors $\dfrac{\partial g^i(u)}{\partial u}$, $i \in I$, are linearly independent and therefore the numbers a^i, $i \in I$, are uniquely defined by the vectors (x, ψ) as well. Then we can speak about the function $a^I(x, \psi)$ defined on W_I for every I admissible.

While the existence of an $a^I \geqslant 0$ and the validity of (24) for $u \in U_I$ are a necessary and sufficient condition for $(x, \psi) \in W_I$ in the next lemma we prove that the existence of an $a^I \geqslant 0$ and the validity of (24) are a necessary condition for $(x, \psi) \in \bar{W}_I$ as well.

Lemma 6. *Let I be admissible, let $(x, \psi) \in \bar{W}_I - W_I$. Then there exists a J admissible, $J \supset I$, such that $(x, \psi) \in W_J$ and the function a^J defined for J has the property $a^i(x, \psi) = 0$ for $i \in J - I$.*

Proof. Denote

$$\tilde{U}_I = \{u \in R^m / g^I(u) \leqslant 0\}.$$

Note that $U_I \subset \tilde{U}_I$ and $U \subset \tilde{U}_I$ as well. Consider the maximum condition

$$(30) \qquad M(x, \psi, z) = \max_{u \in \tilde{U}_I} M(x, \psi, u).$$

Denote $\tilde{W}_I$ the set of all $(x, \psi) \in R^n \times R_0^{n+1}$ for which there exists the solution $z(x, \psi)$ of (30).

First we prove $W_I \subset \tilde{W}_I$. Let $(\hat{x}, \hat{\psi}) \in W_I$. Then $w(x, \psi) \in U_I$ and the Kuhn--Tucker conditions (26)—(29) for the problem (22) at the point (x, ψ) are satisfied. These conditions are the Kuhn-Tucker conditions for the problem (30) at $(\hat{x}, \hat{\psi})$ as well. Since (30) is a problem of convex programing, these conditions are sufficient conditions as well. From this it follows that $w(\hat{x}, \hat{\psi}) = z(\hat{x}, \hat{\psi})$, so $(\hat{x}, \hat{\psi}) \in \hat{W}_I$.

Now we prove that if $(x, \psi) \in \bar{W}_I - W_I$, then $(x, \psi) \in \tilde{W}_I$. Let (x_k, ψ_k), $k = 1, 2, \ldots$ be a sequence of points of W_I such that $(x_k, \psi_k) \to (x, \psi) \in \bar{W}_I - W_I$. Denote $u_k = w(x_k, \psi_k)$. Since w is continuous we have $w(x, \psi) = \lim_{k \to \infty} u_k$. As proved above we have $(x_k, \psi_k) \in \tilde{W}_I$ and $z_k = z(x_k, \psi_k) = u_k$. Let u be an arbitrary point of U_I. Then $M(x, \psi, w) = \lim_{k \to \infty} M(x_k, \psi_k, u_k) = \lim_{k \to \infty} M(x_k, \psi_k, z_k) \geqslant$

$\geqslant \lim_{k \to \infty} M(x_k, \psi_k, u) = M(x, \psi, u)$. From this it follows that $w(x, \psi)$ is a solution of (30). Therefore $(x, \psi) \in \tilde{W}_I$ and $w(x, \psi) = z(x, \psi)$.

Let $(x, \psi) \in \bar{W}_I - W_I$ and $w(x, \psi) = z(x, \psi) = u$. According to the Kuhn--Tucker theorem applied to the problem (30) at (x, ψ) there exists an $\tilde{a}^I \geqslant 0$ such that

$$(31) \qquad K(x, \psi, u) + \sum_{i \in I} \tilde{a}^i \frac{\partial g^i(u)}{\partial u} = 0.$$

Since $(x, \psi) \in \bar{W}_I$ we have $u \in \bar{U}_I$ and according to Lemma 2 there exists a $J \supset I$ such that $u \in U_J$ and so $(x, \psi) \in W_J$. Then, because of Lemma 5 (used for J admissible) there exists an an $a^J \geqslant 0$ such that

$$(32) \qquad K(x, \psi, u) + \sum_{j \in J} a^j \frac{\partial g^j(u)}{\partial u} = 0$$

From the conditions (31), (32), the inclusion $I \subset J$ and from the linear independence of vectors $\dfrac{\partial g^J(u)}{\partial u}$ it follows that for a given point $(x, \psi) \in \bar{W}_I - W_I$ we have $\tilde{a}^I = a^I$ and $a^{J-I} = 0$.

Lemma 7. *Let $I \subset P$, I admissible. Let $\hat{x} \in R^n$, $\hat{\psi} = (\hat{\psi}^0, \hat{\eta}) \in R_0^{n+1}$, $\hat{u} \in U$, $\hat{a}^I \geqslant 0$ satisfies the conditions*

$$K(\hat{x}, \hat{\psi}, \hat{u}) + \sum_{i \in I} \hat{a}^i \frac{\partial g^i(\hat{u})}{\partial u} = 0 \tag{33}$$

$$g^I(\hat{u}) = 0. \tag{34}$$

Then there exists a neighbourhood O of $(\hat{x}, \hat{\psi})$ and analytic functions $\alpha^i(x, \psi)$, $i \in I$, $u(x, \psi)$ defined on O satisfying $\alpha^I(\hat{x}, \hat{\psi}) = \hat{a}^I$, $u(\hat{x}, \hat{\psi}) = \hat{u}$ such that the equations

$$K(x, \psi, u(x, \psi)) + \sum_{i \in I} \alpha^i(x, \psi) \frac{\partial g^i(u(x, \psi))}{\partial u} = 0 \tag{35}$$

$$g^I(u(x, \psi)) = 0 \tag{36}$$

hold for every $(x, \psi) \in O$.

Proof. The lemma will be proved using the implicit function theorem. Let the function

$$F(x, \psi, a^I, u): R^n \times R_0^{n+1} \times R^{|I|} \times R^m \to R^m \times R^{|I|}$$

be given by

$$F(x, \psi, a^I, u) = \begin{pmatrix} K(x, \psi, u) + \sum_{i \in I} a^i \frac{\partial g^i(u)}{\partial u} \\ g^I(u) \end{pmatrix}.$$

There holds $F(\hat{x}, \hat{\psi}, \hat{a}^I, \hat{u}) = 0$. Denote

$$M_I = -\hat{\psi}^0 \frac{\partial^2 f^0(\hat{x}, \hat{u})}{\partial u^2} + \sum_{i \in I} \hat{a}^i \frac{\partial^2 g^i(\hat{u})}{\partial u^2}.$$

This matrix is of type $m \times m$ and $M_I > 0$. Since the vectors $\frac{\partial g^I(u)}{\partial u}$ are linearly independent and the matrix M_I is positive definite, using the formula for the determinant of a block matrix [4] we obtain

$$\det \frac{\partial F(\hat{x}, \hat{\psi}, \hat{a}^I, \hat{u})}{\partial(a^I, u)} = \det \begin{pmatrix} M_I & \frac{\partial g^I(\hat{u})}{\partial u} \\ \left(\frac{\partial g^I(\hat{u})}{\partial u}\right)^* & 0 \end{pmatrix} =$$

$$\det M_I . \det\left(-\left(\frac{\partial g^I(\hat{u})}{\partial u}\right)^* M_I^{-1} \frac{\partial g^I(\hat{u})}{\partial u}\right) \neq 0 .$$

By the implicit function theorem there exists a neighbourhood O of (x, ψ) and analytic functions $\alpha^i(x, \psi)$, $i \in I$, $u(x, \psi)$ defined on O satisfying $\alpha^I(\hat{x}, \hat{\psi}) = \hat{a}^I$, $u(\hat{x}, \hat{\psi}) = \hat{u}$ such that the equations (35), (36) hold for every $(x, \psi) \in O$.

Lemma 8. *For every I admissible* int $W_I \neq \emptyset$.

Proof. for every I admissible we denote

$$V_I = \left\{ v \in R^m / \text{there exist } u \in U_I,\ a^I \geqslant 0 \text{ such that } v = u + a^I \frac{\partial g^I(u)}{\partial u} \right\}.$$

First, we prove that the sets V_I, I admissible, form a partition of R^m.

It is easy to see that for every $z \in R^m$ there exists a unique $v(z) \in U$ such that

$$z^* v(z) - \frac{1}{2} v(z)^* v(z) = \max_{u \in U} \left(z^* u - \frac{1}{2} u^* u \right).$$

The sets Z_I, I admissible, where $Z_I = \{z \in R^m / v(z) \in U_I\}$, form a partition of R^m. Analogously as in Lemma 5 we obtain $z \in Z_I$ if and only if there exist $a^I \geqslant 0$, $u \in U_I$ such that

$$z - u = a^I \frac{\partial g^I(u)}{\partial u}$$

$$g^I(u) = 0$$

and $u = v(z)$. Therefore $Z_I = V_I$ and V_I form a partition of R^m.

Now, we prove int $V_I \neq \emptyset$. Let $\hat{u} \in U_I$ and $\hat{a}^I > 0$. Then $\hat{v} = \hat{u} + \hat{a}^I \dfrac{\partial g^I(\hat{u})}{\partial u} \in V_I$.

Applying the implicit function theorem to the function $f\colon R^m \times R^m \times R^{|I|} \to$ $\to R^m \times R^{|I|}$

$$f(v, u, a^I) = \begin{pmatrix} v - u - a^I \dfrac{\partial g^I(u)}{\partial u} \\ g^I(u) \end{pmatrix}$$

similarly as in Lemma 7 we conclude that in some neighbourhood O of $\hat{v}$, $u \in U_I$ and $a^I > 0$ can be expressed as analytic functions of v from the equation $f(v, u, a^I) = 0$. This proves $O \subset V_I$.

Define a function $G\colon R^n \times R_0^{n+1} \to R^m$ by the formula

$$G(x, \psi) = w(x, \psi) + \frac{\psi^0\, \partial f^0(x, w(x, \psi))}{\partial u} + \eta^* B.$$

Since $w(x, \psi)$ is continuous, the function $G(x, \psi)$ is continuous as well.

In the end, we prove $G^{-1}(U_I) \subset W_I$. Let $(x, \psi) \in G^{-1}(V_I)$. Then

$$z = w(x, \psi) + \frac{\psi^{\circ}\, \partial f^{\circ}(x, w(x, \psi))}{\partial u} + \eta^* B \in V_I.$$

If $(x, \psi) \notin W_I$, then there exist J admissible, $J \neq I$, and $a^J \geqslant 0$ such that $(x, \psi) \in \in W_J$, $w(x, \psi) \in U_J$ and

$$\psi^{\circ} \frac{\partial f^{\circ}(x, w(x, \psi))}{\partial u} + \eta^* B = a^J \frac{\partial g^J(w(x, \psi))}{\partial u}.$$

Therefore $z \in V_J \cap V_I$ and this is in contradiction with the fact that V_I form a partition of R^m.

Theorem 3. *For every W_I, I admissible, there exists a neighbourhood B_I of the set $\bar{W}_I$, I and an analytic function $w_I(x, \psi)$ defined on B_I such that $w_I(x, \psi) = = w(x, \psi)$ for every $(x, \psi) \in \bar{W}_I$.*

Proof. Let $(\hat{x}, \hat{\psi}) \in W_I$. Then according to Lemma 5 there exist $\hat{u} \in U_I$, $\hat{a}^I \geqslant 0$ such that for $\hat{x}$, $\hat{\psi}$, $\hat{u}$, $\hat{a}^I$ the condition (30) holds. According to Lemma 7 there exists a neighbourhood O_1 of $(\hat{x}, \hat{\psi})$ and analytic functions $a^I(x, \psi)$, $u(x, \psi)$ defined on O_1 such that $a^I(\hat{x}, \hat{\psi}) = \hat{a}^I$, $u(\hat{x}, \hat{\psi}) = \hat{u}$ and conditions (35) and (36) hold on O_1. Since $g^{P-I}(u) < 0$ we are able to choose a neighbourhood O_1 so small that $g^{P-I}(u(x, \psi)) < 0$ for every $(x, \psi) \in O_1$. Therefore $u(x, \psi) \in U_I$ for every $(x, \psi) \in O_1$. If $\hat{a}^I > 0$ then there exists a neighbourhood $O_2 \subset O$ such that $a^I(x, \psi) > 0$ on O_2 and thus according to Lemma 5 we have $O_2 \subset W_I$ and $u(x, \psi) = w(x, \psi)$ for every $(x, \psi) \in O_2$.

If $\hat{a}^I \ngtr 0$ then there exists an J admissible such that $J \subset I$, $\hat{a}^J = 0$, $\hat{a}^{I-J} > 0$. Then there exists a neighbourhood O_1 of $(\hat{x}, \hat{\psi})$ and analytic functions $a^I(x, \psi)$, $u(x, \psi)$ defined on O_1 such that (35) and (36) hold on O_1 and $u(x, \psi) \in U_I$. The neighbourhood O_1 can be chosen so small that $a^{I-J}(x, \psi) > 0$ holds for every $(x, \psi) \in O_1$. Denote $A = \{(x, \psi) \in O_1 / a^J(x, \psi) \geqslant O\}$. According to Lemma 5 there holds that $A = O_1 \cap W_I$ and hence $u(x, \psi) = w(x, \psi)$ for every $(x, \psi) \in A$.

Let now $(\hat{x}, \hat{\psi}) \in \bar{W}_I - W_I$. Then according to Lemma 6 there exists a $J \supset I$, $\hat{u} \in U_J$, $a^I \geqslant 0$ such that $\hat{x}$, $\hat{\psi}$, $\hat{u}$, $\hat{a}^I$ satisfy (24) for given I and $g^I(\hat{u}) = 0$. According to Lemma 7 there exists a neighbourhood O_1 of $(\hat{x}, \hat{\psi})$ and analytic functions $a^I(x, \psi)$, $u(x, \psi)$ defined on O_1 such that the conditions (35), (36) are satisfied on O_1. Denote

$$A = \{(x, \psi) \in O_1 / a^I(x, \psi) \geqslant 0, g^{J-I}(u(x, \psi)) \leqslant 0\}.$$

From the uniqueness of the maximum condition solution it follows that if $(x, \psi) \in A$ then $u(x, \psi) = w(x, \psi)$. We prove that for every $(x, \psi) \in O_1 \cap \bar{W}_I$ we have $u(x, \psi) = w(x, \psi)$. Since the functions $u(x, \psi)$ and $w(x, \psi)$ are continuous it is sufficient to prove that $u(x, \psi) = w(x, \psi)$ on $O_1 \cap W_I$. Let $(x, \psi) \in O_1 \cap W_I$, then $(x, \psi) \in A$ since $w(x, \psi) \in U_I$ and $\alpha^I(x, \psi) = a^I(x, \psi) \geqslant 0$ and therefore $u(x, \psi) = w(x, \psi)$.

Thus, for every $(x, \psi) \in \bar{W}_I$ we proved the existence of a neighbourhood O of (x, ψ) and an analytic function $u(x, \psi)$ defined on O such that $u(x, \psi) = w(x, \psi)$ on $O \cap \bar{W}_I$.

From Lemma 1 (uniqueness of solution of (22)), from the theorem of uniqueness of the extension of real analytic functions to open connected sets and from Lemma 8 the existence of a neighbourhood B_I of the set $\bar{W}_I$ and an analytic function $w_I(x, \psi)$ defined on B_I follows such that $w(x, \psi) = w_I(x, \psi)$ on $\bar{W}_I$.

8. Existence of regular synthesis

The sets $\bar{W}_I$ and the corresponding functions $w_I(x, \psi)$ defined and analytic on the neighbourhood B_I have such properties which are very similar to those required in the assumptions of the existence theorem from [2]. Replacing the studied LK problem by the problem with free time we increase the dimension of the space of the state variables and adjoint variables by one. Because of this we shall define the functions w', w'_I and the sets N_I in the space $R^n \times R \times R_0^{n+1} \times R$ of the variables $(x, x^{n+1}, \psi, \psi^{n+1})$ using the functions w, w_I and the sets N_I to have the properties of the original functions and the sets on $R^n \times R_0^{n+1}$.

Let the function w': $R^n \times R \times R_0^{n+1} \times R \rightarrow R^m$ be given by

$$w'(x, x^{n+1}, \psi, \psi^{n+1}) = w(x, \psi).$$

For every I admissible denote

$$N_I = \{(x, x^{n+1}, \psi, \psi^{n+1}) \in R^n \times R \times R_0^{n+1} \times R \,/\, (x, \psi) \in W_I\}$$

and analogously

$$C_I = \{(x, x^{n+1}, \psi, \psi^{n+1}) \in R^n \times R \times R_0^{n+1} \times R \,/\, (x, \psi) \in B_I\}.$$

Let the function w'_I: $C_I \rightarrow R^m$ be given by

$$w'_I(x, x^{n+1}, \psi, \psi^{n+1}) = w_I(x, \psi).$$

Instead of (x, x^{n+1}) and (ψ, ψ^{n+1}) we shall write $\tilde{x}$ and $\tilde{\psi}$. The sets N_I are subanalytic and they form a partition of $R^n \times R \times R_0^{n+1} \times R$; the set C_I is a neighbourhood of $\bar{N}_I$ for every I admissible. The functions w'_I are analytic on C_I,

$w'_I(\tilde{x}, \tilde{\psi}) = w'(\tilde{x}, \tilde{\psi})$ for every $(\tilde{x}, \tilde{\psi}) \in N_I$ and $w'(\tilde{x}, \tilde{\psi})$ is a solution of the maximum condition for the free time optimal control problem associated with the original fixed time control problem.

Let us note that the control system from the existence theorem from [2] is in our case of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{x}^{n+1} &= 1 \\ \dot{\psi}^{\circ} &= 0 \\ \dot{\psi} &= -\left(\frac{\partial f^{\circ}(x, u)}{\partial x}\right)^{*} \psi^{\circ} - A^{*}\eta \\ \dot{\psi}^{n+1} &= 0 \\ u &= w'_I(\tilde{x}, \tilde{\psi}). \end{aligned} \tag{33}$$

Theorem 4. *The* LK$'(y, T)$ *problem for* $(y, -T) \in \operatorname{int} K_I$ *admits a regular synthesis in the sense of Definition* 1.

Proof.

1. As the function f from [2] we take the function which associates with every $(x, x^{n+1}) \in R^n \times R$ the point $(Ax + Bu, 1) \in R^n \times R$; as the function f° we take the function f° from (2). The functions f and f° are analytic by assumption.
2. As the sets N_i we take the family of all such N_I, I admissible, for which $\bar{N}_I \cap (K_1 \times R_0^{n+1} \times R) \neq \emptyset$. The functions $w'_I(\tilde{x}, \tilde{\psi})$ are defined and analytic on some neighbourhood of $\bar{N}_I$ for all I and satisfy the maximum condition (22).
3. The first part of Assumption 3 of the existence theorem is proved in LK6b). Now let $(x, -x^{n+1}) \in K_1$ be such that there exists an extremal control response $\xi_x(t)$ satisfying $\xi_x(0) = x$, $\xi_x(x^{n+1}) = 0$. Let $\psi \in \psi_x$(ψ_x is defined in[2]). According to [6, Theor. II] applied to the system (33) and the partition of $R^{n+1} \times R_0^{n+2}$ into sets N_I, I admissible, the solution $\tilde{x}(t)$, $\tilde{\psi}(t)$ of the system (33) has a finite number of switchings in a neighbourhood of $(\tilde{x}(0), \tilde{\psi}(0)) = (\tilde{x}, \tilde{\psi})$. Therefore there exists I admissible and $t_1 \in [0, T]$ such that $(\tilde{x}(t), \tilde{\psi}(t)) \in N_I$ for every $t \in (0, t_1)$.

In Assumption 3 of [2] it was required that $(\tilde{x}(t), \tilde{\psi}(t)) \notin \bar{N}_J$ for small $t \geqslant 0$ for any $J \neq I = \mu(x)$; in order to meet this requirement in the LQ problem a normality condition had to be assumed in [3]. This stronger unicity, however, is not needed once w_i can be analytically extended to a neighbourhood of $\bar{N}_i$ for each i (which is true in our case). Indeed, the only changes in the proof of the theorem of [2] one has to make is to define $i = \mu(x)$ to be such that $(\xi_x(t), \psi(t)) \in N_i$ for $t > 0$ (insead of $t \geqslant 0$) and define H' by

$$H' = \{\Phi^i_r(x, \psi) \,/\, (x, \psi) \in D(S)\}$$

where

$$r = \min\{\eta, \inf\{t \,/\, \Phi_s^i(x, \psi) \in N_i^\circ, \ t < s < 0\}\}$$

(η is from Assumption 7)

Now we return to the verification of the assumptions of the existence theorem. As the sets N_i we take the sets N_I, I admissible. As the $\mu(x)$ we take the index determined by the set N_I, for which $(\tilde{x}(t), \tilde{\psi}(t)) \in N_I$, $t \in (0, t_1)$. Now, $\mu(x)$ is determined uniquely since the sets N_I, I admissible, form a partition of $R^{n+1} \times R_0^{n+2}$. The independence of $\mu(x)$ from the choice of $\psi \in \Psi_x$ follows directly from the fact that U_I form a partition of U.

4. We want to prove that for every compact subset K of $K_1 \cup \{\hat{x}\}$ there exists a $\nu = \nu(K) > 0$ such that every extremal control u_x for $x \in \operatorname{int} K_1$ has at most $\nu(K)$ switching points, i.e., points t such that $\mu(\xi_x(s)) \neq \mu(\xi_x(t))$ for $s > t$, s near t. To this aim it suffices to prove that for K there exists a compact subset $K' \subset K \times R_0^{n+2}$ such that if $\tilde{x} \in K$, then $(\tilde{\xi}_x(t), \tilde{\psi}_x(t)) \in K'$ for every $t \in [0, T(x)]$ and every $\psi \in \Psi_x$ such that $|\psi| = 1$.

Take the closure of the set $(\xi_x(t), \psi(t)) \in K_1 \times R_0^{n+2} / (x, -T) \in K, |\psi(0)| = 1$, $t \in (0, T)\}$ as the set K'. Then using the [6, Theor. II] and according to the foregoing steps of this proof the trajectories $(\tilde{x}(t), \tilde{\psi}(t)) = (\tilde{\xi}_x(t), \tilde{\psi}(t))$ have at most $N(K')$ switching points and therefore $\xi_x(t)$ must have at most $N(K) =$ $= N(K')$ such points t in which $\mu(\xi_x(s)) \neq \mu(\xi_x(t))$ for $s > t$, s near t, for every $x \in K$.

5. The continuity of the performance index suffices to prove only for $x \in \operatorname{int} K_1$ and therefore the proof is such as the proof of continuity of the performance index for the linear-quadratic problem from [3]. The Lipschitz continuity of $w(x,\psi)$ follows from [6].

6. The validity of assumption 6 follows from the fact that time appears as a state variable in our case.

7. It suffices to prove that for every compact $K \subset K_1$ and for every N_I, I admissible there exists an $\eta_I(K) > 0$ such that the solution $(\tilde{x}(t), \tilde{\psi}(t))$ of the system (33) with $\tilde{x}(0) \in K$, $|\tilde{\psi}(0)| = 1$, $(\tilde{x}(0), \tilde{\psi}(0)) \in N_I$ exists on the interval $[-\eta_I(K), 0]$ and satisfies $\tilde{x}(t) \in K_1$ for $t \in [-\eta_I(K), 0]$.

Denote $A = \{(\tilde{x}', \tilde{\psi}') / (\tilde{x}', \tilde{\psi}') \in \bar{N}_I, \tilde{x}' \in K, |\tilde{\psi}'| = 1\}$. The set $\bar{A}$ is compact. According to Theorem 3, definition of w' and compactness of $\bar{A}$ there exist numbers $r_I > 0$, $k_I > 0$ such that

a) if $(\tilde{x}', \tilde{\psi}') \in A$, then $C_I \supset G((\tilde{x}', \tilde{\psi}'), r_I)$, where $G((\tilde{x}', \tilde{\psi}'), r_I) = \{(\tilde{x}', \tilde{\psi}') /$ $/ |(\tilde{x}', \tilde{\psi}') - (\tilde{x}', \tilde{\psi}')| < r_I\}$;

b) $|w'(\tilde{x}, \tilde{\psi})| \leqslant k_I$ for every $(\tilde{x}, \tilde{\psi}) \in G((\tilde{x}', \tilde{\psi}'), r_I)$.

The required statement then easily follows from the Gronwall theorem.

Now we prove satisfying the assumption E of the modified regular synthesis existence theorem. Let $\bar{u}(t)$, $t \in [0, T]$ be the admissible control which steers the system (1') from $(x, -T) \in \operatorname{int} K_1$ to 0. Let $\tilde{x}(t) = (x(t), -t)$ be its response. Let $\delta_n \to 0$. We define

$$u_n = \begin{cases} 0, & t \in [0, \delta_n) \\ u(T-t), & t \in (\delta_n, T] \end{cases}$$

for every n. Consider the responses $x_n(t)$ of (6) and $u_n(t)$ satisfying the initial condition $x_n(0) = 0$. Because of $0 \in \operatorname{int} U$ we have $x_n(t) \in \operatorname{int} K(t)$ on $(0, T]$. The points $x_n = x_n(T) \to x$ satisfy the assumption E. This completes the proof of the theorem.

11. Acknowledgement

The author would like to thank P. Brunovský for discussions which stimulated this research and for critical comments.

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Received June 6, 1985

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РЕГУЛЯРНЫЙ СИНТЕЗ ДЛЯ ЛИНЕЙНО-ВЫПУКЛОЙ ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ С ВЫПУКЛЫМИ ОГРАНИЧЕНИЯМИ НА УПРАВЛЕНИЕ

Margaréta Halická

Резюме

В работе рассматривается линейная задача оптимального управления с интегральным выпуклым критерием качества и с выпуклыми аналитическими ограничениями на управление. Показано, что экстремальные траектории при переходе в начало координат проходят по границе множества достижимости. Поэтому сделана модификация определения регулярного синтеза и доказано существование регулярного синтеза для рассматриваемой задачи в смысле модифицированного определения.