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ON EXTENSION OF MAPS WITH VALUES IN ORDERED SPACES

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The aim of this paper is to present an extension of maps defined on a lattice A and having their values in an ordered commutative group G. Following [5] we postulate properties of the lattice A in such a way that theorems about the extension of measures and the Daniell integral are obtained as consequences of the main theorem.

The construction is a modification of the well-known Daniell scheme which, modified in another way, has been used also in [1]. In his paper [1] D. H. Fremlin gave a direct proof of the Matthes—Wright integral extension theorem which states that the condition of weak σ-distributivity of the range of an integral is necessary as well as sufficient.

After our presentation of the general construction we derive several results from the main theorem and discuss the relation of the conditions weak σ-distributivity and g-regularity in the case of σ-complete vector lattices.

I

Notations and notions

An ordered commutative group G is a commutative group with a reflexive, antisymmetric and transitive relation connected with the group structure of G by the condition: $x \leqq y$ implies $x+z \leqq y+z$ for all x, y and z in G. A group is said to be monotone complete (σ-complete) if, for each upper bounded, upward directed family (x_λ) in G (monotone increasing sequence (x_n) in G), there exists a least upper bound $\vee x_\lambda$ $(\vee x_n)$ in G.

In analogy with the notion of the order separable Riesz space (= vector lattice) we call an ordered group G o-separable if every non-empty subset $E \subset G$ possessing a supremum contains an at most countable subset possessing the same supremum as E. It is clear that the monotone σ-complete commutative group G is Archimedean. When an ordered group is a lattice group, then σ-completeness implies not only that it is Archimedean but also commutative.

In [7] we introduced the notion of the regularity of a lattice group, but this notion has a meaning also in an ordered group. We recall that a monotone σ-complete ordered commutative group G is said to be regular if there holds: If $(a_{n,k})_{n,k\in\mathbf{N}}$ is an order-bounded double sequence in G such that $a_{n,k}\searrow 0$ $(k\to\infty)$ for each $n\in\mathbf{N}$, then there exists $\varphi_0\in\mathbf{N}^{\mathbf{N}}$ ($\varphi_0:\mathbf{N}\to\mathbf{N}$ is a function) such that the sequence $\left(\sum_{n=1}^{m} a_{n,\varphi_0(n)}\right)_{m\in\mathbf{N}}$ is bounded from above and if $b\in G$, $b\geqslant 0$ is an element for which $b\leqslant\bigvee_{m=1}^{\infty}\left(\sum_{n=1}^{m} a_{n,\varphi(n)}\right)$ for all $\varphi\in\mathbf{N}^{\mathbf{N}}$, then $b=0$.

As the notion of regular Riesz spaces (= regular vector lattices) is reserved for Archimedean Riesz spaces possessing the diagonal property or another equivalent property, see § 70 [3], we shall call a vector lattice, resp. a partially ordered vector space V, g-regular if the group $(V,+)$ is a regular in the above sense.

Examples

There are many spaces which are regular groups, resp. g-regular vector lattices. Here are several of them:

(a) Real numbers, or course, resp. R^n with a pointwise ordering.

(b) σ-complete regular Riesz spaces. o-convergence is stable and such spaces have the σ-property, in other words, every sequence of o-convergent sequences has a common regulator of convergence (§ 5, Ch. 6 [8]). If $a_{n,k}\searrow 0$ $(k\to\infty)$ for each $n\in\mathbf{N}$ and u is the common regulator for $(a_{n,k})$, then for every positive real ε there exists $\varphi\in\mathbf{N}^{\mathbf{N}}$ such that $a_{n,\varphi(n)}\leqslant\frac{\varepsilon}{2^n}u$ and we obtain $b\leqslant\varepsilon u$ for all $\varepsilon>0$, assuming that b is a lower bound of $\left\{\bigvee_{m=1}^{\infty}\sum_{n=1}^{m} a_{n,\varphi(n)}:\varphi\in\mathbf{N}^{\mathbf{N}}\right\}$.

(c) Let s be the Riesz space of all real sequences and the ordering is coordinatewise. Let F be the space of all real sequences having only a finite number of non-zero terms. Since F is an ideal in s, o-convergence is pointwise. If $c\in F$ bounds the double sequence $(a_{n,k})_{n,k\in\mathbf{N}}$, $a_{n,k}\in F$, $n,k\in\mathbf{N}$, such that $a_{n,k}\searrow 0$ $(k\to\infty)$, the problem is reduced on a finite number of coordinates and so F is g-regular. It is well known that F has not the σ-property, i.e. is not a regular Riesz space.

As a next example of a non regular Riesz space which is g-regular is the space L^{p+0}, p real, $p>1$ see for a detail discussion §6 Ch. VII. [8].

(d) Every commutative, σ-complete, linearly ordered group G is regular. Example of a regular group which is not a lattice group is the multiplicative group G of reals with ordering $\leqq$ associated with a semigroup $\{x\in G: x\geqslant 1$ in natural ordering$\}$, i.e. $x\leqq y$ iff $y\,.\,x^{-1}\geqq 1$. Note that this ordering is not directed.

II

Construction

In this part of the paper we consider a relative σ-complete and σ-continuous lattice X in which there are given two binary operations $+$ and $/$, satisfying the following conditions:

(i) $+$ is commutative.

(ii) If $x, y, z \in X$, $x \leqslant y$, then $x+z \leqslant y+z$, $x/z \leqslant y/z$, $z/y \leqslant z/x$.

(iii) If $x_n, y_n \in X$, $n=0, 1, 2, \ldots$, $x_n \nearrow x_0$, $y_n \nearrow y_0$, then $x_n + y_n \nearrow x_0 + y_0$ and $x_n / y_0 \nearrow x_0 / y_0$.

(iv) If $x_n \in X$, $n=1, 2, \ldots$, $x_n \searrow x_0$, $y \in X$, then $y/x_n \nearrow y/x_0$.

(v) If x, $y \in X$ and $x \leqslant y$, then $y = x + y/x$.

It is clear that a σ-complete Boolean algebra or a σ-complete Reisz space are examples of the given structure. The interpretation of operations $+$ and / is evident in both cases. Now we start with a triple (A, T_0, G), where

A denotes a sublattice of a lattice X which is closed under the operations $+$, / and members of A dominate the elements of X, i.e. for all $x \in X$ there exist elements $u, v \in A$ such that $u \leqslant x \leqslant v$,

G denotes a monotone complete, o-separable, regular group and

$T_0 : A \to G$ is a map satisfied

(i) If $x \leqslant y$, then $T_0(x) \leqslant T_0(y)$ and $T_0(y) = T_0(x) + T_0(y/x)$.

(ii) If $x, y \in A$, then $T_0(x) + T_0(y) = T_0(x \vee y) + T_0(x \wedge y)$ and $T_0(x+y) \leqslant T_0(x) + T_0(y)$.

(iii) If $x_n \in A$, $n=1, 2, \ldots$, $x_n \nearrow x_0$ in X, then $T_0(x_0) = \vee T_0(x_n)$.

Let us denote

$A_1 = \{y \in X : \exists (x_n)_{n \in \mathrm{N}} \text{ in } A,\ x_n \nearrow y \text{ in } X\}$.
$A_2 = \{y \in X : \exists (x_n)_{n \in \mathrm{N}} \text{ in } A,\ x_n \searrow y \text{ in } X\}$.

Just as in [5] def. 5 we find that T_0 may be extended to A_1 by writing $T_1 : A_1 \to G$,

$T_1(y) = \vee T_0(x_n)$, whenever $(x_n)_{n \in \mathrm{N}}$ is sequence in A increasing to $y \in A_1$.

Observe that the supremum of $(T_0(x_n))_{n \in \mathrm{N}}$ exists; as y is dominated by some $v \in A$, so does $T_0(v)$ bounds $(T_0(x_n))_{n \in \mathrm{N}}$. It is easy to see that the map T_1 has the following properties:

(i) T_1 is unambiguously defined and is a monotone map.

(ii) If $x, y \in A_1$, then $T_1(x) + T_1(y) = T_1(x \vee y) + T_1(x \wedge y)$ and $T_1(x+y) \leqslant T_1(x) + T_1(y)$.

(iii) If $x_n \in A_1$, $n=1, 2, \ldots$, $x_0 \in X$ and $x_n \nearrow x_0$ in X, then $x_0 \in A_1$ and $T_1(x_0) = \vee T_1(x_n)$.

We define $T^*: X \to G$

$$T^*(x) = \wedge\{T_1(y): x \leq y \in A_1\}.$$

The infimum always exists because the set $\{T_1(y): x \leq y \in A_1\}$ is bounded from below and downward directed (A_1 is a lattice and T_1 is a monotone map).

Proposition 1. *The map T^* has the following properties:*

(i) T^* is an extension of T_0 and T^* is an increasing map.

(ii) If $x, y \in X$, then $T^*(x \vee y) + T^*(x \wedge y) \leq T^*(x) + T^*(y)$ and $T^*(x+y) \leq T^*(x) + T^*(y)$.

(iii) If $x_n \in X$, $n = 1, 2, \ldots$, $x \in X$, $x_n \nearrow x$ in X, then $T^*(x) = \vee T^*(x_n)$

Proof. We prove only (iii). It is clear from (i) that $T^*(x) \geq T^*(x_n)$. G is o-separable and hence there exist sequences $(x_{n,k})_{k \in \mathbf{N}}$, $x_{n,k} \in A_1$, $n, k \in \mathbf{N}$ such that for every $n \in \mathbf{N}$ $x_{n,k} \geq x_{n,k+1}$ for all $k \in \mathbf{N}$ and $(T_1(x_{n,k}) - T^*(x_n)) \searrow 0$ $(k \to \infty)$. *Let* $\varphi \in \mathbf{N}^{\mathbf{N}}$. Using mathematical induction and the properties (i) and (ii) of T_1 we have $T_1\left(\bigvee_{n=1}^{m} x_{n,\varphi(n)}\right) \leq T_1(x_{m,\varphi(m)}) + \sum_{n=1}^{m-1} (T_1(x_{n,\varphi(n)}) - T^*(x_n))$. From this we obtain

$$T_1\left(\bigvee_{n=1}^{m} x_{n,\varphi(n)}\right) = T_1\left(\bigvee_{n=1}^{m} x_{n,\varphi(n)}\right) - T^*(x_m) + T^*(x_m) \leqq \sum_{n=1}^{m} [T_1(x_{n,\varphi(n)}) - T^*(x_n)] + $$
$$+ T^*(x_m),$$

and so

$$\bigvee_{m=1}^{\infty} T_1\left(\bigvee_{n=1}^{m} x_{n,\varphi(n)}\right) \leq \bigvee_{m=1}^{\infty}\left(\sum_{n=1}^{m} [T_1(x_{n,\varphi(n)}) - T^*(x_n)]\right) + \bigvee_{n=1}^{\infty} T^*(x_n).$$

We have

$$0 \leq T^*(x) - \bigvee_{n=1}^{\infty} T^*(x_n) \leq T_1\left(\bigvee_{n=1}^{\infty} x_{n,\varphi(n)}\right) - \bigvee_{n=1}^{\infty} T^*(x_n) \leq$$

$$\leq \bigvee_{m=1}^{\infty} T_1\left(\bigvee_{n=1}^{m} x_{n,\varphi(n)}\right) - \bigvee_{n=1}^{\infty} T^*(x_n) \leq \bigvee_{m=1}^{\infty}\left(\sum_{n=1}^{m} [T_1(x_{n,\varphi(n)}) - T^*(x_n)]\right)$$

and with respect to the regularity of G, $T^*(x) = \vee T^*(x_n)$.

Proposition 2.

(a) *Let* $x \leq y$, $x \in A$ $(x \in A_1$, *resp.* $x \in A_2)$, $y \in A_1$, *then* $T^*(x) + T^*(y/x) = \mathrm{T}_1(\mathrm{y})$.

(b) If $x_n \in A_2$, $n = 1, 2, \ldots$, $x_n \searrow x$ in X, then $T^*(x) = \wedge T^*(x_n)$.

Proof.

(a) Let $x \in A$, $u_n \in A$, $u_n \geq x$, $n = 1, 2, \ldots$, $u_n \nearrow y$, $y \in A_1$. We have $T_1(y) = \vee T_0(u_n)$ and $T_0(x) + T_0(u_n/x) = T_0(u_n)$ with respect to (i) from preporties of T_0. Now $u_n/x \nearrow y/x$ and $T_0(x) + T_1(y/x) = T_1(y)$. Let $x \in A_1$ and $x_n \in A$, $n = 1, 2,$

..., $x_n \nearrow x$. $T^*(y/x) \leqslant T^*(y/x_n) = T_1(y) - T_0(x_n)$, which implies $T_1(x) + T^*(y/x) \leqslant T_1(y)$. Finally, let $x \in A_2$. There exists a sequence (x_n), $x_n \in A_1$, $x_n \searrow x$ and $T^*(x) = \wedge T_1(x_n)$ (G is o-separable). Since $x \leqslant y \in A_1$, we can manage $x_n \leqslant y$, $n = 1, 2, \ldots$, and result follows from the above.

(b) We can put $x_0 \in A$, $x_0 \geqslant x_1$. Now we have $x_0/x_n \nearrow x_0/x$ and $T^*(x_0/x) = \vee T^*(x_0/x_n) = \vee(T^*(x_0) - T^*(x_n)) = T^*(x_0) - \wedge T^*(x_n)$, using Prop. 1 (iii). According to (a) $T^*(x_0/x) + T^*(x) = T^*(x_0)$, and so result follows.

Denote by L the set of all $x \in X$ for which

$$\vee\{T^*(y) : x \geqslant y \in A_2\} = \wedge\{T^*(z) : x \leqslant z \in A_1\}.$$

Proposition 3.

(a) If $x \in L$, $y \in X$, $x \leqslant y$, then $T^*(x) + T^*(y/x) = T^*(y)$.

(b) If $x_n \in L$, $n = 1, 2, \ldots, x_n \searrow x$ in X, then $T^*(x) = \wedge T^*(x_n)$.

Proof.

(a) Let $x \in A_2$, $y_1 \in A_1$, $y \leqslant y_1$. With respect to Prop. 2 (a) $T^*(y/x) \leqslant T^*(y_1/x) = T^*(y_1) - T^*(x)$ and $T^*(x) + T^*(y/x) \leqslant T^*(y)$. If $x \in L$, we have for all $u \in A_2$, $u \leqslant x$ $T^*(y/x) \leqslant T^*(y/u) \leqslant T^*(y) - T^*(u)$ according to the above.

We omit the proof of part (b) as a consequence of part (a) and Prop. 1. (ii), (iii).

Let K be a subset of X. K is said to be a σ-monotone subset of X if K contains suprema and infima of convergent monotone sequences of elements of K, i.e. if (x_n) is a sequence in K, $x \in X$ and $x_n \nearrow x$, then $x \in K$ and dually.

We shall prove in the next Proposition that L is a σ-monotone subset of X. Denote by Z the intersection of all σ-monotone subsets of X which contain A.

Proposition 4. *L is a σ-monotone subset of X, and so $Z \subset L$. Z is, in fact, the smallest σ-monotone subset of X which contains A.*

Proof. We shall prove only that L is a σ-monotone subset of X. It is clear that L contains A. According to Prop. 1 (iii) if $x_n \in L$, $x_n \nearrow x$ in X, we have $x \in L$. Let $y_n \in L$, $y_n \searrow y$ in X. $T^*(y) = \wedge T^*(y_n)$ with respect to prop. 3 (b). It is sufficient to show that $T^*(y) = \vee\{T^*(u) : y \geqslant u \in A_2\}$.

We use dual arguments as in the proof of (iii) Prop. 1. Let $u_{n,k} \in A_2$, $u_{n,k} \leqslant u_{n,k+1} \leqslant y_n$, $n, k \in \mathbf{N}$ such that $\bigvee_{k=1}^{\infty} T^*(u_{n,k}) = T^*(y_n)$. For all $\varphi \in \mathbf{N}^{\mathbf{N}}$ we have

$$T^*(y) - \vee\{T^*(u) : y \geqslant u \in A_2\} \leqslant T^*(y) - T^*\Big(\bigwedge_{m=1}^{\infty} \bigwedge_{n=1}^{m} u_{n,\varphi(n)}\Big) \leqslant$$

$$\leqslant T^*(y) - \bigwedge_{m=1}^{\infty} T^*\Big(\bigwedge_{n=1}^{m} u_{n,\varphi(n)}\Big) = \bigvee_{m=1}^{\infty} \Big(T^*(y) - T^*\Big(\bigwedge_{n=1}^{m} u_{n,\varphi(n)}\Big)\Big) \leqslant$$

$$\leqslant \bigvee_{m=1}^{\infty} \Big(\sum_{n=1}^{m} [T^*(y_n) - T^*(u_{n,\varphi(n)})]\Big).$$

Since $T^*(y_n)-T^*(u_{n,k}))\searrow 0$ $(k\to\infty)$ for all $n\in\mathbf{N}$ and G is a regular group, the result follows.

Denote by T the restriction of T^* to Z.

Theorem 1. *The set Z is a sublattice of X. The map T is the extension of T_0 and has the following properties:*

(i) If $x, y\in Z$, $x\leq y$, then $T(x)\leq T(y)$, $T(x)+T^*(y/x)=T(y)$.

(ii) If $x, y\in Z$, then $T(x\vee y)+T(x\wedge y) = T(x)+T(y)$ and $T^*(x+y) \leq T(x)+T(y)$.

(iii) T is continuous from above and below, i.e. if $x_n\in Z$, $n\in\mathbf{N}$, $x\in X$, $x_n\nearrow x$ in X, then $x\in Z$ and $T(x) = \vee T(x_n)$ and dually.

(iv) If $I:Z\to G$ is a map which satisfied (iii) and is the extension of T_0, then $I=T$.

Proof. Denote by $B_x = \{y\in Z: z\vee y,\ x\wedge y\in Z,\ T(x) + T(y) = T(x\vee y) + T(x\wedge y)\}$, where $x\in Z$. Since B_x is a σ-monotone subset of X and contains A, $B_x\supset Z$. (i) is evident, (ii) has just been proved, resp. Prop. 1. (ii); (iii) is clear from the definition Z and Prop. 1 (iii), resp. Prop. 3 (b). Finally $\{x\in Z: I(x) = T(x)\}$ is a σ-monotone subset of X and contains A which implies (iv).

Remark 1. The latest Theorem is a generalization of Theorem 7 in [5] in two directions. We abandon the two structures of the range of the map — a linear and a lattice one. On the other hand if we consider the Reisz space as the range of the map the examples in (c) part I show that our assumptions are weaker than the ones in [5].

III

Consequences

As the first consequence of Theorem 1 we obtain the theorem about an extension of monotonic group homomorphisms. A mapping f from one l-group H to another G is called a monotonic homomorphism if it is a group homomorphism and preserves an ordering, i.e. if $x\leq y$ in H, then $f(x)\leq f(y)$ in G.

We shall work with σ-complete l-group. It is known that they are commutative, Archimedean, relatively σ-complete and σ-continuous lattices. Let H and G be σ-complete l-groups and f be a monotonic homomorphism from H to $G\cdot f$ is said to be sequentially smooth if $(x_n)_{n\in\mathbf{N}}$, $x_n\in H$, $x_n\searrow 0$ implies $f(x_n)\searrow 0$ in G.

Theorem 2. *Let H be a σ-complete l-group and G be an o-separable and regular l-group. Let A be a sub l-group of H such that every element of H is dominated by some element of A. Let f_0 be a monotic sequentially smooth homomorphism from A to G. Then there exists a sub l-group B of a group H* which is a *σ-monotone subset of H and a monotonic homomorphism g from B to G which is an extension of f_0 and sequentially smooth.*

Proof. Arguing as in Theorem 23.6 page 129 [3], it can be verified that the σ-complete o-separable l-group G is a complete l-group. From now on observe that all assumptions of the construction of part II are fulfilled, interpreting the binary operation / as the minus in l-group, i.e. $x/y := x + (-y)$. We can put $B = Z$, where Z is the sublattice from Theorem 1 and define by g the restriction f^* to B (f^* is an analogy of T^*). Now we have to prove that B is a subgroup of H and g preserves $+$.

From the properties of f^* we have $f^*(x)+f^*(y) \geqslant f^*(x+y)$ for all $x, y \in H$. We show the reverse inequality for the elements $x, y \in B$. There exist sequences (x_n), (y_n) in A_2 such that $x_n \leqslant x$, $y_n \leqslant y$, for each $n \in \mathbf{N}$, $f^*(x) = \vee f^*(x_n)$, $f^*(y) = \vee f^*(y_n)$, because G is o-separable.

According to Prop. 2 (b) we have $f^*(x_n)+f^*(y_n) = f^*(x_n+y_n)$ for each $n \in \mathbf{N}$. Finally $f^*(x)+f^*(y) = \vee(f^*(x_n)+f^*(y_n)) = \vee f^*(x_n+y_n) \leqslant f^*(x+y)$. The fact that B is a subgroup of H follows without difficulty.

As the second consequence of the main theorem is the theorem concerning an extension of a measure defined on an algebra $\mathscr{A}$ and having values in an ordered group G to a measure on the smallest σ-algebra containing $\mathscr{A}$.

Let m be a set function on an algebra $\mathscr{A}$ of subsets of a fixed set Y and having values in a monotome σ-complete, comutative group G. m is said to be a measure on $\mathscr{A}$ with values in G iff

(i) $m(A) \geqslant 0$ for every $A \in \mathscr{A}$, $m(\emptyset) = 0$.

(ii) $m(A) = \bigvee_{k=1}^{\infty}\left(\sum_{n=1}^{k} m(A_n)\right)$ for every disjoint sequence (A_n) of elements $\mathscr{A}$ whose union is A.

It is easy to observe that a measure m has the following properties:

(iii) $m(A) \leqslant m(B)$ whenever $A, B \in \mathscr{A}$, $A \subset B$.

(iv) $m(A)+m(B) = m(A \cup B) + m(A \cap B)$ for every $A, B \in \mathscr{A}$.

(v) m is continuous from above (below) on $\mathscr{A}$.

Theorem 3. *If m is a measure on an algebra $\mathscr{A}$ with values in a monotone complete, o-separable, regular group G, then m has a unique extension m^* on a σ-algebra $\mathscr{S}$ generated by an algebra $\mathscr{A}$.*

Proof. We use result from part Ii in an obvious way. The system 2^Y with set theoretical operations $\cup$, $\cap$ and $-$ (set theoretical difference) has all the properties of the lattice X from part II. It is clear that a measure m, resp. an algebra $\mathscr{A}$, have the properties of T_0 and A in part II. Consider the system Z from Theorem 1. Z is a σ-monotone system and the extension m^* of m has all the properties of a G-valued measure on Z. By its definition Z is the smallest monotone system of sets which cantains $\mathscr{A}$, i.e. the smallest σ-algebra containing $\mathscr{A}$.

Remark 2. The above result should be compared with Theorem 3 in [7], where the method of measurable sets was used. The case vector valued measure is discussed, in fact, at the end of this paper.

In his paper [1] D. H. Fremlin, using a direct method, proved the well-known Daniell integral scheme for a linear map T, whose domain is a Riesz subspace F of a Dedekind σ-complete Riesz space E and having values in a weakly σ-distributive Dedekind σ-complete Riesz space G.

Let G be a Dedekind σ-complete Riesz space. Then G is said to be weakly σ-distributive if and only if whenever $(a_{n,k})_{n,k\in\mathbf{N}}$ is an order-bounded double sequence such that $a_{n,k}\searrow 0$ $(k\to\infty)$ for each $n\in\mathbf{N}$, then

$$\wedge\left\{\bigvee_{n=1}^{\infty} a_{n,\varphi(n)}:\varphi\in\mathbf{N}^{\mathbf{N}}\right\}=0.$$

First let us formulate our result of an extension of a linear map having values in an ordered vector space V and then consider the condition of g-regularity in case when V is a Riesz space.

Theorem 4. *Let X be a Dedekind σ-complete Riesz space and A be a Riesz subspace of X such that every element of X is dominated by some member of A. Let V be a monotone complete, o-separable and g-regular vector space. Let $T: A\to V$ be linear, monotone and sequentially continuous. Then T has an extension T^* to a linear, monotone and sequentially continuous map from Z to V, where Z is the smallest σ-monotone sublattice of X containing A.*

Proof. The above assumptions imply that we may use the result of Theorem 1. It will be sufficient to realize that the extension T^* is a linear map and Z (the σ-monotone sublattice of X from Theorem 1) is a vector subspace of X. The desired result follows from the fact that for real $\alpha>0$ $T^*(\alpha x) = \alpha T^*(x)$, for all $x\in A_1\cup A_2$ and $T^*(-x) = -T^*(x)$ for all $x\in Z$, resp. T^* is additive on Z.

Let us discuss the conditions of g-regularity and weak σ-distributivity of a Riesz space V.

Let $(a_{n,k})$ be an order-bounded double sequence in V such that $a_{n,k}\searrow 0$ $(k\to\infty)$ for each $n\in\mathbf{N}$. Since for each $\varphi\in\mathbf{N}^{\mathbf{N}}$ $\bigvee_{m=1}^{\infty}\left(\sum_{n=1}^{m} a_{n,\varphi(n)}\right) = \bigvee_{n=1}^{\infty} a_{n,\varphi(n)}$, g-regularity implies weak σ-distributivity of V.

Conversely we prove that if V is weak σ-distributive, then V is relatively g-regular, i.e. if whenever $(a_{n,k})$ is an order-bounded double sequence in V such that $a_{n,k}\searrow 0$ $(k\to\infty)$ for each $n\in\mathbf{N}$ and such that there exists $\varphi_0\in\mathbf{N}^{\mathbf{N}}$ that $\left(\sum_{n=1}^{m} a_{n,\varphi_0(n)}\right)_{m\in\mathbf{N}}$ is bounded, then $\wedge\left\{\bigvee_{m=1}^{\infty}\left(\sum_{n=1}^{m} a_{n,\varphi(n)}\right):\varphi\in\mathbf{N}^{\mathbf{N}}\right\}=0.$

Let c be a positive element in a Dedekind σ-complete Riesz space V. Denote $V[c] = \{b\in V: -\alpha c\leqslant b\leqq\alpha c,\ \text{real}\ \alpha>0\}$. It is obvious that $V[c]$ is also a Dedekind σ-complete Riesz space and has an order unit c. By the fundamental Krein—Kakutani vector lattice representation theorem there exists a compact Hausdorff space S such that $V[c]$ is isometric and lattice isomorphic to $C(S)$. It is well known that in this case ($C(S)$ is σ-complete) S is totally disconnected and the closure of a countable union of clopen subsets of S is clopen. The proofs of the following lemmas are known and therefore may be omitted (see [10]).

Lemma 1. *When $C(S)$ is a Dedekind σ-complete and (f_n) is a sequence in $C(S)$ which is bounded below, then*

$$\left\{s \in S : \inf f_n(s) > \left(\bigwedge_{n=1}^{\infty} f_n\right)(s)\right\}$$

is a countable union of closed nowhere dense Baire sets.

Lemma 2. *If $C(S)$ is weakly σ-distributive, then each subset of the union of a countable family of closed nowhere dense Baire sets is nowhere dense.*

Proposition 5. *When a Riesz space V is weakly σ-distributive, then V is relatively g-regular.*

Proof. Let $(a_{n,k})$ be a double sequence in V bounded by an element $c \in V$. Let $a_{n,k} \searrow 0$ $(k \to \infty)$ for each $n \in \mathbf{N}$, $\varphi_0 \in \mathbf{N}^{\mathbf{N}}$ and $d \in V$ such that $d \geqslant \sum_{n=1}^{m} a_{n,\varphi_0(n)}$ for all $m \in \mathbf{N}$. $V[c \vee d]$ may be identified with $C(S)$, where $C(S)$ is weakly σ-distributive. Denote by $\mathscr{C}$ a system of all clopen subsets of S. Let $e \in V$, $e \geqslant 0$ be a lower bound of

$$\left\{\bigvee_{m=1}^{\infty} \sum_{n=1}^{m} a_{n,\varphi(n)} : \varphi \in \mathbf{N}^{\mathbf{N}}\right\}.$$

We have $e \in V[c \vee d] \sim C(S)$. Let $e \neq 0$, i.e. (after identification) there exists an $x_0 \in S$, $e(x_0) > 0$. S is totally disconnected, then there exist $\varepsilon > 0$ real and $C \in \mathscr{C}$, $C \neq \emptyset$ such that $\varepsilon \cdot \chi_C(x) \leqslant e(x)$ for all $x \in C$. $a_{n,k} \in V[c \vee d] \sim C(S)$ for all $n, k \in \mathbf{N}$. Since $\bigwedge_{k=1}^{\infty} a_{n,k} = 0$, according to Lemma 1 there exist Baire sets A_n such that $a_{n,k}(x) \searrow 0$ $(k \to \infty)$ for all $x \in S - A_n$. Let $A = \bigcup_{n=1}^{\infty} A_n$. With respect to Lemma 2 A^- (the closure of A) is nowhere dense. Let $C_1 \neq \emptyset$, $C_1 \in \mathscr{C}$, $C_1 \subset C - A^-$. Sequences of continuous functions $(a_{n,k})_{k \in \mathbf{N}}$, monotonously converge on C_1, pointwise so, by Dini's theorem the zero function is the uniform limit of $(a_{n,k})_{k \in \mathbf{N}}$ for each $n \in \mathbf{N}$. For every $n \in \mathbf{N}$ there exists $\varphi(n) \in \mathbf{N}$ such that $\varphi(n) \geqslant \varphi_0(n)$ and $a_{n,\varphi(n)}(x) \leqslant \frac{\varepsilon}{2^{n+1}}$ for all $x \in C_1$. Hence $\sum_{n=1}^{\infty} a_{n,\varphi(n)}(x) \leqslant \frac{\varepsilon}{2}$ for all $x \in C_1$. On the other hand a set

$$B = \left\{x \in C_1 : \left(\bigvee_{m=1}^{\infty} \sum_{n=1}^{m} a_{n,\varphi(n)}\right)(x) > \sum_{n=1}^{\infty} a_{n,\varphi(n)}(x)\right\}$$

is nowhere dense, according to the lemmas above. In this way there exists a non-empty $C_2 \in \mathscr{C}$, $C_2 \subset C_1 - B^-$. We have

$$e(x) \leqslant \left(\bigvee_{m=1}^{\infty} \sum_{n=1}^{m} a_{n,\varphi(n)}\right)(x) = \sum_{n=1}^{\infty} a_{n,\varphi(n)}(x) \leqslant \frac{\varepsilon}{2} < e(x)$$

for all $x \in C_2$, a contradiction.

Remark 3. A simple example shows that we cannot prove more because the space m of all bounded real sequences with coordinatewise ordering is weakly σ-distributive but not g-regular. Indeed, if $b \in m$, denote by $b(1), b(2), b(3), \dots$ its coordinates. Hence $b = (b(1), b(2), b(3), \dots)$. Now we can set $b_k(i) = 0$ if $i < k$ and $b_k(i) = 1$ whenever $i \geqslant k$. It is clear that $b_k \searrow 0$ $(k \to \infty)$ in m but if we define $a_{n,k} = b_k$ for all $n \in \mathbf{N}$, we have a sequence $\left(\sum_{n=1}^{m} a_{n,\varphi(n)}\right)_{m \in \mathbf{N}}$ unbounded for all $\varphi \in \mathbf{N}^{\mathbf{N}}$.

Remark 4. There are several papers discussing an integral with values in ordered spaces, however, from other points of view. See, for example [9], [4], [6].

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О ПРОДОЛЖЕНИИ ОПЕРАТОРОВ С ЗНАЧЕНИЯМИ В ПОЛУУПОРЯДОЧЕННЫХ ПРОСТРАНСТВАХ

Петер Волауф

Резюме

В первой части излагается общая теория — продолжение огератора определенного на подмножестве A можества X со значениями в регулярной $л$-группе Γ (σ-полная $л$-группа Γ называется регулярной, если выполняется:

$$c \geqq a_{n,k} \searrow 0 \; (k \to \infty) \; n = 1, 2, 3, \ldots, \Rightarrow \wedge \left\{ \bigvee_{m=1}^{\infty} \sum_{n=1}^{m} a_{n,\varphi(n)} : \varphi \in \mathbf{N}^{\mathbf{N}} \right\} = 0).$$

Следствием этой теории являются теорема о продолжении изотонного гомоморфизма σ-полной $л$-группы X в регулярную $л$-группу Γ, теорема о продолжении интеграла Даниела и теорема о продолжении меры со значениями в герулярной $л$-группе Γ. В последной части работы выясняется условие регулярности в том случае когда Γ линейная структура.