Mathematica Slovaca

Grażyna Kwiecińska

Measurability of real functions defined on the product of metric spaces

Mathematica Slovaca, Vol. 36 (1986), No. 3, 267--281

Persistent URL: http://dml.cz/dmlcz/129033

Math. Slovaca 36, 1986, No. 3, 267—281

MEASURABILITY OF REAL FUNCTIONS DEFINED ON THE PRODUCT OF METRIC SPACES

GRAŻYŃA KWIECIŃSKA

Let $(T, d, \mathcal{K}, \lambda)$ be a complete metric space with a metric d, with a σ-finite G_δ-regular complete measure λ defined over a σ-field $\mathcal{K}$ of subsets of T.

Denote by λ^* the outer measure corresponding to λ.

(1) Let $\mathcal{A}$ be a family of λ-measurable sets with nonempty interiors of a positive and finite measure λ, the boundaries of which are of λ measure zero.

Definition 1. *The sequence $\{I_k\}_{k=1}^{\infty} \subset \mathcal{A}$ is said to converge to the point $t_0 \in T$ iff $t_0 \in \operatorname{Int}(I_k)$ for $k = 1, 2, \dots$ and the sequence of diameters $\delta(I_k)$ converge to zero as k approches infinity.*

This will be denoted by $I_k \to t_0$.

Let us note that according to the definition due to Bruckner ([1], p. 30) the pair $(\mathcal{A}, \to)$ forms a differentiation basis for the space $(T, d, \mathcal{K}, \lambda)$.

Definition 2. *Let $A \subset T$ and $t_0 \in T$. The upper (lower) bound of the set of numbers* $\lim\limits_{k\to\infty} \dfrac{\lambda^*(A \cap I_k)}{\lambda(I_k)}$ taken from all the sequences $I_k \to t_0$ (for which this limit exists) is called the upper (lower) external density of A at t_0 with respect to $\mathcal{A}$ and is denoted by $D_u^*(t_0, A)(D_l^*(t_0, A))$.

If $D_u^*(t_0, A) = D_l^*(t_0, A)$, then their common value is called the external density of A at t_0 with respect to $\mathcal{A}$ and is denoted by $D^*(t_0, A)$.

If $A \in \mathcal{K}$, then the respective external densities are called densities with respect to $\mathcal{A}$ and denoted by $D_u(t_0, A)$, $D_l(t_0, A)$ and $D(t_0, A)$, respectively.

A point t_0 is called a density point of the set A with respect to $\mathcal{A}$ if there exists a set $B \in \mathcal{K}$ such that $B \subset A$ and $D(t_0, B) = 1$.

Assume that

(2) the family $\mathcal{A}$ is countable and for every $t_0 \in T$ there is a sequence of sets $\{I_k\}_{k=1}^{\infty}$ from $\mathcal{A}$ converging to t_0.

Moreover assume that

(3) $\mathscr{A}$ has the density property, i.e. for every set $A \subset T$ the λ measure of set $\{t \in A: D_1^*(t, A) < 1\}$ is equal to zero.

Definition 3. *The function g: $T \to R$ is called approximately upper (lower) semicontinuous at the point $t_0 \in T$ with respect to $\mathscr{A}$ iff for every $a \in R$ if $f(t_0) < a$ ($f(t_0) > a$), then there exists a set $F \in \mathscr{K}$ such that $F \subset \{t \in T: f(t) < a\}$ ($F \subset \{t \in T: f(t) > a\}$) and $D(t_0, F) = 1$.*

A function that is simultaneously approximately lower and upper semicontinuous at $t_0 \in T$ with respect to $\mathscr{A}$ is called approximately continuous at t_0 with respect to $\mathscr{A}$.

A function that is approximately continuous (approximately lower semicontinuous) (approximately upper semicontinuous)) in any point $t_0 \in T$ with respect to $\mathscr{A}$ is called approximately continuous (approximately lower semicontinuous) (approximately upper semicontinuous)) with respect to $\mathscr{A}$.

Lemma 1. *If the function g: $T \to R$ is λ-measurable, then g is λ-almost everywhere approximately continuous with respect to $\mathscr{A}$.*

Proof. Indeed, by Lusin's theorem for every positive ε there exists a closed set $F \subset T$ such that the function $g|_F$ is continuous and $\lambda(T - F) < \varepsilon$. Since $\mathscr{A}$ has the density property almost every point of the set F is the density point of this set with respect to $\mathscr{A}$. Therefore the function g is λ-almost everywhere approximately continuous with respect to $\mathscr{A}$.

Definition 4. *The λ-measurable function g: $T \to R$ is said to be degenerate (positively degenerate) at the point $t_0 \in T$ with respect to $\mathscr{A}$ when there exists a open interval $U \subset R$ such that $g(t_0) \in U$ and the upper (lower) density of the counterimage $g^{-1}(U)$ at t_0 with respect to $\mathscr{A}$ is equal to zero.*

Definition 5. ([4], definition 4). *The function g: $T \to R$ has the property (G) with respect to $\mathscr{A}$ iff for every positive ε there exists a set $I \in \mathscr{A}$ such that $\lambda(A \cap I) > 0$ and $\operatorname{osc}_U g \leq \varepsilon$, where U is the set of density points of $A \cap I$ with respect to $\mathscr{A}$ belonging to $A \cap I$.*

Theorem 1. *Let the λ-measurable function g: $T \to R$ be positively nondegenerate at every point of the closed set $A \subset T$. Then the λ-measurable function*

$$f(x) = \begin{cases} g(x) & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

has the property (G) with respect to $\mathscr{A}$.

Proof. Let $E \in \mathscr{K}$ be a set of a positive λ measure and let $\varepsilon > 0$ be fixed.

Assume that $\lambda(E - A) > 0$. Then there is a point $t_0 \in T$ such that $t_0 \in E - A$ and

$D(t_0, E-A)=1$. As the set A is closed, it follows from property (2) of the family $\mathscr{A}$ that there exists a set $I\in\mathscr{A}$ such that $t_0\in \mathrm{Int}\,(I)$ and $I\cap A=\emptyset$. Therefore for $t\in T$ we have $f(t)=0$. Hence $\operatorname*{osc}_I f=0\leqslant\varepsilon$ and $\lambda(E\cap I)>0$.

Assume now that $\lambda(E-A)=0$. Then we notice that all density points of E bolong to A. In order to show that

A6 1) there exists a set $I\in\mathscr{A}$ such that $\lambda(I\cap E\cap A)>0$ and $\operatorname*{osc}_V f\leqslant\varepsilon$, where V is the set of density points of $I\cap E\cap A$ with respect to $\mathscr{A}$ belonging to $I\cap E\cap A$, assume that 1) does not hold. Then we have:

2) if for the set $J\in\mathscr{A}$ the inequality $\lambda(J\cap E\cap A)>0$ holds, then $\operatorname*{osc}_W f>\varepsilon$, where W is the set of density points of $J\cap E\cap A$ with respect to $\mathscr{A}$ belonging to $J\cap E\cap A$.

We shall construct a sequence of points $\{t_k\}_{k=1}^{\infty}\subset E\cap A$ and a sequence $\{I_k\}_{k=1}^{\infty}\subset\mathscr{A}$ such that the condition 2) leads to a contradiction.

Let $t_1\in E\cap A$ be a point such that

3) $D(t_1, E\cap A)=1$ and

4) the function f is approximately continuous at t_1 with respect to $\mathscr{A}$.

The existence of point t_1 follows from the density property of $\mathscr{A}$ and from lemma 1.

Let $I_1\in\mathscr{A}$ be the set such that

5) $t_1\in\mathrm{Int}\,(I_1)$ and

6) $\dfrac{\lambda(I_1\cap E\cap A)}{\lambda(I_1)}>\dfrac{1}{2}$ and $\dfrac{\lambda\left(I_1\cap\left\{t\in E\cap A\colon\ |f(t)-f(t_1)|<\frac{\varepsilon}{8}\right\}\right)}{\lambda(I_1)}>\dfrac{1}{2}$.

The existence of the set I_1 follows from 3) and 4).

Let $G_1=\left\{t\in E\cap A\colon\ |f(t)-f(t_1)|\geqslant\frac{\varepsilon}{2}\right\}$. Then

7) $\lambda(G_1)>0$.

Indeed. Assume that

8) $\lambda(G_1)=0$.

Then for points $t\in(I_1\cap E\cap A)-G_1$ the inequality $|f(t)-f(t_1)|<\frac{\varepsilon}{2}$ holds and therefore $\operatorname*{osc}_{(I_1\cap E\cap A)-G_1} f<\varepsilon$.

If $|f(t)-f(t_1)|\leqslant\frac{\varepsilon}{2}$ for the points $t\in I_1\cap E\cap A\cap G_1$ such that $D(t, I_1\cap E\cap A)=1$, then $\operatorname{osc} f\leqslant\varepsilon$ on the set of the density points of the set $I_1\cap E\cap A$, which contradicts 2). Therefore there exists a point $s_1\in I_1\cap E\cap A\cap G_1$ such that $D(s_1, I_1\cap E\cap A)=1$ and $|f(s_1)-f(t_1)|>\frac{\varepsilon}{2}$. But the function f is positively nondegenerate at the point $s_1(s_1 \in A)$ and $D(s_1, I_1 \cap E \cap A)=1$, thence $\lambda\left(\left\{t \in I_1 \cap E \cap A\colon\ |f(t_1)-f(t)|>\frac{\varepsilon}{2}\right\}\right)>0$, which is contradictory with 8). Therefore 7) holds true.

Let $t_2 \in G_1 \cap \mathrm{Int}\,(I_1)$ be a point such that

9) $D(t_2, G_1)=1$ and

10) f is approximately continuous at t_2 with respect to $\mathscr{A}$.

Again the existence of point t_2 follows from the density property of $\mathscr{A}$ and from lemma 1.

Let $I_2 \in \mathscr{A}$ be such that

11) $t_2 \in \mathrm{Int}\,(I_2)$, $\mathrm{Cl}\,(I_2) \subset \mathrm{Int}\,(I_1)$, $\delta(I_2)<\frac{1}{2}$ and

12) $$\frac{\lambda(I_2 \cap E \cap A)}{\lambda(I_2)}>\frac{2}{3} \quad \text{and} \quad \frac{\lambda\left(I_2 \cap\left\{t \in E \cap A\colon\ |f(t_2)-f(t)|<\frac{\varepsilon}{8}\right\}\right)}{\lambda(I_2)}>\frac{2}{3}.$$

The existence of set I_2 follows from 9) and 10). Similarly as before the set $G_2=\left\{t \in I_2 \cap E \cap A\colon\ |f(t_2)-f(t)| \geqslant \frac{\varepsilon}{2}\right\}$ is λ-measurable and has a positive measure λ.

Proceeding analogously we define the sequence $\{I_k\}_{k=1}^{\infty}$ of the sets from $\mathscr{A}$ and the sequence $\{t_k\}_{k=1}^{\infty}$ such that

13) $t_k \in G_{k-1} \cap \mathrm{Int}\,(I_{k-1})$, $D(t_k, G_{k-1})=1$ and f is approximately continuous at the point t_k with respect to $\mathscr{A}$, where

$$G_{k-1}=\left\{t \in I_{k-1} \cap E \cap A\colon\ |f(t_{k-1})-f(t)| \geqslant \frac{\varepsilon}{2}\right\},$$

14) $t_k \in \mathrm{Int}\,(I_k)$, $\mathrm{Cl}\,(I_k) \subset \mathrm{Int}\,(I_{k-1})$, $\delta(I_k)<\frac{1}{2^{k-1}}$ and

15) $$\frac{\lambda(I_k \cap E \cap A)}{\lambda(I_k)}>\frac{k}{k+1} \quad \text{and}$$

$$\frac{\lambda\left(I_k \cap\left\{t \in E \cap A\colon\ |f(t_k)-f(t)|<\frac{\varepsilon}{8}\right\}\right)}{\lambda(I_k)}>\frac{k}{k+1}$$

for $k=1, 2, \ldots$.

Since $t_k \in G_{k-1}$, we have

16) $|f(t_{k-1})-f(t_k)| \geqslant \frac{\varepsilon}{2}$ for $k=1, 2, \ldots$.

The set $\bigcap_{k=1}^{\infty} I_k$ consists of one point t_0. As the function f is positively nondegenerate at t_0 with respect to $\mathscr{A}$ $\left(t_0 \in \bigcap_{k=1}^{\infty} I_k \cap A\right)$ we have shown that

$$D_1\left(t_0, \left\{t:\ |f(t_0)-f(t)|<\frac{\varepsilon}{8}\right\}\right)>0.$$

Denote by α this density. Moreover the sequence of sets $\{I_k\}_{k=1}^{\infty}$ is a convergence to t_0, hence there exists a natural number n such that for $k>n$

$$\frac{\lambda\left(I_k\cap\left\{t:\ |f(t_0)-f(t)|<\frac{\varepsilon}{8}\right\}\right)}{\lambda(I_k)}>\frac{\alpha}{2} \quad \text{and}$$

$$\frac{\lambda\left(I_k\cap\left\{t\in E\cap A:\ |f(t_k)-f(t)|<\frac{\varepsilon}{8}\right\}\right)}{\lambda(I_k)}>1-\frac{\alpha}{2}.$$

Therefore for every $k>n$

$$\left\{t:\ |f(t_0)-f(t)|<\frac{\varepsilon}{8}\right\}\cap\left\{t\in E\cap A:\ |f(t)-f(t_k)|<\frac{\varepsilon}{8}\right\}\cap I_k\neq\emptyset.$$

Thence for $k>n$ the following inequality holds $|f(t_0)-f(t_k)|<\frac{\varepsilon}{8}$, which contradicts 16). Thus the negation of 1) leads to a contradiction. Therefore 1) holds true. The proof of the theorem is completed.

Lemma 2 ([2], lemma 2). *Let $(X, \mathcal{M}, \mu)$ be a measurable space with the σ-finite measure μ. Let g: $X\to R$ be such that for any $\varepsilon>0$ for a class of sets $\mathscr{D}_\varepsilon = \{D\in\mathcal{M}:\ \operatorname{osc}_D g\leq\varepsilon\}$ satisfies the following condition:*

(d) *for any set $B\in\mathcal{M}$ with a positive measure there exists a set $D\in\mathscr{D}_\varepsilon$ such that $D\subset B$ and $\mu(D)>0$.*

Then the function g is $\bar{\mu}$-measurable, where $\bar{\mu}$ stands for the completion of μ.

(Davies has proved the lemma under the assumption that is finite, whereas σ-finiteness is sufficient).

Let for every $i=i, \ldots, n (X_i, \varrho_i, \mathcal{M}_i, \mu_i)$ be a space as $(T, d, \mathscr{K}, \lambda)$ was, i.e. let every $(X_i, \varrho_i, \mathcal{M}_i, \mu_i)$ be a complete space with a σ-finite G_δ-regular complete measure μ_i defined over the σ-field $\mathcal{M}_i$ of subsets of X_i.

Moreover let for every $i=1, \ldots, n$ $\mathscr{F}_i\subset\mathcal{M}_i$ be a family which satisfies the conditions (1), (2) and (3) of family $\mathscr{A}$.

Let $(X, \varrho, \mathcal{M}, \mu)=(X_1\times\ldots\times X_n, \varrho_1\times\ldots\times\varrho_n, \overline{\mathcal{M}_1\times\ldots\times\mathcal{M}_n}, \overline{\mu_1\times\ldots\times\mu_n})$ where

$\overline{\mu_1 \times \ldots \times \mu_n}$ denotes the completion of the measure $\mu_1 \times \ldots \times \mu_n$. Moreover let

$$\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n = \{F\colon\ F = F_1 \times \ldots \times F_n,\ F_i \in \mathcal{F}_i \text{ for } i = 1, \ldots, n\}.$$

We note that $\mathcal{F}$ has the density property because every family $\mathcal{F}_i$ has the density property (see [1], p. 2 and 34).

Let $A \subset X = X_{i-1)} \times X_i \times X_{(i+1}$, where $X_{i-1)} = X_1 \times \ldots \times X_{i-1}$ and $X_{(i+1} = X_{i+1} \times \ldots \times X_n$. Then the sets $A_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n} = \{x_i \in X_i\colon (x_1, \ldots, x_n) \in A\}$ and

$$A_{\ldots, \bullet, x_i, \bullet, \ldots} = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{i-1)} \times X_{(i+1}\colon\ (x_1, \ldots, x_n) \in A\}$$

are called a section of the set A with respect to $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and a section of the set A with respect to x_i respectively.

Lemma 3. *Let $A \in \mathcal{M}$. For every fixed $i = 1, \ldots, n$ there exists a set $B \subset A$ and $B \in \mathcal{M}$ such that $\mu(A - B) = 0$, every point $(x_1, \ldots, x_n) \in B$ is the density point of B with respect to $\mathcal{F}$ and for every point $(x_1, \ldots, x_n) \in B$*

(i) $D(x_i, B_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}) = 1$ *and*

(ii) $D((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), B_{\ldots, \bullet, x_i, \bullet, \ldots}) = 1$.

The proof of this lemma is analogous to the proof of lemma 2 of [7].

Lemma 4. *Let $A \in \mathcal{M}$. There exists a set $B \subset A$ and $B \in \mathcal{M}$ such that $\mu(A - B) = 0$ and for every point $(x_1, \ldots, x_n) \in B$*

(i) $D((x_1, \ldots, x_n), B) = 1$,

(ii) *for every $i = 1, \ldots, n$* $D(x_i, B_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}) = 1$,

(iii) $D((x_2, \ldots, x_n), B_{x_1, \bullet, \ldots}) = 1$.

Proof. In accordance with lemma 3 ($i = 1$) for the set A there exists a μ-measurable subset A_1 of A such that $\mu(A - A_1) = 0$, for every point $(x_1, \ldots, x_n) \in A_1$ $D((x_1, \ldots, x_n), A_1) = 1$, $D(x_1, (A_1)_{\bullet, x_2, \ldots, x_n}) = 1$ and $D((x_2, \ldots, x_n), (A_1)_{x_1, \bullet, \ldots}) = 1$. Again in accordance with lemma 3 ($i = 2$) for the set A_1 there exists a μ-measurable subset A_2 of A_1 such that $\mu(A_1 - A_2) = 0$ and for every point $(x_1, \ldots, x_n) \in A_2$ $D((x_1, \ldots, x_n), A_2) = 1$, $D(x_2, (A_2)_{x_1, \bullet, x_3, \ldots, x_n}) = 1$ and

$$D((x_1, x_3, \ldots, x_n), (A_2)_{\bullet, x_2, \bullet, \ldots}) = 1.$$

Let $C_1 = A_1 - A_2$. It is clear that

$$\overline{\mu_2 \times \ldots \times \mu_n}(\{(x_2, \ldots, x_n)\colon\ \mu_1^*(C_1)_{\bullet, x_2, \ldots, x_n}) > 0\}) = 0.$$

Let $D_1 = \{(x_2, \ldots, x_n)\colon\ \mu_1^*(c_1)_{\bullet, x_2, \ldots, x_n}) > 0\}$ and let

$$F_1 = \{(x_3, \ldots, x_n)\colon\ \mu_2^*(D_1)_{\bullet, x_3, \ldots, x_n}) > 0\}(\overline{\mu_3 \times \ldots \times \mu_n}(F_1) = 0)$$

and $H_1 = \{x_1\colon\ \overline{\mu_2 \times \ldots \times \mu_n^*}((C_1)_{x_1, \bullet, \ldots}) > 0\}$ $(\mu_1(H_1) = 0)$. For B_1 take $B_1 = A_2 - [(X_1 \times D_1) \cup (X_1 \times X_2 \times F_1) \cup (H_1 \times X_2 \times \ldots \times X_n)]$. Evidently for every point

$(x_1, \ldots, x_n) \in B_1$ we have $D((x_1, \ldots, x_n), B_1) = 1$, $D(x_1, (B_1)_{\bullet, x_2, \ldots, x_n}) = 1$, $D(x_2, (B_1)_{x_1, \bullet, x_3, \ldots, x_n}) = 1$ and $D((x_2, \ldots, x_n), (B_1)_{x_1, \bullet, \ldots}) = 1$.

As a sequel to the set B_1 in accordance with lemma 3 ($i = 3$) there exists a μ-measurable subset A_3 of B_1 such that $\mu(B_1 - A_3) = 0$ and for every point $(x_1, \ldots, x_n) \in A_3$ $D((x_1, \ldots, x_n), A_3) = 1$, $D(x_3, (A_3)_{x_1, x_2, \bullet, x_4, \ldots, x_n}) = 1$ and $D((x_1, x_2, x_4, \ldots, x_n), (A_3)_{\bullet, \bullet, x_3, \bullet, \ldots}) = 1$. Let $C_2 = B_1 - A_3$ and let $D_{2,1} = \{(x_2, \ldots, x_n): \mu_1^*((C_2)_{\bullet, x_2, \ldots, x_n}) > 0\}$ and $D_{2,2} = \{(x_1, x_3, \ldots, x_n): \mu_2^*((C_2)_{x_1, \bullet, x_3, \ldots, x_n}) > 0\}$. $\overline{\mu_2 \times \ldots \times \mu_n}(D_{2,1}) = 0$ and $\overline{\mu_1 \times \mu_3 \times \ldots \times \mu_n}(D_{2,2}) = 0$ because $\mu(C_2) = 0$.

Let $F_{2,1,1} = \{(x_3, \ldots, x_n): \mu_2^*((D_{2,1})_{\bullet, x_3, \ldots, x_n}) > 0\}$,

$$F_{2,1,2} = \{(x_2, x_4, \ldots, x_n): \mu_3^*((D_{2,1})_{x_2, \bullet, x_4, \ldots, x_n}) > 0\},$$

$$F_{2,2,1} = \{(x_3, \ldots, x_n): \mu_1^*((D_{2,2})_{\bullet, x_3, \ldots, x_n}) > 0\},$$

$$F_{2,2,2} = \{(x_1, x_4, \ldots, x_n): \mu_3^*((D_{2,2})_{x_1, \bullet, x_4, \ldots, x_n}) > 0\}$$

and

$$H_2 = \{x_1: \overline{\mu_2 \times \ldots \times \mu_n^*}((A_1 - A_3)_{x_1, \bullet, \ldots}) > 0\}.$$

Evidently all these sets are of respective measure zero.

Let $B_2 = A_3 - [(X_1 \times D_{2,1}) \cup \{(x_1, \ldots, x_n): (x_1, x_3, \ldots, x_n) \in D_{2,2}$ and $x_2 \in X_2\} \cup (X_1 \times X_2 \times F_{2,1,1}) \cup \{(x_1, \ldots, x_n): x_1 \in X_1$ and $x_3 \in X_3$ and $(x_2, x_4, \ldots, x_n) \in F_{2,1,2}\} \cup (X_1 \times X_2 \times F_{2,2,1}) \cup \{(x_1, \ldots, x_n): (x_1, x_4, \ldots, x_n) \in F_{2,2,2}$ and $(x_2, x_3) \in X_2 X_3\} \cup (H_2 \times X_2 \times \ldots \times X_n)]$.

For every point $(x_1, \ldots, x_n) \in B_2$ $D((x_1, \ldots, x_n), B_2) = 1$, $D(x_1, (B_2)_{\bullet, x_2, \ldots, x_n}) = 1$, $D(x_2, (B_2)_{x_1, \bullet, x_3, \ldots, x_n}) = 1$, $D(x_3, (B_2)_{x_1, x_2, \bullet, x_4, \ldots, x_n}) = 1$, $D((x_2, \ldots, x_n), (B_2)_{x_1, \bullet, \ldots}) = 1$. Proceeding analogously in accordance with lemma 3 ($i = n$) we define for the set B_{n-2} a μ-measurable set $A_n \subset B_{n-2}$ such that $\mu(B_{n-2} - A_n) = 0$ and for every point $(x_1, \ldots, x_n) \in A_n$ $D((x_1, \ldots, x_n), A_n) = 1$, $D(x_n, (A_n)_{x_1, \ldots, x_{n-1}, \bullet}) = 1$ and $D((x_1, \ldots, x_{n-1}), (A_n)_{\ldots, \bullet, x_n}) = 1$. Let $C_{n-1} = B_{n-2} - A_n$. Evidently $\mu(C_{n-1}) = 0$. Let

$$D_{n-1,1} = \{(x_2, \ldots, x_n): \mu_1^*((C_{n-1})_{\bullet, x_2, \ldots, x_n}) > 0\},$$

$$D_{n-1,2} = \{(x_1, x_3, \ldots, x_n): \mu_2^*((C_{n-1})_{x_1, \bullet, x_3, \ldots, x_n}) > 0\},$$

....

$$D_{n-1,n-1} = \{(x_1, \ldots, x_{n-2}, x_n): \mu_{n-1}^*((C_{n-1})_{x_1, \ldots, x_{n-2}, \bullet, x_n}) > 0\}.$$

Evidently all these sets are of respective measure zero.

Moreover the sets

$F_{n-1,1,1} = \{(x_3, \ldots, x_n): \mu_2^*((D_{n-1,1})_{\bullet, x_3, \ldots, x_n}) > 0\}$,

$F_{n-1,1,2} = \{(x_2, x_4, \ldots, x_n): \mu_3^*((D_{n-1,1})_{x_2, \bullet, x_4, \ldots, x_n}) > 0\}$,

....

$F_{n-1,1,n-1} = \{(x_2, \ldots, x_{n-1}): \mu_n^*((D_{n-1,1})_{x_2, \ldots, x_{n-1}, \bullet}) > 0\}$ and

$F_{n-1,2,1}=\{(x_3, \ldots, x_n):\ \mu_1^*((D_{n-1,2})_{\bullet, x_3, \ldots, x_n})>0\}$,

$F_{n-1,2,2}=\{(x_1, x_4, \ldots, x_n):\ \mu_3^*((D_{n-1,2})_{x_1\bullet, x_4, \ldots, x_n})>0\}$,

....

$F_{n-1,2,n-1}=\{(x_1, x_3, \ldots, x_n):\ \mu_n^*((D_{n-1,2})_{x_1, \ldots, x_{n-1},\bullet})>0\}$

....

$F_{n-1,n-1,1}=\{(x_2, \ldots, x_{n-2}, x_n):\ \mu_1^*((D_{n-1,n-1})_{\bullet, x_2, \ldots, x_{n-2}, x_n})>0\}$,

$F_{n-1,n-1,2}=\{(x_1, x_3, \ldots, x_{n-2}, x_n):\ \mu_2^*((D_{n-1,n-1})_{x_1,\bullet, x_3, \ldots, x_{n-2}, x_n})>0\}$

....

$F_{n-1,n-1,n-2}=\{(x_1, \ldots, x_{n-3}, x_n):\ \mu_{n-2}^*((D_{n-1,n-1})_{x_1, \ldots, x_{n-3},\bullet, x_n})>0\}$

$F_{n-1,n-1,n-1}=\{(x_1, \ldots, x_{n-2}):\ \mu_n^*((D_{n-1,n-1})_{x_1, \ldots, x_{n-2},\bullet})>0\}$ and

$H_{n-1}=\{x_1: \overline{\mu_2\times\ldots\times\mu^*_n}((A_1-A_n)_{x_1,\bullet,\ldots})>0\}$ are of respective measure zero too. Let

$B=A_n-[(X_1\times D_{n-1,1})\cup\{(x_1, \ldots, x_n): (x_1, x_3, \ldots, x_n)\in D_{n-1,2}$ and $x_2\in X_2\}\cup\ldots\cup\{(x_1, \ldots, x_n): (x_1, \ldots, x_{n-2}, x_n)\in D_{n-1,n-1}$ and $x_{n-1}\in X_{n-1}\}\cup(X_1\times X_2\times F_{n-1,1,1})\cup\{(x_1, \ldots, x_n):\ x_1\in X_1,\ x_3\in X_3$ and $(x_2, x_4, \ldots, x_n)\in F_{n-1,1,2}\}\cup\ldots\cup(X_1\times F_{n-1,1,n-1}\times X_n)\cup(X_1\times X_2\times F_{n-2,2,1})\cup\{(x_1, \ldots, x_n): (x_1, x_4, \bullet, x_n)\in F_{n-1,2,2}$ and $(x_2, x_3)\in X_2\times X_3\}\cup\ldots\cup\{(x_1, \ldots, x_n):\ (x_1, x_3, \ldots, x_n)\in F_{n-1,2,n-1}\times X_n$ and $x_2\in X_2\}\cup\ldots\cup\{(x_1, \ldots, x_n): x_1\in X_1$ and $(x_2, \ldots, x_{n-2}, x_n)\in F_{n-1,n-1,1}$ and $x_{n-1}\in X_{n-1}\}\cup\{(x_1, \ldots, x_n):\ (x_1, x_3, \ldots, x_{n-2}, x_n)\in F_{n-1,n-1,2},\ x_2\in X_2$ and $x_{n-1}\in X_{n-1}\}\cup\ldots\cup\{(x_1, \ldots, x_n):\ (x_1, \ldots, x_{n-3})\in F_{n-1,n-1,n-2}$ and $(x_{n-2}, x_{n-1})\in X_{n-2}X_{n-1}\}\cup(F_{n-1,n-1,n-1}\times X_{n-1}\times X_n)\cup(H_{n-1}\times X_{(2)})]$.

By this definition B satisfies all the conditions of the lemma and this completes the proof.

Let f: $X\to R$ be a function. Then the function $f_{x_1, \ldots, x_{i-1},\bullet, x_{i+1}, \ldots, x_n}(x_i)=f(x_1, \ldots, x_n)$ is called as usually a section of f with respect to $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

Let $\Phi(f)=\{(x_1, \ldots, x_n):\ \exists_{i=1, \ldots, n}\ f_{x_1, \ldots, x_{i-1},\bullet, x_{i+1}, \ldots, x_n}$ is not approximately continuous at $x_i\in X_i$ with respect to $\mathcal{F}_i\}$.

Lemma 5 ([7], lemma 5). *Let f: $X\to R$ be a μ-measurable function. Then $\mu(\Phi(f))=0$.*

For the function f: $X\to R$ we denote by $A(f)=\{(x_1, \ldots, x_n):\ \exists_{i=2, \ldots, n-1}\ f_{x_1, \ldots, x_{i-1},\bullet, x_{i+1}, \ldots, x_n}$ is positively degenerate at the point x_i with respect to $\mathcal{F}_i$ or the section $f_{x_i, \ldots, x_{n-1},\bullet}$ is degenerate at the point x_n with respect to $\mathcal{F}_n\}$.

Theorem 2. *Let f: $X\to R$ be a function such that all its sections $f_{x_1, \ldots, x_{i-1},\bullet, x_{i+1}, \ldots, x_n}$ are μ_i-measurable $(i=2, \ldots, n)$ and all its sections $f_{\bullet, x_2, \ldots, x_n}$ have* *the property* (G) *with respect to* $\mathscr{F}_1$. *Then the function f is* μ*-measurable iff* $\mu(A(f))=0$.

Proof. This theorem holds true for $n=2$ (see [4], theorem 4).

Assume that

(*) if for the function f: $X_2\times\ldots\times X_n\to R$ all its sections $f_{x_2,\ldots,x_{i-1},\bullet,x_{i+1},\ldots,x_n}$ are μ_i-measurable for every $i=3,\ldots,n$ and all its sections $f_{\bullet,x_3,\ldots,x_n}$ have the property (G) with respect to $\mathscr{F}_2$, then f is $\overline{\mu_2\times\ldots\times\mu_n}$-measurable iff $\overline{\mu_2\times\ldots\times\mu_n}\,(A_i(f))=0$ where

$$A_1(f)=\{(x_2,\ldots,x_n):\ \underset{i=3,\ldots,n-1}{\exists}\ f_{x_2,\ldots,x_{i-1},\bullet,x_{i+1},\ldots,x_n}$$

is positively degenerate at the point $x_i\in X_i$ with respect to $\mathscr{F}_i$ or $f_{x_2,\ldots,x_{n-1},\bullet}$ is degenerate at x_n with respect to $\mathscr{F}_n\}$.

Let f be such as in this theorem. If f is the μ-measurable function, then $\mu(A(f))=0$ because $A(f)\subset\Phi(f)$ and in accordance with lemma 5 $\mu(\Phi(f))=0$.

Assume that $\mu(A(f))=0$. It is sufficient to show that the function f satisfies the assumptions concerning the function g of lemma 2.

Let $E\in\mathscr{M}$, $\mu(E)>0$, $\varepsilon>0$ and let $\{I_k\}_{k=1}^{\infty}$ be the sequence of all sets belonging to $\mathscr{F}_1$ and let $\{K_k\}_{k=1}^{\infty}$ be the sequence of all closed intervals with rational ends and lengths smaller then ε.

Let $Q=\{(x_2,\ldots,x_n):\ (x_2,\ldots,x_n)\in X_{(2},\ E_{\bullet,x_2,\ldots,x_n}\in\mathscr{M}_1$ and $\mu 1(E_{\bullet,x_2,\ldots,x_n})>0\}$.

The set Q is $\overline{\mu_2\times\ldots\times\mu_n}$-measurable and $\overline{\mu_2\times\ldots\times\mu_n}(Q)>0$. Let $Q_{r,s}$ be a set of points $(x_2,\ldots,x_n)\in Q$ such that

(i) $\mu_1(I_r\cap E_{\bullet,x_2,\ldots,x_n})>0$

(ii) if $D(x_1,\ I_r\cap E_{\bullet,x_2,\ldots,x_n})=1$ and $x_1\in I_r\cap E_{\bullet,x_2,\ldots,x_n}$, then $f(x_1,\ldots,x_n)\in K_s$.

Evidently $Q\supset\bigcup_{r,s}Q_{r,s}$. Moreover $Q\subset\bigcup_{r,s}Q_{r,s}$ because all sections $f_{\bullet,x_2,\ldots,x_n}$ have the property (G) with respect to $\mathscr{F}_1$. Therefore $Q=\bigcup_{r,s}Q_{r,s}$. Thus there exists a couple of positive integers (r_0,s_0) such that $\overline{\mu_2\times\ldots\times\mu_n^*}\,(Q_{r_0,s_0})>0$ because $\overline{\mu_2\times\ldots\times\mu_n}\,(Q)>0$. Let

$$P=\{(x_2,\ldots,x_n):\ D^*((x_2,\ldots,x_n),\ Q_{r_0,s_0})=1\}.$$

The measure $\overline{\mu_2\times\ldots\times\mu_n}$ is G_δ regular and $\mathscr{F}_2\times\ldots\times\mathscr{F}_n$ has the density property, thence $P\in\mathscr{M}_2\times\ldots\times\mathscr{M}_n$ and $\overline{\mu_2\times\ldots\times\mu_n}\,(P)=\overline{\mu_2\times\ldots\times\mu_n^*}\,(Q_{r_0,s_0})>0$.

Let $F=E\cap(I_{r_0}\times P)$. Evidently $F\in\mathscr{M}$ and $\mu(F)>0$ because for all points $(x_2,\ldots,x_n)\in Q_{r_0,s_0}$ $\overline{\mu_2\times\ldots\times\mu_n}\,(F_{\bullet,x_2,\ldots,x_n})>0$. Let $M=F-A(f)$. For the set M, in accordance with lemma 4, there exists a set $H \subset M$ such that $\mu(M-H)=0$, for every point $(x_1, \ldots, x_n) \in H$

$$D((x_1, \ldots, x_n), H)=1 \text{ and for every } i=1, \ldots, n$$
$$D(x_i, H_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n})=1 \text{ and}$$
$$D((x_2, \ldots, x_n), H_{x_1, \bullet, \ldots})=1.$$

Evidently $H \subset E$ and $\mu(H)>0$. To prove the theorem, in accordance with lemma 2 it is sufficient to show that $f(x_1, \ldots, x_n) \in K_{s_0}$ for every point $(x_1, \ldots, x_n) \in H$.

Let $(x_1^0, \ldots, x_n^0)$ be a point of the set H such that $f(x_1^0, \ldots, x_n^0) \in K_{s_0}$. Every point of $H_{x_1, \bullet, \ldots}$ is the density point of $H_{x_1, \bullet, \ldots}$, therefore

$$H_{x_1, \bullet, \ldots} \in \mathcal{M}_2 \times \ldots \times \mathcal{M}_n \quad \text{and} \quad \overline{\mu_2 \times \ldots \times \mu_n}\,(H_{x_1, \bullet, \ldots})>0.$$

Moreover every subset of $H_{x_1, \bullet, \ldots}$ of positive measure and the set Q_{r_0, s_0} have common points. Let $f_{x_1^0, \bullet, \ldots}: X_{x_1^0, \bullet, \ldots} \to R$. For every $i=2, \ldots, n$ $f_{x_1^0, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}$ is the μ_i-measurable function. Moreover by theorem 1 all sections $f_{x_1^0, \bullet, x_3, \ldots, x_n}$ have the property (G) with respect to $\mathcal{F}_2$ because the functions $f_{x_1^0, \bullet, x_3, \ldots, x_n}$ are positively nondegenerate at every point x_2 with respect to $\mathcal{F}_2$ $((x_1^0, x_2, \ldots, x_n) \notin A(f))$. Notice that $\overline{\mu_2 \times \ldots \times \mu_n}\,(H_{x_1^0, \bullet, \ldots} \cap A(f)_{x_1^0, \bullet, \ldots})=0$. Then if we assume that

$$f(x_2, \ldots, x_n)=\begin{cases} f_{x_1^0, \bullet, \ldots}(x_2, \ldots, x_n) & \text{for} \quad (x_2, \ldots, x_n) \in H_{x_1^0, \bullet, \ldots} \\ 0 & \text{for} \quad (x_2, \ldots, x_n) \notin H_{x_1^0, \bullet, \ldots} \end{cases},$$

then, according to (*), the function $f_{x_1^0, \bullet, \ldots}$ is $\overline{\mu_2 \times \ldots \times \mu_n}$-measurable. In result the set

$$(f_{x_1^0, \bullet, \ldots})^{-1}(K_{s_0}) \in \mathcal{M}_2 \times \ldots \times \mathcal{M}_n \quad \text{and as}$$
$$f_{x_1^0, \bullet, \ldots}(Q_{r_0, s_0}) \subset K_{s_0} \quad \text{then}$$

$$(**) \qquad \overline{\mu_2 \times \ldots \times \mu_n}\,(H_{x_1^0, \bullet, \ldots}-(f_{x_1^0, \bullet, \ldots})^{-1}(K_{s_0}))=0.$$

On the other hand $f(x_1^0, \ldots, x_n^0) \in K_{s_0}$ and the function $f_{x_1^0, \bullet, x_3^0, \ldots, x_n^0}$ is positively nondegenerate at the point x_2^0 with respect to $\mathcal{F}_2$, thence we infer that

$$\mu_2^*(H_{x_1^0, \bullet, x_3^0, \ldots, x_n^0} \cap (f_{x_1^0, \bullet, x_3^0, \ldots, x_n^0})^{-1}(R-K_{s_0}))>0.$$

For every point

$$x_2 \in H_{x_1^0, \bullet, x_3^0, \ldots, x_n^0} \cap f(_{x_1^0, \bullet, x_3^0, \ldots, x_n^0})^{-1}(R-K_{s_0})$$

the sections $f_{x_1^0, x_2, \bullet, x_4^0, \ldots, x_n^0}$ are nondegenerate at x_3^0 with respect to $\mathcal{F}_3$, thence

$$\overline{\mu_2 \times \mu_3^*} (H_{x_1^0, \bullet, \bullet, x_4^0, \ldots, x_n^0} \cap (f_{x_1^0, \bullet, \bullet, x_4^0, \ldots, x_n^0})^{-1} (R - K_{s_0})) > 0.$$

Proceeding analogously we infer that for every point

$$(x_2, \ldots, x_{n-1}) \in H_{x_1^0, \bullet, \ldots, \bullet, x_n^0} \cap (f_{x_1^0, \bullet, \ldots, \bullet, x_n^0})^{-1} (R - K_{s_0})$$

the sections $f_{x_1^0, x_2, \ldots, x_{n-1}, \bullet}$ are nondegenerate at the point x_n^0 with respect to $\mathcal{F}_n$, therefore

$$\overline{\mu_2 \times \ldots \times \mu_n^*} (H_{x_1^0, \bullet, \ldots} \cap (f_{x_1^0, \bullet, \ldots})^{-1} (R - K_{s_0})) > 0,$$

which contradicts (* *). The function f of n variables is μ-measurable. Thence by the mathematical induction theorem 2 holds true.

Remark 1. The following theorem is not true:

Theorem ([5], theorem 1). Let the function f: $R^n \to R$ be such that all its sections $f_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}$ $(i = 1, \ldots, n)$ are measurable in the sense of Lebesque and all its sections $f_{\bullet, x_2, \ldots, x_n}$ have the property (G). Then the function f is measurable in the sense of Lebesque iff

$$m_n(R^n - D(f)) = 0$$

where m_n denoted the Lebesque measure in R^n and

$$D(f) = \{(x_1, \ldots, x_n)\colon \text{ for } i = 1, \ldots, n \ f_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n} \text{ is nondegenerate at the point } x_i\}$$

This is stated in the example given in the paper [6] by Z. Grande. Indeed, the theorem:

Theorem 3 ([6] theorem 1). *Assume that the continuum hypothesis holds. Then there exists a function* F: $R \times R \to R$ *of Lebesque nonmeasurable such that all its sections* $F_{\bullet, x_2}$ *and* $F_{x_1, \bullet}$ *are of Lebesque measurable and nondegenerate at any point* $t \in R$.

It is sufficient to take the function f: $R^3 \to R$ such that

$$f(x_1, x_2, x_3) = F(x_2, x_3).$$

Let f: $X \to R$ be a function such that all its sections $f_{\bullet, x_2, \ldots, x_n}$ are μ_1-measurable. Denote by $B(f) = \{(x_1, \ldots, x_n) \in X$: $f_{\bullet, x_2, \ldots, x_n}$ is not approximately continuous with respect to $\mathcal{F}_1$ at $x_1 \in X_1\}$ and $C(f) = \{(x_1, \ldots, x_n) \in X$: $f_{\bullet, x_2, \ldots, x_n}$ is positively degenerate at $x_1 \in X_1$ with respect to $\mathcal{F}_1\}$.

Theorem 4. *Let* f: $X \to R$ *be a function such that for* $i = 1, \ldots, n$ *all its sections* $f_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}$ *are* μ_i*-measurable.*

Then the conditions:

(i) *the function* f *is* μ*-measurable,*

(ii) $\mu(A(f)\cup B(f))=0$ *and*
(iii) $\mu(A(f)\cup C(f))=0$
are equivalent.

Proof. If the function f is μ-measurable, then $\mu(A(f)\cup B(f))=0$ because $A(f)\cup B(f)\subset \Phi(f)$ and by lemma 5 (i) implies (ii). Also (ii) implies (iii) because $A(f)\cup C(f)\subset A(f)\cup B(f)$. It is sufficient to show that (iii) implies (i).

Let $\mu(A(f)\cup C(f))=0$ and let $A=X-[A(f)\cup C(f)]$. The measure μ is G_δ regular and $\mathcal{F}$ has the density property, thence there exists a sequence $\{A_k\}_{k=1}^{\infty}$ of closed sets of positive and finite measure such that $A_k\subset A_{k+1}$ and $\mu\left(A-\bigcup_{k=1}^{\infty} A_k\right)=0$.

Let

$$f_k(x_1, \ldots, x_n)=\begin{cases} f(x_1, \ldots, x_n) & \text{for } (x_1, \ldots, x_n)\in A_k \\ 0 & \text{for } (x_1, \ldots, x_n)\notin A_k \end{cases}.$$

As almost everywhere $\lim_{k\to\infty} f_k(x_1, \ldots, x_n)=f(x_1, \ldots, x_n)$ with respect to the measure μ, it is sufficient to show that the functions f_k satisfy the assumptions of theorem 2. According to the assumption all sections $(f_k)_{\bullet, x_2, \ldots, x_n}$ are μ_1-measurable and at almost every point of the closed set $(A_k)_{\bullet, x_2, \ldots, x_n}$ are positively nondegenerate with respect to $\mathcal{F}_1$ because $(x_1, \ldots, x_n)\notin C(f)$. Here we infer from theorem 1 that the function f has the property (G) with respect to $\mathcal{F}_1$. Moreover $\mu(A(f)\cup C(f))=0$, therefore $\mu(A(f_k))=0$. Thence by theorem 2 the functions f_k are μ-measurable. The proof of the theorem 4 is completed.

Returning to our space $(T, d, \mathcal{K}, \lambda)$ let $\mathcal{K}$ be a σ-field enclosing Borel sets of T.

Definition 6. *The function g: $T\to R$ has the property* (H) *with respect to $\mathcal{A}$ iff for every point $t\in T$ there exist two open and nonempty sets $U(t)$ and $V(t)$ such that $D_u(t, U(t))>0$, $D_u(t, V(t))>0$, $f|_{U(t)\cup\{t\}}$ is upper semicontinuous and $f|_{V(t)\cup\{t\}}$ is lower semicontinuous at t.*

Theorem 5. *The function g: $T\to R$ which has property* (H) *with respect to $\mathcal{A}$ is λ-almost everywhere continuous.*

Proof. Denote by D_g the set of points of discontinuity of the function g. Assume that $\lambda(D_g)>0$. We can assume that g is bounded. Let $A=\{t\in D_g\colon D(t, D_g)=1\}$ and let $B\subset A$ be a closed set such that:
(a) for every $I\in\mathcal{A}$: Int $(I)\cap B\neq\emptyset\Rightarrow\lambda(I\cap B)>0$. Denote by m the essential infimum of g on the set B. Let $t_1\in B$ be a point such that $D(t_1, B)=1$ and $g(t_1)<m+\frac{1}{4}$. The function g has the property (H) with respect to $\mathcal{A}$, therefore for the point t_1 there exists a open nonempty set $U(t_1)$ such that $D_u(t_1, U(t_1))>0$ and $g|_{U(t_1)\cup\{t_1\}}$ is upper semicontinuous at s_1. Therefore $g(t)-g(t_1)<\frac{1}{4}$ for $t\in U(t_1)$. As

$D_u(t_1, U(t_1))>0$ and $D(t_1, B)=1$, there exists $I_1 \in \mathscr{A}$ such that $\mathrm{Cl}\,(I_1) \subset U(t_1)$ and $B \cap \mathrm{Int}\,(I_1) \neq \emptyset$. Evidently

$$g(t) < g(t_1) + \frac{1}{4} < m + \frac{1}{4} + \frac{1}{4} = m + \frac{1}{2} \quad \text{for} \quad t \in I_1.$$

Let $s_1 \in B \cap \mathrm{Int}\,(I_1)$ be a point such that $D(s_1, B \cap \mathrm{Int}\,(I_1))=1$. The existence of point s_1 follows from (a). As g has the property (H) with respect to $\mathscr{A}$, for the point s_1 there exists an open monempty set $V(s_1) \subset \mathrm{Int}\,(I_1)$ such that $D_u(s_1, V(s_1))>0$ and $g|_{V(s_1) \cup \{s_1\}}$ is lower semicontinuous at s_1. Therefore $g(s_1) - g(t) < \frac{1}{4}$ for $t \in V(s_1)$.

As $D(s_1, B \cap \mathrm{Int}\,(I_1))=1$ and $D_u(s_1, V(s_1))>0$, there exists a set $J_1 \in \mathscr{A}$ such that $\mathrm{Cl}\,(J_1) \subset V(s_1)$, $B \cap \mathrm{Int}\,(J_1) \neq \emptyset$ and $\delta(J_1)<1$. Evidently $\operatorname*{osc}_{J} g < 1$, because $g(t) < m + \frac{1}{2}$ and $g(t) > g(s_1) - \frac{1}{4}$. Therefore we have a set $J_1 \in \mathscr{A}$ such that $B \cap \mathrm{Int}\,(J_1) = 0$, $\delta(J_1)<1$ and osc $g<1$ on the set J_1.

Proceeding analogously we define the sequence $\{J_k\}_{k=1}^{\infty}$ of the sets from $\mathscr{A}$ such that

(i) $\mathrm{Cl}\,(J_k) \subset \mathrm{Int}\,(J_{k-1})$

(ii) $B \cap \mathrm{Int}\,(J_k) \neq \emptyset$

(iii) $\delta(J_k) < \frac{1}{k}$ and osc $g < \frac{1}{k}$ on the set J_k.

The set $B \cap \bigcap_{k=1}^{\infty} \mathrm{Cl}\,(J_k) \neq \emptyset$. Let $t_0 \in \bigcap_{k=1}^{\infty} B \cap \mathrm{Cl}\,(J_k)$. As for $k=1, 2, \ldots$ $t_0 \in \mathrm{Int}\,(J_k)$, the oscillation of the function g at the point t_0 is equal to zero i.e. $t_0 \notin D_g$. On the other hand $t_0 \in B$, therefore $t_0 \in D_g$, which is contradictory with $t_0 \in D_g$. The proof of the theorem is completed. Theorem 5 is a generalization of theorem 1 of [3].

Remark 2. Let $S \subset T$ be a countable dense set. If the function g: $T \to R$ has the property (H) with respect to $\mathscr{A}$, then: (R) $\liminf_{\substack{t \to s \\ t \in S}} g(t) \leq g(s) \leq \limsup_{\substack{t \to s \\ t \in S}} g(t)$ for every $s \in S$.

Theorem 6. *Let f: $X \to R$ be a function such that all its sections $f_{\bullet, x_2, \ldots, x_k}$ are μ_1-measurable and all its sections $f_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n}$ have the property* (H) *with respect to $\mathscr{F}_i$ for every $i=2, \ldots, n$.*
Then f is a μ-measurable function.

Proof. This theorem for $n=2$ holds by the theorem given in the paper [8] by E. Marczewski and Cz. Ryll-Nardzewski.

Theorem 7 ([8], theorem 2). *Let f: $Y \times T \to R$, where Y is a space with* *a measure $\varkappa$, be a function such that all its sections $f_{\bullet, t}$ are $\varkappa$-measurable and all its sections $f_{y, \bullet}$ are λ — almost everywhere continuous and satisfy the condition* (R).

Then the function f is $\bar{\mu}$ — measurable, where $\bar{\mu} = \overline{\varkappa \times \lambda}$.

Assume that if g: $X_1 \times \ldots \times X_{n-1} \to R$ is a function such that all its sections $g_{\bullet, x_2, \ldots, x_{n-1}}$ are μ_1 — measurable and all its sections $g_{x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_{n-1}}$ have the property (H) with respect to $\mathscr{F}_i$ for $i = 2, \ldots, n-1$, then g is $\overline{\mu_1 \times \ldots \times \mu_{n-1}}$ — measurable. Let f: $X_1 \times \ldots \times X_n \to R$ satisfy the condition of theorem 6. Then the function

$$f_{\ldots, \bullet, x_n}(x_1, \ldots, x_{n-1}) = g(x_1, \ldots, x_{n-1}) \quad \text{is} \quad \overline{\mu_1 \times \ldots \times \mu_{n-1}}$$

measurable. Therefore f: $X_{n-1)} \times X_n \to R$ as the function of two variables is μ — measurable. The proof of the theorem is completed.

REFERENCES

[1] BRUCKNER, A. M.: Differentiation of integrals. Amer. Math. Monthly 78, 1971, 1—54.

[2] DAVIES, R. O.: Separate approximate continuity implies measurability. Proc. Comb. Phil. Soc. 73, 1973, 461—465.

[3] GRANDE, E.: Sur les fonctions ayant la proprieté (H). Demonstratio Mathematicae, 1, 1983, 125—131.

[4] GRANDE, Z.: La mesurabilité des fonctions de deux variables et de la superposition $F(x, f(x))$. Dissertationes Mathematicae, 159, 1978.

[5] GRANDE, Z.: Sur la mesurabilité des fonctions de plusieurs variables. Math. Slovaca 28, 1978, 113—118.

[6] GRANDE, Z.: Deux exemples de fonctions non mesurables. Colloq. Math. 40, 1979, 305—309.

[7] KWIECIŃSKA, G.: On the measurability of real functions defined on product-spaces. Math. Slovaca 31, 1981, 319—331.

[8] MARCZEWSKI, E.—RYLL-NARDZEWSKI, Cz.: Sur la mesurabilité des fonctions de plusieurs variables. Ann. Soc. Polon. Math., 25, 1953.

Received June 4, 1984

Universytet Gdanski
Instytut Matematyki
ul. Wita Stwosza 57
80-952 Gdansk
POLAND

ИЗМЕРИМОСТЬ ДЕИСТВИТЕЛЬНЫХ ФУНКЦИЙ, ЗАДАННЫХ НА ДЕКАРТОВОМ ПРОИЗВЕДЕНИИ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

Grazyna Kwiecińska

Резюме

Эта работа состоит из двух части. В первой части находятся необходимое и достаточное условия измеримости действительных функций, заданных на декартовом произведении n ($n>2$) метрических пространств с мерами, которые удовлетворяют некоторым дополнительным условиям. Вторая часть содержит теорему, которая связана с теоремой Лебега о измеримости функции двух переменных.