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Mathematica Slovaca, Vol. 26 (1976), No. 4, 309--315

Persistent URL: http://dml.cz/dmlcz/128994

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ON THE FRATTINI IDEAL IN COMPACT SEMIGROUPS

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An algebraic semigroup S which is also a Hausdorff space is called a topological semigroup if the multiplication is (jointly) continuous. A non-empty subset I of S is called an ideal of S if $IS \subset I$ and $SI \subset I$. The Frattini ideal of S, denoted by $\Phi(S)$, is defined to be the intersection of all maximal ideals of S. According to Š. Schwarz [10], $\Phi(S)$ is always nonempty, provided that S has proper maximal ideals.

The studies of the Frattini ideal in a semigroup were made by several authors, namely, J. E. Kuczkowski [7], Š. Schwarz [10], P. A. Grillet [4], R. Fulp [3] and others. In his paper [10], Š. Schwarz remarks that some results concerning the Frattini ideal in commutative rings can be transformed analogously to (noncommutative) semigroups. The purpose of the present paper is to extend a topological version of Schwarz's results from algebraic semigroups to compact semigroups. We shall prove that, under certain conditions, the Frattini ideal $\Phi(S)$ of a compact semigroup S will coincide with the intersection of all open prime ideals containing $\Phi(S)$.

Throughout this paper, the symbol S will always denote a topological semigroup. The reader is referred to [9] for definitions not explicitly given here.

Definition. *A non-empty ideal P of a semigroup S is said to be prime if $AB \subset P$ implies that $A \subset P$ or $B \subset P$, A, B being ideals of S.*

An ideal T is *completely prime* if $ab \in T$ implies that $a \in T$ or $b \in T$, a, b being elements of S. An ideal which is completely prime is prime. But the converse need not be true. These concepts coincide when S is a *normal* semigroup, that is, $aS = Sa$ for all elements of S.

An ideal Q is *completely semiprime* if $a^2 \in Q$ implies that $a \in Q$, a being an element of S. Clearly, a completely prime ideal is also completely semiprime, but not conversely. For instance, let $S = \{0, a, b\}$ be a semigroup with zero in which $ab = ba = 0$, $a^2 = a$ and $b^2 = b$, then the ideal $\{0\}$ is completely semiprime, but not completely prime.

The author wishes to thank Dr H. L. Chow for his enthusiastic discussions.

An ideal M of S is called *g-maximal* if M is a proper maximal ideal of S and $S\backslash M$ is a group.

Definition. *An idempotent e of S is said to be a g-maximal idempotent if and only if $J_0(S\backslash e)$, the maximal ideal contained in the set $S\backslash e$ being a g-maximal ideal.*

Let I be an ideal of S. We define an idempotent $e \notin I$ to be *I-primitive* if e is the only idempotent contained in $eSe\backslash I$.

Definition. *A semigroup S is said to be a quasi-normal semigroup if and only if the set of all idempotents E of S forms a semilattice. In other words, S is a quasi-normal semigroup if and only if its idempotents are mutually commutative with each other under multiplication.*

For $e, f \in E$, define $e \leqslant f$ if and only if $ef = fe = e$ It is clear that $\leqslant$ is a partial ordering in E. If S is an arbitrary semigroup and I is an ideal of S, then the atoms of the partially ordered set $E \cap (S\backslash I)$ (if it exists) are all I-primitive idempotents of S. We u ually denote the set $E \cap (S\backslash I)$ by $E(I)$.

Definition. *An ideal I of a semigroup S is defined to be an E-recognizable ideal if $E(I) \neq \emptyset$* and $\overline{E(I)} \cap I = \emptyset$, where $\overline{E(I)}$ is the closure of the set $E(I)$.

If I is an open ideal of a semigroup S with $E(I) \neq \emptyset$, then I is always E-recognizable. But conversely, an E-recognizable ideal need not be open. For example let $S = [\frac{1}{2}, 1]$ with multiplication $*$ defined by $x * y = \max\{\frac{1}{2}, xy\}$ for all $x, y \in S$, then $\{\frac{1}{2}\}$ is an E-recognizable ideal, but $\{\frac{1}{2}\}$ is not open. The following theorem shows that (among other things) under certain conditions, the E-recognizable Frattini ideal of a semigroup is open.

Theorem 1. *Let S be a compact quasi-normal semigroup with zero. If every maximal ideal of S is g-maximal, and if the Frattini ideal $\Phi(S)$ is an E-recognizable nil ideal of S, then $\Phi(S)$ is an open completely semiprime ideal of S.*

Conversely, if $\Phi(S)$ is an open completely semiprime ideal and if $E(\Phi(S)) = E \cap (S\backslash \Phi(S))$ contains only $\Phi(S)$-primitive idempotents, then $\Phi(S)$ can be expressed as the intersection of g-maximal ideals of S.

Remark: In general, if S is an arbitrary semigroup, then $\Phi(S)$ may be neither open nor closed as can be seen in example 3 on page 74 in [10].

The following lemmas are needed for the proof of Theorem 1.

Lemma A. *Let S be a compact semigroup. If each maximal ideal of S is completely semiprime, then the Frattini ideal $\Phi(S)$ can be expressed as the intersection of open prime ideals containing $\Phi(S)$, and in fact, $\Phi(S) = J_0(S\backslash E(\Phi(S))$.*

The proof of lemma A is in [12]. We notice that Corollary 12 in [12] is a generalized form of Lemma A.

Lemma B. *Let I be an open completely semiprime ideal of a compact semigroup S. For each $e^2 = e \in S \backslash I$, define* $\mathrm{Tod}_I e$ *to be the set* $\{x \in S | ex \in I\}$. *Then* $\mathrm{Tod}_I e = \{x \in S | xe \in I\}$, *and* $\mathrm{Tod}_I e$ *is an open ideal of S containg I. Morever, if e_i is I-primitive, then* $\mathrm{Tod}_I e_i = J_0(S \backslash e_i)$ *is an open completely prime ideal of S.*

The proof of lemma B can be also found in [12].

The lemma below slightly generalizes lemma 13 (iii) in [12]. It can be derived immediately from lemma 13 (ii) in [12], but for the sake of completeness, we provide a proof.

Lemma C. *Every g-maximal idempotent of S is $\Phi(S)$-primitive.*

Proof. Let e be a g-maximal idempotent of S. Then e is the unique idempotent in $S \backslash M_\alpha = P_\alpha$ for some maximal ideal M_α. Consider $f^2 = f \in eSe \backslash \Phi(S)$, then $f = exe$ for some $x \in S$, and so $f = ef$. Suppose $f \neq e$. Then, since $f \notin \Phi(S)$, $f \in S \backslash M_\beta = P_\beta$ for some maximal ideal M_β of S. Because e is g-maximal, $P_\alpha \neq P_\beta$ and so by Schwarz [10] $f = ef \in P_\alpha P_\beta \subset \Phi(S)$, which is a contradiction to $f \notin \Phi(S)$. Hence $f = e$ and e is therefore $\Phi(S)$-primitive.

Lemma D. *Let S be a quasi-normal semigroup with zero. If I is an E-recognizable nil ideal of S, then the set of all I-primitive idempotents of S is* closed.

Proof. Let $\tilde{E(I)}$ denote the set of all I-primitive idempotents of S. Take e in the closure of $\tilde{E(I)}$ and there exists a net $\{e_\alpha\}$ in $\tilde{E(I)}$ such that $e_\alpha \to e$. Since I is an E-recognizable ideal, then $e \in E \backslash I$. Now let $f \in E \backslash I$ such that $f \leqslant e$, that is, $f = ef$. Consider $f_\alpha = e_\alpha f$. Clearly $e_\alpha f \to ef = f$ gives $f_\alpha \to f$. Since S is quasi-normal and I is also a nil ideal of S, hence $f_\alpha = e_\alpha f$ is an idempotent not in I. However, $F_\alpha \leqslant e_\alpha$ and e_α is I-primitive, thus it follows that the only possible cluster points of $\{f_\alpha\}$ is e. Consequently, $f = e$. This means that $e \in \tilde{E(I)}$, completing the proof.

Remark 1: If we replace in lemma D the E-recognizable nil ideal I by an E-recognizable completely prime ideal, then the result of lemma D is still valid.

Remark 2: If I is a completely prime ideal of S and $\tilde{E(I)} \neq \emptyset$, the set $E(I)$ is a singleton. Let e f be idempotents in $E(I)$: then, because S is a quasi-normal semigroup, we have $(ef)^2 = ef$ and $ef = eef = efe \in eSe$. As $e \in E(I)$ then $ef = e$ or $ef \in I$. Similarly, $ef = f$ or $ef \in I$. Since I is a completely prime ideal of S, $ef \notin I$. Therefore we must have $e = f$.

Remark 3: Let $\tilde{E}$ be the set of all primitive idempotents (for a definition of primitive idempotents see [5]) of a compact semigroup. Whether or not the set $\tilde{E}$

must be closed is an open problem proposed by R. J. Koch in 1954 [page 831; 5]. By applying the same arguments as used in the proof of lemma 4. we can easily prove that $\bar{E}$ is closed if S is a quasi-normal semigroup. Thus a partial answer to Koch's problem is obtained.

We now turn to prove Theorem 1.

Suppose that each maximal ideal of S is g-maximal. They by lemma A, $\Phi(S)$ is completely semiprime and $\Phi(S)=J_0(S\backslash E(\Phi(S)))=\cap\{J_0(S\backslash e_i)|e_i\in E(\Phi(S))\}$. The proof will be complete if we can prove that $E(\Phi(S))$ is a closed subset of S. Since every idempotent in $E(\Phi(S))$ is g-maximal then, by lemma C, every idempotent in $E(\Phi(S))$ is $\Phi(S)$-primitive. As $\Phi(S)$ is assumed to be an E-recognizable nil ideal then, by applying lemma D, it follows that $E(\Phi(S))$ is closed. Thus $\Phi(S)=J_0(S\backslash E(\Phi(S)))$ is open.

For the converse part, let $\Phi(S)$ be an open completely semiprime ideal of a quasi-normal semigroup S; then by Theorem 3.4 in [13], $\Phi(S)=\cap\{J_0(S\backslash e_i)|e_i\in E(\Phi(S))\}$. Since $E(\Phi(S))$ consists of $\Phi(S)$-primitive idempotents only, then, by lemma B, each $J_0(S\backslash e_i)$ is a completely prime ideal of S. Now, by Schwarz [10], each of these open completely prime ideals is a maximal ideal of S, and so it follows that $S\backslash J_0(S\backslash e_i)$ is a disjoint union of groups [2]. Applying remark 2 of lemma D, we know that $S\backslash J_0(S\backslash e_i)$ contains only a unique idempotent e_i and therefore must be a group. Thus each $J_0(S\backslash e_i)$ is a g-maximal ideal, completing the proof.

Remark: In the necessity part of Theorem 1, if $\Phi(S)$ is assumed to be an E-recognizable completely prime ideal instead of an E-recognizable nil ideal of S, then we can prove easily that $\Phi(S)$ itself is a g-maximal ideal of S.

Corollary. If $E(\Phi(S))$ contains only $\Phi(S)$-primitive idempotents of S, then any open completely semiprime ideal of a compact semigroup can be expressed as an intersection of g-maximal ideals if and only if it contains $\Phi(S)$.

Proof. Clearly, every ideal which is the intersection of g-maximal ideals of S contains $\Phi(S)$. Conversely, let A be an open completely semiprime ideal containing $\Phi(S)$. Then there exists at least one idempotent $e_i^2=e_i\in S\backslash A$, and hence $A\subset J_0(S\backslash e_i)$. Thus, by theorem 1, each $J_0(S\backslash e_i)$, $e_i\in S\backslash A$ is a g-maximal ideal of S. Let $E(A)$ denote $E\cap(S\backslash A)$. Suppose, if possible, that $A\subsetneqq J_0(S\backslash E(A))=\cap\{J_0(S\backslash e_i)|e_i\in E(A)\}$. Then we can pick $y\notin A$, $y\in J_0(S\backslash E(A))$. Hence there is an idempotent f such that $f\in\Gamma(y)=\overline{\{y^n\}_{n=1}^{\infty}}\subset J_0(S\backslash E(A))$, which implies that $f\in A$. However, since A is an open completely semiprime ideal of S. then $y\notin A$ implies $f\notin A$, a contradiction. Thus $A=\cap\{J_0(S\backslash e_i)|e_i\in E(A)\}$, completing the proof

Theorem 2. *Let S be a compact semigroup with $S^2=S$, then the Frattini ideal $\Phi(S)$ of S is the intersection of all open prime ideals containing $\Phi(S)$. Moreover, $\Phi(S)=J_0(S\backslash E(\Phi(S))$.*

Proof. Since $S^2=S$, then by Schwarz [10], every maximal ideal of S is a prime ideal containing $\Phi(S)$. Moreover, since S is compact, each maximal ideal is open [6], and so each maximal ideal of S is an open prime ideal containing $\Phi(S)$. On the other hand, each open prime ideal containing $\Phi(S)$ must be a proper maximal ideal of S. (This was proved by Schwarz in [10]). Hence, there is a 1-1 correspondence between the set of all proper maximal ideals of S and the set of all open prime ideals containing $\Phi(S)$. Therefore we conclude that $\Phi(S)$ is the intersection of all open prime ideals containing $\Phi(S)$. Moreover, by Numakura [8], each open prime ideal containing $\Phi(S)$ has the form $J_0(S\backslash e_i)$ with $e_i \notin \Phi(S)$. Hence, $\Phi(S)= \cap\{J_0(S\backslash e_i)|e_i \in E(\Phi(S))\} = J_0(S\backslash E(\Phi(S)))$.

Corollary 1. *Let S be a compact connected semigroup with $S^2=S$; then $\Phi(S)$ is a connected ideal of S. Moreover, $\Phi(S)$ is open if and only if $E(\Phi(S))$ is non-empty and compact.*

Corollary 2. (*Schwarz* [10].) *Let S be a compact semigroup with $S^2=S$. If $\Phi(S)$ is a proper ideal of S and if every open prime ideal of S contains $\Phi()$, then $\Phi(S)$ does not contain any idempotents which are not contained in the kernel K of S.*

Proof. Let Q^* denote the intersection of all prime ideals of S. As $\Phi(S)$ is a proper ideal of S, $Q^* \neq \emptyset$ and so by Theorem 2, we have $K \subset Q^* \subset \Phi(S)$. By Schwarz [10], Q^* contains exactly those idempotents which are contained in K. We only need to show that there exists no idempotent in $\Phi(S)\backslash Q^*$. Suppose $f^2=f\in \Phi(S)\backslash Q^*$. Then by Numakura [8], $J_0(S\backslash f)$ is an open prime ideal of S and hence $f\in \Phi(S) \subset J_0(S\backslash f)$, which is a contradiction. The proof is completed.

Let S be a compact semigroup with zero. Let $N=\{x\in S|0\in \Gamma(x)=\overline{\{x^n\}_{n=1}^{\infty}}\}$. Then $N_1=J_0(N)$ is called the nil radical of S.

Corollary 3. *Let S be a compact semigroup with zero. If $S^2=S$ and if $\Phi(S)$ is the intersection of all open prime ideals of S, then the Frattini ideal of S coincides with the nil radical of S.*

Proof. By Corollary 2, we know immediately that $E(\Phi(S))=E(N_1)$, where $E(N_1)=S\cap(S\backslash N_1)$. Hence it follows that $J_0(S\backslash E(\Phi(S))=J_0(S\backslash E(N_1))$. We now have by Theorem $\Phi(S)=J_0(S\backslash E(\Phi(S))$, and also by Theorem 2.3 in [11], we have $N_1=J_0(S\backslash E(N_1)$. Thus $\Phi(S)=N_1$.

A semigroup with zero is called an N-semigroup if the set of all nilpotent elements of S, denoted by N, is an open subset of S. K. P. Shum and C. S. Hoo [11] have shown that N is an ideal of S in the case of S, being a compact abelian semigroup. Recently, H. L. Chow [1] has pointed out that the abelian condition can be weakened. He shows that the result of Shum and Hoo is still valid if S is a compact weakly normal semigroup, that is, $eS=Se$ for all idempotent $e\in S$. Thus the following facts follow verbatim from Corollary 3.

Corollary 4. *Let S be a compact weakly normal semigroup with zero satisfying $S^2=S$. If $\Phi(S)$ is the intersection of all open prime ideals of S then S s an N-semigroup if and only if its Frattini ideal is open.*

Corollary 5. *Let S be a compact weakly normal N-semigroup with $S^2=S$ If $\Phi(S)$ is the intersection of all open prime ideals of S and if e is a $\Phi(S)$-primitive idempotent not in $\Phi(S)$, then $eS\backslash\Phi(S)$ is a compact group.*

Proof. By Corollary 4 we know that $\Phi(S)=N$. Since e is a $\Phi(S)$-primitive idempotent not in $\Phi(S)$, then by R. J. Koch [5], $eS\backslash\Phi(S)=eS\backslash N$ is a disjoint union of compact groups. Now, let $f^2=f\neq e$ such that $f\in eS\backslash\Phi(S)=Se\backslash\Phi(S)$. Then there exists elements x and $y\in S$ such that $f=ex$ and $f=ye$. Consequently, $f=ef=fe$ and so $ef=f\leqslant e$. Because $f\notin\Phi(S)$ and e are $\Phi(S)$-primitive, $f=e$. Hence, we conclude that $eS\backslash\Phi(S)$ is a group.

Remarks:

(I) Theorem 2 is a generalized result of S. Schwarz in [10]. The reader is referred to Theorem 6 in [10].

(II) A compact semigroup with $S^2=S$ does not imply that every open prime ideal of S is completely open prime. For instance, see example on page 51 in [9].

(iii) A compact semigroup with $S^2=S$ does not imply that $\Phi(S)$ is the intersection of all open prime ideals of S. For instance, let S be a min-thread, then $\Phi(S)=[0, 1)$. Clearly, $\Phi(S)$ is not the intersection of all open prime ideals of S

(IV) The hypothesis $S^2=S$ cannot be dropped in proving the necessity part for Corollary 1. For instance, the example 3 in [10] shows that $E(\Phi(S))$ is non-empty and compact, but $\Phi(S)$ is neither open nor closed.

(V) Corollary 3 is analogous to the following well-known result in the Ring Theory: let R be an arbitrary commutative ring with identity, then the set of all nilpotent elements of R coincides with the intersection of all the prime ideals of S.

(VI) Let S be a compact semigroup with the kernel K. An element $x\in S$ is called K-potent if there is an integer $p>0$ such that $x^p\in K$. We denote by N^*_l the set of all K-potent elements of S, N^*_1 the largest ideal contained in N^*, then our Corollary 3 can be restated as follows: Let S be a compact semigroup with a kernel satisfying $S^2=S$. If $\Phi(S)$ is the intersection of all open prime ideals of S and if N^* is open, then $\Phi(S)=N^*_1$. Thus, Corollary 3 is a generalized version of Theorem 9 in [10].

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Received October 2, 1974

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ОБ ИДЕАЛЕ ФРАТТИН В КОМПАКТНЫХ ПОЛУГРУППАХ

Кар-Пинг Шум

Резюме

В работе доказывается что в определенных условиах идеал Фраттини $\Phi(S)$ в компактной полугруппе равен пересечению всех открытых простых идеалов содержащих $\Phi(S)$.