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VARIANCE COMPONENTS ESTIMATORS IN A REPLICATED REGRESSION MODEL

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Introduction

The locally and uniformly best estimators for the function $\gamma = \mathrm{tr}\,(\mathbf{D}\boldsymbol{\beta}\boldsymbol{\beta}') + \mathrm{tr}\,(\mathbf{C}\boldsymbol{\Sigma})$ in a replicated regression model

$$\boldsymbol{Y} = (\boldsymbol{1} \otimes \mathbf{X})\boldsymbol{\beta} + \boldsymbol{\varepsilon}\,, \tag{1}$$

where $E[\boldsymbol{\varepsilon}] = 0$, $E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'] = \mathbf{I} \otimes \boldsymbol{\Sigma}$, $\boldsymbol{1} = (1, \ldots, 1)'$, $\boldsymbol{Y} = (\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_m')'$ — is a $m \cdot n$ random vector whose components $\boldsymbol{Y}_i$; $i = 1, \ldots, m$ are assumed to be independent, $N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$ distributed random vectors, are given in the paper [5]. These quadratic estimators are based on $\bar{\boldsymbol{Y}} = 1/m \sum_{i=1}^{m} \boldsymbol{Y}_i$ and

$$\mathbf{S} = \frac{1}{m-1} \sum_{i=1}^{m} (\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})'\,.$$

The aim of our paper is to study some (unbiased) invariant estimators for the fuction $\gamma = \mathrm{tr}\,(\mathbf{C}\boldsymbol{\Sigma})$. This approach covers the problem of estimation of a covariance function of a stationary time series, the mean value of which is given by the usual linear regression model, on the base of repeated independent observations $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$. Each observation is of the length n. If we denote by $\boldsymbol{\Sigma}$ the covariance matrix of any observation $\boldsymbol{Y}_i$; $i = 1, \ldots, m$ of the stationary time series Y_t; $t = 0, 1, \ldots$ having the covariance function $R(\tau)$; $\tau = 0, 1, \ldots$, then it can be written: $R(\tau) = \frac{1}{n-\tau}\,\mathrm{tr}\,(\mathbf{A}(\tau)\boldsymbol{\Sigma})$; $\tau = 0, \ldots, n-1$, where

$$\mathbf{A}(\tau)_{ij} = \begin{cases} 1/2 & \text{if } |i-j| = \tau \\ 0 & \text{elsewhere} \end{cases}; \quad \mathrm{i, j} = 1, \ldots, \mathrm{n}; \quad \tau = 0, \ldots, \mathrm{n} - 1\,.$$

Thus the problem of estimation of a covariance function of a stationary time series with an unknown mean value given by the linear regression model on the basis of repeated independent observations is a special case of estimation of the function $\gamma = \mathrm{tr}\,(\mathbf{C}\boldsymbol{\Sigma})$.

1. Unbiased invariant estimators for the function $\gamma = \mathrm{tr}\,(\mathbf{C\Sigma})$

Let $\mathbf{P}$ be any $n \times n$ matrix. Let us denote by

$$\tilde{\mathbf{\Sigma}} = \frac{1}{m-1} \sum_{i=1}^{m} (\mathbf{Y}_i - \mathbf{P}\bar{\mathbf{Y}})(\mathbf{Y}_i - \mathbf{P}\bar{\mathbf{Y}})'.$$

We show that the random matrix $\tilde{\mathbf{\Sigma}}$ can be expressed with the help of the matrix $\mathbf{S}$ and some other matrix depending on the random vectory $\bar{\mathbf{Y}}$. Hence we have:

$$(m-1)\tilde{\mathbf{\Sigma}} = \sum_{i=1}^{m} (\mathbf{Y}_i - \bar{\mathbf{Y}} + \bar{\mathbf{Y}} - \mathbf{P}\bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}} + \bar{\mathbf{Y}} - \mathbf{P}\bar{\mathbf{Y}})' =$$

$$= \sum_{i=1}^{m} [(\mathbf{Y}_i - \bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}})' + (\bar{\mathbf{Y}} - \mathbf{P}\bar{\mathbf{Y}})(\bar{\mathbf{Y}} - \mathbf{P}\bar{\mathbf{Y}})'] =$$

$$= (m-1)\mathbf{S} + m\mathbf{M}\bar{\mathbf{Y}}\bar{\mathbf{Y}}'\mathbf{M}', \text{ where } \mathbf{M} = \mathbf{I} - \mathbf{P}.$$

Thus we can write:

$$\tilde{\mathbf{\Sigma}} = \mathbf{S} + \frac{m}{m-1} \mathbf{M}\bar{\mathbf{Y}}\bar{\mathbf{Y}}'\mathbf{M}'. \tag{2}$$

Let us denote

$$\tilde{\gamma} = \mathrm{tr}\left(\left(\mathbf{C} - \frac{1}{m}\mathbf{M}'\mathbf{C}\mathbf{M}\right)\tilde{\mathbf{\Sigma}}\right). \tag{3}$$

This random variable can be regarded as an estimator for the function $\gamma = \mathrm{tr}\,(\mathbf{C\Sigma})$. The following theorem describes the properties of $\tilde{\gamma}$.

Theorem 1. *Let the matrix* $\mathbf{P}$ *be such that* $\mathbf{P}^2 = \mathbf{P}$ *and* $\mathbf{PX} = \mathbf{X}$. *Then the estimator* $\tilde{\gamma}$ *given by (3) is unbiased and invariant for the function* $\gamma = \mathrm{tr}\,(\mathbf{C\Sigma})$. *It has the dispersion given by*

$$D_{\mathbf{\Sigma}}(\tilde{\gamma}) = \frac{2}{m-1}\,\mathrm{tr}\,(\mathbf{C\Sigma})^2 - \frac{2}{m(m-1)}\,[\mathrm{tr}\,(\mathbf{C\Sigma})^2 - \mathrm{tr}\,((\mathbf{C} - \mathbf{M}'\mathbf{CM})\mathbf{\Sigma})^2]. \tag{4}$$

Proof. The condition $\mathbf{PX} = \mathbf{X}$ guarantees that the random matrix $\tilde{\mathbf{\Sigma}}$ is invariant with respect to the mean value $\mathbf{X}\boldsymbol{\beta}$ of the random vectors $\mathbf{Y}_i$; $i = 1, \ldots, m$ and thus the estimator $\tilde{\gamma}$ is invariant too. The condition $\mathbf{P}^2 = \mathbf{P}$ implies that $\mathbf{M}^2 = \mathbf{M}$ and $\mathbf{M}'^2 = \mathbf{M}'$. Using these factors and (2) we can write:

$$\tilde{\gamma} = \mathrm{tr}\left(\left(\mathbf{C} - \frac{1}{m}\mathbf{M}'\mathbf{CM}\right)\left(\mathbf{S} + \frac{m}{m-1}\mathbf{M}\bar{\mathbf{Y}}\bar{\mathbf{Y}}'\mathbf{M}'\right)\right),$$

from which we have

$$\tilde{\gamma} = \mathrm{tr}\left(\left(\mathbf{C} - \frac{1}{m}\mathbf{M}'\mathbf{CM}\right)\mathbf{S}\right) + \bar{\mathbf{Y}}'\mathbf{M}'\mathbf{CM}\bar{\mathbf{Y}}. \tag{5}$$

Thus we can write, using (5):

$$E_{\Sigma}[\tilde{\gamma}]=\operatorname{tr}\left(\left(\mathbf{C}-\frac{1}{m}\mathbf{M}'\mathbf{C}\mathbf{M}\right)\Sigma\right)+\frac{1}{m}\operatorname{tr}(\mathbf{M}'\mathbf{C}\mathbf{M}\Sigma)=\operatorname{tr}(\mathbf{C}\Sigma)\,.$$

The last two equalities are consequences of the known facts that

$$E_{\Sigma}[\operatorname{tr}(\mathbf{AS})]=\operatorname{tr}(\mathbf{A}\Sigma)\ \text{ and }\ E_{\Sigma}[\boldsymbol{Y}'\mathbf{B}\boldsymbol{Y}]=\boldsymbol{\beta}'\mathbf{X}'\mathbf{B}\mathbf{X}\boldsymbol{\beta}+\operatorname{tr}(\mathbf{B}\Sigma)$$

for $\mathbf{A}, \mathbf{B}$ any symmetric matrices and of the fact that $\mathbf{MX}=\mathbf{0}$ if $\mathbf{PX}=\mathbf{X}$. The dispersion of $\tilde{\gamma}$ can be computed using the known relations (see [3])

$$D_{\Sigma}[\operatorname{tr}(\mathbf{AS})]=\frac{2}{m-1}\operatorname{tr}(\mathbf{A}\Sigma)^2\ \text{ and }\ D_{\Sigma}[\boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Y}]=2\operatorname{tr}(\boldsymbol{B}\Sigma)^2\ \text{ if }\ \mathbf{BX}=\mathbf{0}.$$

From these expresions, using the independence of $\bar{\boldsymbol{Y}}$ and $\mathbf{S}$ and (5), we get:

$$D_{\Sigma}[\tilde{\gamma}]=D_{\Sigma}\left[\operatorname{tr}\left(\left(\mathbf{C}-\frac{1}{m}\mathbf{M}'\mathbf{C}\mathbf{M}\right)\mathbf{S}\right)+\bar{\boldsymbol{Y}}'\mathbf{M}'\mathbf{C}\mathbf{M}\bar{\boldsymbol{Y}}\right]=$$

$$=\frac{2}{m-1}\operatorname{tr}\left(\left(\mathbf{C}-\frac{1}{m}\mathbf{M}'\mathbf{C}\mathbf{M}\right)\Sigma\right)^2+\frac{2}{m}\operatorname{tr}(\mathbf{M}'\mathbf{C}\mathbf{M}\Sigma)^2=$$

$$=\frac{2}{m-1}\operatorname{tr}(\mathbf{C}\Sigma)^2-\frac{2}{m(m-1)}\operatorname{tr}((2\mathbf{C}-\mathbf{M}'\mathbf{C}\mathbf{M})\Sigma\mathbf{M}'\mathbf{C}\mathbf{M}\Sigma)=$$

$$=D_{\Sigma}[\operatorname{tr}(\mathbf{CS})]-\frac{2}{m(m-1)}[\operatorname{tr}(\mathbf{C}\Sigma)^2-\operatorname{tr}((\mathbf{C}-\mathbf{M}'\mathbf{C}\mathbf{M})\Sigma)^2].$$

Remarks:

1. If we set $\mathbf{P}=\mathbf{I}$, then $\tilde{\Sigma}=\mathbf{S}$ and $\tilde{\gamma}=\operatorname{tr}(\mathbf{CS})$.

2. $\mathbf{P}$ can be equal to any projector on the space $\mathcal{M}(\mathbf{X})$, the subspace of E^n generated by the columns of the matrix $\mathbf{X}$. Especially the estimator

$$\hat{\gamma}=\operatorname{tr}\left(\left(\mathbf{C}-\frac{1}{m}\mathbf{MCM}\right)\hat{\Sigma}\right) \tag{6}$$

given by (3) with $\mathbf{P}=\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$,

$$\tilde{\Sigma}=\hat{\Sigma}=\frac{1}{m-1}\sum_{i=1}^{m}(\boldsymbol{Y}_i-\mathbf{X}\hat{\boldsymbol{\beta}})(\boldsymbol{Y}_i-\mathbf{X}\hat{\boldsymbol{\beta}})',\ \ \hat{\boldsymbol{\beta}}=(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\bar{\boldsymbol{Y}}$$

being the usual least squres estimator of $\boldsymbol{\beta}$ from the model (1) is unbiased and invariant for the function $\gamma=\operatorname{tr}(\mathbf{C}\Sigma)$.

2. The locally best unbiased invariant estimator for the function $\gamma = \text{tr}\,(\mathbf{C}\boldsymbol{\Sigma})$

It is easy to show that in the model (1) the locally (at $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$) best unbiased estimator $\boldsymbol{\beta}^*$ of the regression vector $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta}^* = \frac{1}{m} \sum_{i=1}^{m} \boldsymbol{\beta}_i^*, \quad \text{where} \quad \boldsymbol{\beta}_i^* = (\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{Y}_i;$$

$i = 1, \ldots, m$. Let $\boldsymbol{\Sigma}^* = \frac{1}{m-1} \sum_{i=1}^{m} (\mathbf{Y}_i - \mathbf{X}\boldsymbol{\beta}^*)(\mathbf{Y}_i - \mathbf{X}\boldsymbol{\beta}^*)'$. It is clear that the matrix $\boldsymbol{\Sigma}^*$ is a special case of the matrix $\hat{\boldsymbol{\Sigma}}$ with $\mathbf{P} = \mathbf{P}_0 = \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}$. Thus the estimator γ^* given by

$$\gamma^* = \text{tr}\Big(\Big(\mathbf{C} - \frac{1}{m}\mathbf{M}_0' C \mathbf{M}_0\Big)\boldsymbol{\Sigma}^*\Big), \quad \mathbf{M}_0 = \mathbf{I} - \mathbf{P}_0$$

is, according to the Theorem 1, an unbiased and invariant estimator for the function $\gamma = \text{tr}\,(\mathbf{C}\boldsymbol{\Sigma})$. We shall prove now the following theorem.

Theorem 2. *The estimator γ^* given by (7) is the locally (at $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$) best unbiased invariant estimator (LBUIE) for the function $\gamma = \text{tr}\,(\mathbf{C}\boldsymbol{\Sigma})$.*

Proof. The LBUIE γ_0^* for $\text{tr}\,(\mathbf{C}\boldsymbol{\Sigma})$ was derived in [5]. It was shown that

$$\gamma_0^* = \text{tr}\,\Big(\Big(\mathbf{C} - \frac{1}{m}\mathbf{M}_0'\mathbf{C}\mathbf{M}_0\Big)\mathbf{S}\Big) + \bar{\mathbf{Y}}'\mathbf{M}_0'\mathbf{C}\mathbf{M}_0\bar{\mathbf{Y}}, \text{ with } \mathbf{M}_0 = \mathbf{I} - \mathbf{P}_0.$$

But using (5) we can see that $\gamma_0^* = \gamma^*$.

Remark: Since γ^* is the LBUIE and $\text{tr}\,(\mathbf{CS})$ is an unbiased and invariant estimator for $\gamma = \text{tr}\,(\mathbf{C}\boldsymbol{\Sigma})$ too, the inequality $D_{\boldsymbol{\Sigma}_0}[\gamma^*] \leqslant D_{\boldsymbol{\Sigma}_0}[\text{tr}\,(\mathbf{CS})]$ holds. From this inequality, using (4), we get the inequality $\text{tr}\,(\mathbf{C}\boldsymbol{\Sigma}_0)^2 \geqslant \text{tr}\,((\mathbf{C} - \mathbf{M}_0'\mathbf{C}\mathbf{M}_0)\boldsymbol{\Sigma}_0)^2$, which holds for any symmetric matrix $\mathbf{C}$ and any covariance matrix $\boldsymbol{\Sigma}_0$.

3. Comparison of some invariant estimators

The LBUIE has the disadvantage that it depends on the matrix $\boldsymbol{\Sigma}_0$ at which we want to minimize the dispersion of the estimator. The LBUIE γ^* given by (7) can have a great dispersion for $\boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$. In this part of the paper we shall compare the estimator $\text{tr}\,(\mathbf{CS})$ with the estimator $\hat{\gamma}$ given by (6). These two estimators do not depend on $\boldsymbol{\Sigma}_0$. Our aim is to show that in some special cases the estimator $\text{tr}\,(\mathbf{CS})$ is not admissible, because the estimator $\hat{\gamma}$ is uniformly better than $\text{tr}\,(\mathbf{CS})$. To prove this, let us begin with the following lemma.

Lemma 1. *The estimator* $\hat{\gamma}$ *given* (6) *is uniformly better than the estimator* tr $(\mathbf{CS})$ *iff for any covariance matrix* $\mathbf{\Sigma}$ *the inequality*

$$\operatorname{tr}(\mathbf{C\Sigma})^2 \geq \operatorname{tr}((\mathbf{C}-\mathbf{MCM})\mathbf{\Sigma})^2 \tag{8}$$

holds, where $\mathbf{M}=\mathbf{I}-\mathbf{P}=\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

Proof: It follows from (4) and from the fact that $D_{\mathbf{\Sigma}}[\operatorname{tr}(\mathbf{CS})] = \frac{2}{m-1}\operatorname{tr}(\mathbf{C\Sigma})^2$.

Consequence: *If* $E[\mathbf{Y}_i]=0$; $i=1, \ldots, m$ $(\mathbf{X}=\mathbf{0})$, *then the estimator* $\hat{\gamma}=\frac{1}{m}\operatorname{tr}\left(\mathbf{C}\sum_{i=1}^{m} Y_i Y_i'\right)$ *is uniformly better than* tr $(\mathbf{CS})$.

Proof: The equality (8) holds trivially for $\mathbf{M}=\mathbf{I}$.

The following theorem can be proved now.

Theorem 3. *Let* $\gamma=\operatorname{tr}\mathbf{\Sigma}$, $(\mathbf{C}=\mathbf{I})$. *Then in the model (1) the unbiased invariant estimator* $\hat{\gamma}$ *given by (6) with* $\mathbf{C}=\mathbf{I}$ *is uniformly better than the unbiased invariant estimator* tr $\mathbf{S}$.

Proof: According to (8) it is enough to prove that for any covariance matrix $\mathbf{\Sigma}$ the inequality $\operatorname{tr}(\mathbf{\Sigma}^2) \geq \operatorname{tr}(\mathbf{P\Sigma})^2$ holds. Here $\mathbf{P}=\mathbf{P}'=\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Because $\operatorname{tr}(\mathbf{AB}')=(\mathbf{A}, \mathbf{B})$ is an inner product in the space of $n \times n$ matrices, we can write (using the Schwarz inequality and the properties $\mathbf{P}^2=\mathbf{P}$, $\mathbf{P}=\mathbf{P}'$ and $\mathbf{\Sigma}=\mathbf{\Sigma}'$):

$$\operatorname{tr}(\mathbf{P\Sigma})^2=(\mathbf{P\Sigma}, \mathbf{\Sigma P}) \leq \|\mathbf{P\Sigma}\|^2 \leq \|\mathbf{P\Sigma}\|^2+\|\mathbf{M\Sigma}\|^2=$$

$$=\operatorname{tr}(\mathbf{P\Sigma}^2\mathbf{P})+\operatorname{tr}(\mathbf{M\Sigma}^2\mathbf{M})=\operatorname{tr}((\mathbf{P}+\mathbf{M})\mathbf{\Sigma}^2)=\operatorname{tr}\mathbf{\Sigma}^2.$$

Now we shall study the problem, whether the estimator $\hat{\gamma}$ given by (6) is admissible in the class of invariant (not necesserily unbiased) estimators for the function $\gamma=\operatorname{tr}(\mathbf{C\Sigma})$. Let $k>0$ be any constant. Then the mean square error (MSE) of the estimator $k \cdot \hat{\gamma}$ is

$$E_{\mathbf{\Sigma}}[k \cdot \hat{\gamma}-\operatorname{tr}(\mathbf{C\Sigma})]^2=\mathrm{k}^2 \cdot \mathrm{D}_{\mathbf{\Sigma}}[\hat{\gamma}]+(1-k)^2 \cdot [\operatorname{tr}(\mathbf{C\Sigma})]^2.$$

Thus the MSE of $k \cdot \gamma$ is uniformly smaller than the MSE of the estimator $\hat{\gamma}$ iff

$$[\operatorname{tr}(\mathbf{C\Sigma})]^2 \leq \frac{1+k}{1-k} D_{\mathbf{\Sigma}}[\hat{\gamma}] \quad \text{for all } \mathbf{\Sigma}.$$

The following lemma is obvious.

Lemma 2. *The invariant estimator* $k \cdot \hat{\gamma}$ *for the function* $\gamma=\operatorname{tr}(\mathbf{C\Sigma})$ *is uniformly better than the unbiased invariant estimator* $\hat{\gamma}$ *given by (6) iff there exists such a constant* d, $1<d<\infty$, *that for every* $\mathbf{\Sigma}$ *the inequality*

$$[\operatorname{tr}(\mathbf{C\Sigma})]^2 \leq d \cdot D_{\mathbf{\Sigma}}[\hat{\gamma}] \quad \textit{holds.} \tag{9}$$

For k and d we have: $k=\frac{d-1}{d+1}$.

Now we are able to prove the following theorem.

Theorem 4. *Let* $\gamma = \operatorname{tr} \boldsymbol{\Sigma}$. *Then the invariant estimator* $\frac{m \cdot n - 1}{m \cdot n + 1} \hat{\gamma}$ *for* γ *is uniformly better than the unbiased invariant estimator* $\hat{\gamma}$ *defined by* (6) *with* $\mathbf{C} = \mathbf{I}$.

Proof:: It is enough to prove that the inequality (9) is true for $d = m \cdot n$. This last inequality is, using (4) with $\mathbf{C} = \mathbf{I}$, equivalent to the inequality

$$(\operatorname{tr} \boldsymbol{\Sigma})^2 \leq 2n \cdot \left[\operatorname{tr} \boldsymbol{\Sigma}^2 + \frac{1}{m-1} \operatorname{tr} (\mathbf{P}\boldsymbol{\Sigma})^2\right].$$

But, using the Schwarz inequality, we get:

$$(\operatorname{tr} \boldsymbol{\Sigma})^2 \leq \|\mathbf{I}\|^2 \cdot \|\boldsymbol{\Sigma}\|^2 = n \cdot \operatorname{tr} \boldsymbol{\Sigma}^2 \leq 2n \cdot \left[\operatorname{tr} \boldsymbol{\Sigma}^2 + \frac{1}{m-1} \operatorname{tr} (\mathbf{P}\boldsymbol{\Sigma})^2\right] = d \cdot D_{\boldsymbol{\Sigma}}[\hat{\gamma}] \quad \text{for any}$$

covariance matrix $\boldsymbol{\Sigma}$.

Remark: For the special case $n = 1$, when Y_i; $i = 1, \ldots, m$ are independent $N(\beta, \sigma^2)$ distributed random variables,

$$\hat{\gamma} = \frac{1}{m-1} \sum_{i=1}^{m} (Y_i - \bar{Y})^2:$$

The estimator $\frac{m-1}{m+1} \hat{\gamma}$ is the uniformly best invariant estimator for $\gamma = \sigma^2$.

Examples.

For $\mathbf{C} \neq \mathbf{I}$ the estimator $\hat{\gamma}$ given by (6) for $\gamma = \operatorname{tr} (\mathbf{C}\boldsymbol{\Sigma})$ is not uniformly better than the estimator $\operatorname{tr} (\mathbf{CS})$. Thus the Theorem 3 is not true for $\mathbf{C} \neq \mathbf{I}$ (see Example 3).

Example 1. Let $n = 2$, $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\boldsymbol{\Sigma} = \begin{pmatrix} \mathrm{R}(0) & \mathrm{R}(1) \\ \mathrm{R}(1) & \mathrm{R}(0) \end{pmatrix}$.

Then $\gamma = \operatorname{tr} (\mathbf{C}\boldsymbol{\Sigma}) = 2 \cdot R(1)$. It is easy to show that $\operatorname{tr} (\mathbf{C}\boldsymbol{\Sigma})^2 = \operatorname{tr} \boldsymbol{\Sigma}^2$ and $\operatorname{tr} ((\mathbf{C} - \mathbf{MCM})\boldsymbol{\Sigma})^2 = \operatorname{tr} (\mathbf{P}\boldsymbol{\Sigma})^2$. Thus from the proof of the Theorem 3 we have that the estimator $\hat{\gamma}$ given by (6) is uniformly better than $\operatorname{tr} (\mathbf{CS})$.

Example 2.

Let $n = 2$,

$$\boldsymbol{\Sigma} = \begin{pmatrix} R(0) & R(1) & R(2) \\ R(1) & R(0) & R(1) \\ R(2) & R(1) & R(0) \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $\mathbf{X} = (1, 1, 1)'$. Then $\operatorname{tr} (\mathbf{C}\boldsymbol{\Sigma}) = 2R(2)$, $\operatorname{tr} (\mathbf{C}\boldsymbol{\Sigma})^2 = 2 \cdot (R(0)^2 + R(2)^2)$ and $\operatorname{tr} ((\mathbf{C} - \mathbf{MCM})\boldsymbol{\Sigma})^2 = \frac{4}{81} \cdot (18R^2(0) - 14R^2(1) + 16R^2(2) + 12R(0)R(1) + 33R(0)R(2) + 16R(1)R(2))$. So, $\operatorname{tr} (\mathbf{C}\boldsymbol{\Sigma})^2 \geq \operatorname{tr} ((\mathbf{C} - \mathbf{MCM})\boldsymbol{\Sigma})^2$ iff the function

$\phi(r_1, r_2) = 49r_2^2 + 28r_1^2 - 24r_1 - 66r_2 - 32r_1r_2 + 45$ is nonnegative for every $r_i = \frac{R(i)}{R(0)}$; $i = 1, 2$ such that $|r_i| \leq 1$. A solution of the equations $\frac{\partial\phi}{\partial r_1} = \frac{\partial\phi}{\partial r_2} = 0$ is $r_1 = r_2 = 1$ and $\phi(1, 1) = 0$.

The matrix $\boldsymbol{K} = \left\{\frac{\partial^2\phi}{\partial r_i\, \partial r_j}\right\}_{i,j=1}^{2} = \begin{pmatrix} 56 & -32 \\ -32 & 98 \end{pmatrix}$ is positive definite and the function ϕ has its minimum at the point (1,1). Thus we have proved that the estimator $\hat{\gamma}$ is uniformly better than tr **(CS)**.

Example 3. Let n, $\boldsymbol{\Sigma}$ and $\boldsymbol{X}$ be the same as in the previous example and let $\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $\operatorname{tr}(\mathbf{C}\boldsymbol{\Sigma}) = 4R(1)$, $\operatorname{tr}(\mathbf{C}\boldsymbol{\Sigma})^2 = 4 \cdot (R^2(0) + 2R^2(1) + R(0)R(2))$ and $\operatorname{tr}((\mathbf{C} - \mathbf{MCM})\boldsymbol{\Sigma})^2 = \frac{4}{81}(45R^2(0) + 82R^2(1) + 4R^2(2) + 120R(0)R(1) + 33R(0)R(2) + 40R(1)R(2))$.

Now let $\boldsymbol{\Sigma}_0 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. Then $\operatorname{tr}(\mathbf{C}\boldsymbol{\Sigma}_0)^2 = 0$, but $\operatorname{tr}((\mathbf{C} - \mathbf{MCM})\boldsymbol{\Sigma}_0)^2 = \frac{64}{81}$.

Thus in this case tr **(CS)** is the locally (at $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$) best unbiased invariant estimator for $\gamma = \operatorname{tr}(\mathbf{C}\boldsymbol{\Sigma})$ with $D_{\boldsymbol{\Sigma}_0}(\operatorname{tr}(\mathbf{CS})] = 0$.

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ОЦЕНКИ КОМПОНЕНТ КОВАРИАЦИОННОЙ МАТРИЦЫ В ПОВТОРЕННОМ РЕГРЕССИОННОМ ЭКСПЕРИМЕНТЕ

František Štulajter

Резюме

Предложены (несмещенные) инвариантные оценки функции $\gamma = \mathrm{tr}(\mathbf{C\Sigma})$, где $\mathbf{\Sigma}$ — ковариационная матрица случайных векторов $\mathbf{Y}_i \sim N_n(\mathbf{X}\boldsymbol{\beta}, \mathbf{\Sigma})$; $i = 1, \ldots, m$, $\mathbf{C}$ — любая симметричная матрица. Эти оценки сравниваются с несмещенной инвариантной оценкой $\mathrm{tr}(\mathbf{CS})$, где

$$\mathbf{S} = \frac{1}{m-1} \sum_{i=1}^{m} (\mathbf{Y}_i - \bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}})'$$

Показано, что для некоторых $\mathbf{C}$ оценка $\mathrm{tr}(\mathbf{CS})$ недопустима.