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ON RADICALS IN SEMIGROUPS

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Let S be a semigroup and $J \subseteq S$ a two-sided ideal of S. All ideals in the following are supposed to be two-sided.

An element $x \in S$ is called nilpotent with respect to J if $x^n \in J$ for some positive integer n. An ideal, or a subsemigroup I of S, is called nilpotent with respect to J if $I^n \subseteq J$ for some positive integer n. An ideal I of S, each element of which is nilpotent with respect to J, is called a nilideal with respect to J. An ideal I, each finitely generated subsemigroup of which is nilpotent with respect to J, is called a locally nilpotent ideal with respect to J. An ideal P of S is called prime if for any two ideals A, B of S $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$. An ideal P of S is called completely prime if for any two elements a, $b \in S$ $ab \in P$ implies that either $a \in P$ or $b \in P$. A subset $M \subseteq S$ is called a filter of S if $P = S - M$ is a completely prime ideal of S or $M = S$. It is known that a subset M of S is a filter of S if and only if $x, y \in M$ is equivalent to $xy \in M$.

The set of all nilpotent elements of S with respect to an ideal J of S will be denoted by $N_J(S)$.

The union $R_J(S)$ of all nilpotent ideals of S with respect to J is called the Schwarz radical of S with respect to J. The union $L_J(S)$ of all locally nilpotent ideals of S with respect to J is called the Ševrin radical of S with respect to J. The union $R^*_J(S)$ of all nilideal of S with respect to J is called the Clifford radical of S with respect to J. The intersection $M_J(S)$ of all prime ideals of S which contain J is called the McCoy radical of S with respect to J. The intersection $C_J(S)$ of all completely prime ideals of S which contain J is called the Luh radical of S with respect to J.

J. Luh [5, Theorem 3,3; 3,4; 3,5; 3,7 and Corollary] proved for a semigroup S with the kernel K (the minimal ideal of S) that $R_K(S) \subseteq M_K(S) \subseteq R^*_K(S) \subseteq C_K(S)$ and for a commutative semigroup $R_K(S) = M_K(S) = R^*_K(S) = C_K(S)$.

Š. Schwarz [6, section II, Theorems 7, 8, 9] studied questions connected with the existence of the kernel of a semigroup S and the relations $R_K(S) \subseteq M_K(S) \subseteq R^*_K(S) \subseteq M^*$ (the last relation if $S^2 = S$ and $M^* \neq \emptyset$), where M^* is the intersection of all maximal ideals of S.

R. Šulka [7, Lemma 19] and J. Bosák [2, Theorem 2] proved that in an arbitrary semigroup S with an ideal $J \subseteq S$ we have

$$R_J(S) \subseteq M_J(S) \subseteq L_J(S) \subseteq R_J^*(S) \subseteq N_J(S) \subseteq C_J(S). \tag{1}$$

In the case of a commutative semigroup S as proved by R. Šulka [7, Theorem 7] and J. Bosák [2, Corollary 1] we have

$$R_J(S) = M_J(S) = L_J(S) = R_J^*(S) = N_J(S) = C_J(S). \tag{2}$$

A semigroup S is called a C_2-semigroup if for all $a, b, c,$ of S $abcba = bacab$. J. E. Kuczowski [3] proved that in a C_2-semigroup S the equalities

$$M_J(S) = L_J(S) = R_J^*(S) = N_J(S) = C_J(S) \quad .$$

hold.

A semigroup S is called quasi-commutative if $ab = b^r a$ for all a, b of S and for some positive integer $r = r(a, b)$. H. Lal [4] proved that in a quasi-commutative semigroup S the equalities of (2) hold.

In the author's paper [8] it is proved that in the semigroup U of all triangular $m \times m$ matrices over a commutative ring we have $R_0(U) = M_0(U) = L_0(U) = R_0^*(U) = N_0(U)$. Theorem 1 of this paper implies that in U the equalities of (2) hold for $J = 0$.

In the present paper we prove (Theorem 1, Corollary 1) that in a semigroup S we have $R_J^*(S) = N_J(S) = C_J(S)$ if and only if the set $N_J(S)$ is an ideal of S. Theorem 1 is analogous to the results of A. Abian [1, Theorem 2] for rings without nilpotent elements. Further we prove that the equalities of (2) in a semigroup S hold if and only if for each $a \in N_J(S)$ the principal ideal (a) is nilpotent with respect to J (Theorem 2). In a finite semigroup S the equalities of (2) are valid if and only if $N_J(S)$ is an ideal of S (Corollary 3).

Lemma 1. *If $a_1, a_2, \ldots, a_n$ are elements of a semigroup S and $a_1 a_2 \ldots a_n \in N_J(S)$, then $a_k \ldots a_n a_1 \ldots a_{k-1} \in N_J(S)$ $(k = 2, \ldots, n)$.*

Proof. Let $ab \in M_J(S)$ and $(ab)^r \in J$ for some positive integer r, then also $(ba)^{r+1} = b(ab)^r a \in J$ and so $ba \in N_J(S)$. By the preceding it follows from $(a_1 \ldots a_{k-1})$ $(a_k \ldots a_n) \in N_J(S)$ that $(a_k \ldots a_n)$ $(a_1 \ldots a_{k-1}) \in N_J(S)$, where $k = 2, \ldots, n$.

Lemma 2. *Let there be $a_i \in S$, $N_J(S)$ be an ideal of S, $a_1 a_2 \ldots a_n \in N_J(S)$ and $i_1, i_2, \ldots, i_n$ an arbitrary permutation of the set $\{1, 2, \ldots, n\}$. Then $a_{i_1} a_{i_2} \ldots a_{i_n} \in N_J(S)$.*

Proof. a) Let $ab \in N_J(S)$. Since $N_J(S)$ is an ideal of S, for each $t \in S$ the product $bat \in N_J(S)$. From $b(at) \in N_J(S)$ by Lemma 1 we have $(at)b \in N_J(S)$. Therefore $ab \in N_J(S)$ implies $satb \in N_J(S)$ for arbitrary $s, t \in S$ (where s, t may be empty symbols).

b) Let there be $b_1 b_2 b_3 \in N_J(S)$ for the elements b_1, b_2, b_3 of S. We shall prove that for each permutation i, j, k of the set $\{1, 2, 3\}$ we have $b_i b_j b_k \in N_J(S)$. From Lemma 1 we at once obtain that $b_2 b_3 b_1 \in N_J(S)$ and $b_3 b_1 b_2 \in N_J(S)$. By a) $b_1(b_2 b_3)$

$\in N_J(S)$ for $s=b_2$, $t=b_3$ implies $sb_1t(b_2b_3) = b_2b_1b_3b_2b_3 \in N_J(S)$. Again by a) from $(b_2b_1b_3b_2)b_3 \in N_J(S)$ for $t'=b_1$ we obtain $(b_2b_1b_3b_2)t'b_3 = (b_2b_1b_3)^2 \in N_J(S)$. Therefore $b_2b_1b_3 \in N_J(S)$. Then Lemma 1 implies that also $b_1b_3b_2 \in N_J(S)$ and $b_3b_2b_1 \in N_J(S)$.

c) Let $a_1 \dots a_{k-1}a_ka_{k+1} \dots a_n \in N_J(S)$ $(k=2, 3, \dots, n)$. We shall show that $a_ka_1 \dots a_{k-1}a_1a_{k+1} \dots a_n \in N_J(S)$.

With respect to b) from $a_1(a_2 \dots a_{k-1})$ $(a_ka_{k+1} \dots a_n) \in N_J(S)$ we have $(a_2 \dots a_{k-1})a_1(a_ka_{k+1} \dots a_n) \in N_J(S)$.

Then $(a_2 \dots a_{k-1}a_1)a_k(a_{k+1} \dots a_n) \in N_J(S)$ by b) implies that $a_k(a_2 \dots a_{k-1}a_1)$ $(a_{k+1} \dots a_n) \in N_J(S)$.

It is well known that the group of all permutations of the set $\{1, 2, \dots, n\}$ is generated by the set of the transpositions $(1, 2)$, $(1, 3)$, ..., $(1, n)$. From this it follows that Lemma 2 is true.

Lemma 3. *If $N_J(S)$ is an ideal of a semigroup S and $a_1a_2 \dots a_n \in N_J(S)$, then for arbitrary elements $s_1, s_2, \dots, s_{n+1}$ (where some of the s_i may be empty symbols) we have $s_1a_1s_2a_2 \dots s_na_ns_{n+1} \in N_J(S)$.*

Proof. Let $a_1a_2 \dots a_n \in N_J(S)$. Since $N_J(S)$ is an ideal of S, for any $s_1, s_2, \dots, s_{n+1}$ of S we have that $s_1s_2 \dots s_{n+1}a_1a_2 \dots a_n \in N_J(S)$. The word $s_1a_1s_2a_2 \dots s_na_ns_{n+1}$ is obtained by means of a suitable rearrangement of the letters in $s_1s_2 \dots s_{n+1}a_1a_2 \dots a_n$. Then by Lemma 2 we have that $s_1a_1s_2a_2 \dots s_na_ns_{n+1} \in N_J(S)$.

Lemma 4. *Let $N_J(S)$ be an ideal of S. Suppose that $a_1a_2 \dots a_n \in N_J(S)$. Denote by $b_1, b_2, \dots, b_r$ $(r \leqq n)$ the different elements in the set $\{a_1, \dots, a_n\}$. Then (in any order) $b_1b_2 \dots b_r \in N_J(S)$.*

Proof. Rearrange the letters in $a_1a_2 \dots a_n$ in such a manner that we obtain a word of the form $b_1^{k_1} \dots b_r^{k_r}$. By Lemma 2 this element is contained in $N_J(S)$. Take an integer $k > k_i$. Then by Lemma 3 $b_1^{k_1}b_1^{k-k_1}b_2^{k_2}b_2^{k-k_2} \dots b_r^{k_r}b_r^{k-k_r} = b_1^kb_2^k \dots b_r^k \in N_J(S)$. By means of a suitable rearrangement of the letters b_i we obtain (by Lemma 2) that $(b_1b_2 \dots b_r)^k \in N_J(S)$ and so $b_1b_2 \dots b_r \in N_J(S)$.

Remark 1. If the set $N_J(S)$ of all nilpotent elements of S is not an ideal, then Lemmas 2—4 need not be true. This shows the next example 1.

Example 1. Let $S_1 = \{0, e_{11}, e_{12}, e_{21}, e_{22}, e\}$ be a semigroup with the multiplication: $e_{ik} \cdot e_{kn} = e_{in}$, $e_{ik} \cdot e_{jn} = 0 \cdot e_{ik} = e_{ik} \cdot 0 = 0$, $e_{ik} \cdot e = e \cdot e_{ik} = e_{ik}$ for $i, j, k, n \in \{1, 2\}$, $j \neq k$. Evidently the set $N_0(S_1) = \{0, e_{12}, e_{21}\}$ is not an ideal of S_1. In S_1 Lemmas 2—4 do not hold, since, e.g., $e_{21}e_{12}e_{11} = 0 \in N_0(S_1)$ but $e_{12}e_{21}e_{11} = e_{11} \notin N_0(S_1)$, $e_{12}^2e_{21}^2 = 0 \in N_0(S_1)$ and the product of all distinct letters $e_{12}e_{21} = e_{11} \notin N_0(S_1)$.

Example 2. The subsemigroup $S_2 = \{0, e_{11}, e_{12}, e_{22}, e\}$ of S_1 with the same multiplication is an example of a non-commutative semigroup in which the set $N_0(S_2) = \{0, e_{12}\}$ is an ideal of S_2. Here, e.g., $J = \{0, e_{12}, e_{22}\}$ is an ideal and $N_J(S_2) = J$ is an ideal of S_2.

Lemma 5. *Let $N_J(S) \neq S$ be an ideal of a semigroup S and let M be a maximal subsemigroup of S which does not meet $N_J(S)$. Then M is a filter of S*

Proof. Let $a \in S - M$. The semigroup $\{M, a\}$ generated by M and a is larger than M, hence it has a non-empty intersection with $N_J(S)$ There exists therefore a product containing the element a and elements $m_i \in M$, which is contained in $N_J(S)$. By rearranging the letters in this product we get a new element of the form $m_1 \ldots m_n a^k$ ($m_i \in M$, $k \geqq 1$) which (by Lemma 2) is again contained in $N_J(S)$. By Lemma 4 there is an element $m \in M$ such that $ma \in N_J(S)$.

We shall prove that M is a filter of S. It is necessary to prove that $ab \in M$ implies $a \in M$ and $b \in M$. We prove it indirectly.

Suppose that $ab \in M$, while either $a \in S - M$ or $b \in S - M$. Let $a \in S - M$. Then by the preceding part of the proof there exists an element $m \in M$ such that $ma \in N_J(S)$. Since $N_J(S)$ is an ideal of S we have $(ma)b = m(ab) \in N_J(S)$ but $m \in M$, $ab \in M$ and so $m(ab) \in M$. This is a contradiction with the assumption that M does not meet $N_J(S)$. Analogously let $b \in S - M$. Then there exists $m' \in M$ such that $m'b \in N_J(S)$ and so by Lemma 3 $m'ab \in N_J(S)$. But $m' \in M$, $ab \in M$ and so $m'(ab) \in M$, which contradicts our assumption that M does not meet $N_J(S)$. Therefore $ab \in M$ implies that $a \in M$ and $b \in M$ and so M is a filter of S.

Theorem 1. *Let S be a semigroup, J an ideal of S and suppose that $N_J(S) \subseteq S$ is an ideal of S. Then $N_J(S) = C_J(S)$.*

Proof. In an arbitrary semigroup S we have $N_J(S) \subseteq C_J(S)$. If $N_J(S) = S$, then obviously $N_J(S) = C_J(S) = S$ and the statement holds.

Suppose that $N_J(S) \neq S$. We prove that $C_J(S) \subseteq N_J(S)$. It is sufficient to show that for any $a \in S$, $a \notin N_J(S)$ there exists a completely prime ideal P which does not contain a. Then $a \notin P$ implies $a \notin C_J(S)$.

Let $a \in S$, $a \notin N_J(S)$ be an arbitrary element. Then the subsemigroup $A = \{a, a^2, a^3, \ldots\}$ of S does not meet $N_J(S)$. From Zorn's lemma it follows that there exists a maximal subsemigroup M of S, $M \supseteq A$ which does not meet $N_J(S)$. Then by Lemma 5 M is a filter of S. Consequently $P = S - M$ is a completely prime ideal of S containing $N_J(S)$ with the property $a \notin P$ and so $a \notin C_J(S)$.

Both relations $N_J(S) \subseteq C_J(S)$ and $C_J(S) \subseteq N_J(S)$ imply $N_J(S) = C_J(S)$.

Corollary 1. *In a semigroup S the equalities $R_J^*(S) = N_J(S) = C_J(S)$ hold if and only if $N_J(S)$ is an ideal of S.*

Proof. Evidently if $R_J^*(S) = N_J(S) = C_J(S)$, then $N_J(S)$ is an ideal of S.

Conversely, let $N_J(S)$ be an ideal of S. By (1) we have $R_J^*(S) \subseteq N_J(S)$. Since $N_J(S)$ is a nilideal of S from the definition of Clifford's radical it follows $N_J(S) \subseteq R_J^*(S)$. Therefore $R_J^*(S) = N_J(S)$. Then by Theorem 1 we obtain $N_J(S) = R_J^*(S) = C_J(S)$.

Clearly $a \in R_J^*(S)$ if and only if the principal ideal (a) is a nilideal of S with respect to J.

Corollary 2. *Let S be a semigroup, J be an ideal of S. Then $R_J^*(S) = N_J(S) = C_J(S)$ holds if and only if for each $a \in N_J(S)$ the principal ideal (a) is a nilideal of S with respect to J.*

Corollary 3. *In a finite semigroup S the equalities (2), i.e., $R_J(S) = M_J(S) = L_J(S) = R_J^*(S) = N_J(S) = C_J(S)$ hold if and only if the set $N_J(S)$ is an ideal of S.*

Proof. Evidently if (2) holds, then $N_J(S)$ is an ideal of S. Conversely, let $N_J(S)$ be an ideal of S. Then from the statement of J. Bosák [2, p. 211, Corollary 2] that in an arbitrary finite semigroup we have $R_J(S) = M_J(S) = L_J(S) = R_J^*(S)$ and from Corollary 1 it follows that (2) holds.

Remark 2. In an infinite semigroup S (2) need not hold. The following example is given by J. Bosák [2, p. 209—210]. Let S be a semigroup generated by $\{0, a, b\}$ with generating relations $0x = x0 = x^3 = 0$ for every word x over the given alphabet. Then for the Ševrin, Clifford and Luh radicals with respect to $J = 0$ we have $L_0(S) \subset R_0^*(S) = N_0(S) = C_0(S) = S$, $L_0(S) \neq R_0^*(S)$.

Theorem 2. *Let S be a semigroup, J be an ideal of S, $N_J(S)$ be the set of all nilpotent elements of S with respect to J. Then* (2), *i.e. the equalities*

$$R_J(S) = M_J(S) = L_J(S) = R_J^*(S) = N_J(S) = C_J(S)$$

hold if and only if for each $a \in N_J(S)$ the principal ideal (a) is nilpotent with respect to J.

Proof. If (2) hold, then evidently for $a \in N_J(S)$ we have $(a) \in R_J(S)$ and the principal ideal (a) is nilpotent with respect to J.

Conversely, suppose that for each $a \in N_J(S)$ the principal ideal (a) of S is nilpotent with respect to J. Then from the relation $R_J(S) \subseteq N_J(S)$ and from the definition of Schwarz's radical we have $R_J(S) = N_J(S)$. Then Theorem 1 implies $N_J(S) = C_J(S)$ and so with respect to (1) we obtain (2).

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О РАДИКАЛАХ ПОЛУГРУПП

Франтишек Кметь

Резюме

Пусть S — произвольная полугруппа, J — идеал полугруппы S, $N_J(S)$ — множество нильпотентных элементов полугруппы S относительно идеала J. Если для всякого $a \in N_J(S)$ главный идеал (a) является нильидеалом (нильпотентным идеалом) полугруппы S относительно идеала J, то радикалы Клиффорда и Луга (радикалы Шварца, Маккойа, Шеврина, Клиффорда и Луга) относительно идеала J равны $N_J(S)$, и наоборот.