Mathematica Slovaca

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Mathematica Slovaca, Vol. 48 (1998), No. 5, 513--525

Persistent URL: http://dml.cz/dmlcz/128783

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LAGUERRE POLYNOMIALS WITH FOUR PARAMETERS

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(*Communicated by Michal Zajac*)

ABSTRACT. The polynomials $P_n(x)$, which are orthonormal in the interval $[0,+\infty)$ with respect to the weight function $P(x)=(a+x)^{2\alpha}(x-b)^{2\beta}\,\mathrm{e}^{-x}$, where $a>0$, $\alpha\in\mathbb{R}$, $b\geq 0$, $\beta>\frac{1}{2}$, are investigated. There are derived relations for coefficients of these polynomials, relations for the sums of kth powers $s_k^{(n)}$ of zeros of these polynomials, the relation giving a connection between the polynomials of different degrees and differential equations for the polynomials $P_n(x)$.

1. Introduction

The Laguerre polynomials $L_n(x)$ may be defined by the relation

$$L_n(x)=x^{-\alpha}\,\mathrm{e}^{x}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{n+\alpha}\,\mathrm{e}^{-x}\right), \tag{1}$$

and $L_0(x)=1$, $\alpha>-1$. The system of polynomials $L_n(x)$ is orthogonal in the interval $I=[0,\infty)$ with respect to the function $L(x)=x^{\alpha}\,\mathrm{e}^{-x}$, so for $n=0,1,\dots$,

$$\int\limits_I L_n^2(x)L(x)\,\mathrm{d}x=n!\,\Gamma(n+\alpha+1)\,,\qquad \int\limits_I L_n(x)L_m(x)L(x)\,\mathrm{d}x=0 \tag{2}$$

for $m\neq n$ (see [1; p. 93]).

In the work [2] J. Korous defined his polynomials $K_n(x)$, which are orthonormal in the interval $[0,+\infty)$ with respect to the weight function $K(x)=x^{\alpha}(a+x)^{\beta}\,\mathrm{e}^{-x}$, where α, β, a are real numbers, $\alpha\geq-\frac{1}{2}$, $a>0$.

AMS Subject Classification (1991): Primary 33C25.
Key words: orthonormal polynomial, Laguerre polynomial, differential equation.

In the present work we will investigate the system of the polynomials $\{P_n(x)\}_{n=0}^{\infty}$, where

$$P_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k}\,, \qquad a_0^{(n)} > 0\,, \tag{3}$$

which are orthonormal in the interval $I = [0, +\infty)$ with respect to the function

$$P(x) = (a+x)^{2\alpha}(x-b)^{2\beta}\,\mathrm{e}^{-x}\,, \tag{4}$$

where $a > 0$, $\alpha \in \mathbb{R}$, $b \geq 0$, $\beta > \frac{1}{2}$, i.e.

$$\int_0^\infty P_n(x)P_m(x)P(x)\,\mathrm{d}x = \delta_{m,n}\,, \tag{5}$$

and $\delta_{n,n} = 1$, $\delta_{m,n} = 0$ for $m \neq n$.

Laguerre polynomials $L_n(x)$ and Korous polynomials $K_n(x)$ are special cases of considered polynomials $P_n(x)$.

In this work we derive some relations for coefficients of the polynomials (3), the relations (19) and (20) for the sums of kth powers $s_k^{(n)}$ of the zeros of these polynomials, the relation (24), which expresses a connection between polynomials of different degrees and differential equations (35) and (37).

2. Basic relations and lemmas for polynomials $P_n(x)$

Notation. For $n = 0, 1, 2, \dots$ we define

$$q_n = \frac{a_0^{(n-1)}}{a_0^{(n)}} \quad \text{for } n > 0\,, \qquad q_n = 0 \quad \text{for } n \leq 0\,, \tag{6}$$

$$r_k^{(n)} = \frac{a_k^{(n)}}{a_0^{(n)}} \quad \text{for } k > 0\,, \qquad r_k^{(n)} = 0 \quad \text{for } k < 0\,, \qquad r_0^{(n)} = 1\,, \tag{7}$$

$$j_n = \int_I x P_n^2(x) P(x)\,\mathrm{d}x\,. \tag{8}$$

Remark 1. $\pi_n(x) = \pi_n$ denotes arbitrary polynomial of at most nth degree.

Remark 2. c and c_i $(i = 1, 2, \dots)$ are positive constants independent of n and x and t, respectively. The numbering of these constants in each lemma and theorem is independent of the numbering in the others.

Lemma 1. (**Reccurence formula for $P_n(x)$**) *Under notations* (3), (6) *and* (8) *for every* n

$$(x - j_n)P_n(x) = q_{n+1}P_{n+1}(x) + q_nP_{n-1}(x)\,. \tag{9}$$

Proof. See [1; p. 77]. □

Lemma 2 (**Christoffel-Darboux formula**) *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be a system of orthonormal polynomials in the interval* $[0,+\infty)$. *Denote*

$$P_n(x,t) = \sum_{k=0}^{n} P_k(x)P_k(t)\,. \tag{10}$$

Then for $x \neq t$

$$P_n(x,t) = q_{n+1}(x-t)^{-1}\{P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)\}\,. \tag{11}$$

Proof. (11) follows from (9). See [1; p. 79]. □

Lemma 3. *Every polynomial* $F_n(x)$ *of the* n*th degree can be expressed in the form*

$$F_n(x) = \sum_{k=0}^{n} \alpha_k P_k(x)\,, \tag{12}$$

where

$$\alpha_k = \int_I F_n(t)P_k(t)P(t)\ \mathrm{d}t\,. \tag{13}$$

Proof is well known.

Lemma 4. *Preserving the notation* (8) *we denote*

$$i_n = 2\alpha a \int_I (x+a)^{-1}P_n^2(x)P(x)\ \mathrm{d}x\,, \tag{14}$$

$$h_n = 2\beta b \int_I (x-b)^{-1}P_n^2(x)P(x)\ \mathrm{d}x\,. \tag{15}$$

Then for $n = 1, 2, \dots$

$$j_n = 2n + 1 + 2\alpha + 2\beta - i_n + h_n\,. \tag{16}$$

Proof. We use integration by parts on (8). Then

$$\begin{aligned} j_n &= \int_I x P_n^2(x) P(x)\, \mathrm{d}x \\ &= -\int_I x P_n^2(x) \big[P(x)\, \mathrm{e}^x\big]\, \mathrm{d}\mathrm{e}^{-x} \\ &= \big[-x P_n^2(x) P(x)\big]_0^\infty + \int_I 2x P_n(x) P_n'(x) P(x)\, \mathrm{d}x \\ &\qquad + \int_I \big\{1 + 2\alpha x (x+a)^{-1} + 2\beta x (x-b)^{-1}\big\} P_n^2(x) P(x)\, \mathrm{d}x \\ &= 2n + 1 + 2\alpha + 2\beta - i_n + h_n\,, \end{aligned}$$

where in the last integral we substitute x by $x = x + a - a$ and by $x = x - b + b$, respectively and we use (5). □

LEMMA 5. *For $n \to +\infty$ the following relations hold*

$$i_n = O(1)\,, \tag{17}$$

$$h_n = O(1)\,. \tag{18}$$

Proof. The relations (17) and (18) follow from (14) and (15). □

LEMMA 6. *Let $s_k^{(n)}$ for $k = 0, 1, 2, \dots$ be the sum of kth powers of zeros of the polynomials $P_n(x)$. Then*

$$s_1^{(n)} = \sum_{k=0}^{n-1} j_k = -r_1^{(n)}\,, \tag{19}$$

$$s_2^{(n)} = \sum_{k=0}^{n-1} \big(2q_k^2 + j_k^2\big)\,, \tag{20}$$

$$-r_1^{(n)} = n^2 + (2\alpha + 2\beta)n + \sigma_n\,, \tag{21}$$

where

$$\sigma_n = \sum_{k=0}^{n-1} (h_k - i_k)\,. \tag{22}$$

Proof.

I. There holds

$$s_1^{(n)} = \sum_{i=1}^{n} x_i = -\frac{a_1(n)}{a_0(n)} = -r_1(n)\,, \tag{a}$$

where $r_k^{(n)}$ is defined in (7).

From recurrence formula (9) comparing coefficients at powers x^n we get

$$a_0^{(n)} r_1^{(n)} - j_n a_0^{(n)} = q_{n+1} a_0^{(n+1)} r_1^{(n+1)} .$$

Dividing this equation by $a_0^{(n)}$ and substituting for q_{n+1} according to (6) we have

$$r_1^{(n+1)} - r_1^{(n)} = -j_n . \tag{b}$$

Let us put for $k = 0, 1, 2, \dots$ $\delta_k^{(n)} = s_k^{(n)} - s_k^{(n-1)}$. From there using (a) and (b) we have

$$\delta_1^{(i)} = s_1^{(i)} - s_1^{(i-1)} = -r_1^{(i)} + r_1^{(i-1)} = j_{i-1} \tag{c}$$

(because $s_1^{(0)} = 0$).

From there

$$\sum_{i=1}^{n} \delta_1^{(i)} = \sum_{i=1}^{n} \left[s_1^{(i)} - s_1^{(i-1)} \right]$$

and thus

$$s_1(n) = \sum_{i=0}^{n-1} j_i .$$

II. From recurrence formula (9) by comparing of coefficients at powers x^n and x^{n-1}, respectively, we get

$$a_0^{(n)} r_1^{(n)} - j_n a_0^{(n)} = q_{n+1} a_0^{(n+1)} r_1^{(n+1)} ,$$
$$a_0^{(n)} r_2^{(n)} - j_n a_0^{(n)} r_1^{(n)} = q_{n+1} a_0^{(n+1)} r_2^{(n+1)} + a_0^{(n-1)} q_n .$$

Dividing these two equations by $a_0^{(n)}$ and substituting for q_n and q_{n+1} according to (6) and, further, using known relation

$$s_2^{(n)} = \sum_{i=1}^{n} x_i^2 = \left[s_1^{(n)} \right]^2 - 2r_2^{(n)} = -r_1^{(n)} s_1^{(n)} - 2r_2^{(n)} ,$$

i.e.

$$s_2^{(n)} + r_1^{(n)} s_1^{(n)} + 2r_2^{(n)} = 0 , \tag{d}$$

we get

$$r_1^{(n+1)} - r_1^{(n)} = -j_n ; \qquad r_2^{(n+1)} - r_2^{(n)} = -q_n^2 - j_n r_1^{(n)} . \tag{e}$$

Further

$$s_2^{(n)} + r_1^{(n)} s_1^{(n)} + 2r_2^{(n)} = 0 ,$$
$$s_2^{(n-1)} + r_1^{(n-1)} s_1^{(n-1)} + 2r_2^{(n-1)} = 0 .$$

Subtracting the second of these equations from the first and using (c) we have

$$\delta_2^{(n)} + r_1^{(n-1)}\delta_1^{(n)} + \big(r_1^{(n)} - r_1^{(n-1)}\big)s_1^{(n)} + 2\big(r_2^{(n)} - r_2^{(n-1)}\big) = 0\,,$$

from where in virtue of (e), (c), (a)

$$\delta_2^{(n)} = -j_{n-1}r_1^{(n-1)} - j_{n-1}r_1^{(n)} + 2j_{n-1}r_1^{(n-1)} + 2q_{n-1}^2 = j_{n-1}^2 + 2q_{n-1}^2\,.$$

Because $s_2^{(0)} = 0$, it follows from there

$$s_2^{(n)} = \sum_{k=0}^{n-1}(2q_k^2 + j_k^2)\,.$$

III. From the relations (19) and (16) we get

$$\begin{aligned}-r_1^{(n)} &= \sum_{k=0}^{n-1} j_k = \sum_{k=0}^{n-1}(2k + 1 + 2\alpha + 2\beta + h_k - i_k)\\ &= n(n-1) + n + 2\alpha n + 2\beta n + \sigma_n = n^2 + (2\alpha + 2\beta)n + \sigma_n\,,\end{aligned}$$

where σ_n is defined in (22). □

LEMMA 7. *For $n \to +\infty$*

$$\sigma_n = O(1)\,. \tag{23}$$

Proof. (23) follows from (17), (18) and (22). □

LEMMA 8. *For $n = 1, 2, \dots$ we have*

$$xP_n'(x) = nP_n(x) + q_nP_{n-1}(x) + \sum_{k=0}^{n-1}\gamma_kP_k(x)\,, \tag{24}$$

where

$$\gamma_k = i_n^{(k)} - h_n^{(k)}\,, \tag{25}$$

$$i_n^{(k)} = 2\alpha a \int_I (x+a)^{-1}P_n(x)P_k(x)P(x)\,\mathrm{d}x\,, \tag{26}$$

$$h_n^{(k)} = 2\beta b \int_I (x-b)^{-1}P_n(x)P_k(x)P(x)\,\mathrm{d}x\,. \tag{27}$$

Proof. According to (12)

$$xP_n'(x) = \sum_{k=0}^{n}\beta_kP_k(x)\,, \tag{a}$$

where

$$\beta_k = \int_I x P_n'(x) P_k(x) P(x) \, \mathrm{d}x \,. \tag{b}$$

We use integration by parts to (b) and then

$$\begin{aligned}\beta_k = {}& \left[x P_n(x) P_k(x) P(x)\right]_0^\infty - \int_I P_n(x) P_k(x) P(x) \, \mathrm{d}x \\ & - \int_I x P_n(x) P_k'(x) P(x) \, \mathrm{d}x + \int_I x P_n(x) P_k(x) P(x) \, \mathrm{d}x \\ & - \int_I x\left[2\alpha(x+a)^{-1} + 2\beta(x-b)^{-1}\right] P_n(x) P_k(x) P(x) \, \mathrm{d}x \\ = {}& \int_I x P_n(x)\left[P_k(x) - P_k'(x)\right] P(x) \, \mathrm{d}x + (-2\alpha - 2\beta) \int_I P_n(x) P_k(x) P(x) \, \mathrm{d}x \\ & - \int_I P_n(x) P_k(x) P(x) \, \mathrm{d}x + i_n^{(k)} - h_n^{(k)} \,. \end{aligned} \tag{c}$$

1. Let $k = n$.

Then in virtue of (8), (16), (14) and (15) from (c) we get (because $i_n^{(n)} = i_n$, $h_n^{(n)} = h_n$)

$$\begin{aligned}\beta_n &= j_n - n - 1 - 2\alpha - 2\beta + i_n - h_n \\ &= 2n + 1 + 2\alpha + 2\beta - i_n + h_n - n - 1 - 2\alpha - 2\beta + i_n - h_n = n \,.\end{aligned} \tag{d}$$

2. Let $k = n - 1$.

Then from (c) in virtue of (6) and (5) there is

$$\beta_{n-1} = q_n + i_n^{(n-1)} - h_n^{(n-1)} \,. \tag{e}$$

3. Let $k < n - 1$.

Then from (c) in virtue of (5) there is

$$\beta_k = i_n^{(k)} - h_n^{(k)} = \gamma_k \,. \tag{f}$$

From (c) – (f) we get (24). □

Lemma 9. *Let* $n = 1, 2, \ldots$ *. Then*

$$q_n^2 - 2\delta_n q_n + r_1^{(n)} = 0 \,, \tag{28}$$

where q_n *is defined in* (6), $r_1^{(n)}$ *in* (7) *and* $2\delta_n = h_n^{(n-1)} - i_n^{(n-1)}$.

P r o o f . There holds

$$\int_I x P_n(x) P_{n-1}(x) P(x) \, \mathrm{d}x = \frac{a_0^{(n-1)}}{a_0^{(n)}} = q_n \,. \tag{a}$$

Integrating left-hand side by parts we get

$$\begin{aligned}
q_n &= -\int_I xP_n(x)P_{n-1}(x)\big[P(x)\,\mathrm{e}^x\big]\,\mathrm{d}\mathrm{e}^{-x} \\
&= \big[-xP_n(x)P_{n-1}(x)P(x)\big]_0^\infty + \int_I xP_n'(x)P_{n-1}(x)P(x)\,\mathrm{d}x \\
&\quad + \int_I x\big[2\alpha(x+a)^{-1} + 2\beta(x-b)^{-1}\big]P_n(x)P_{n-1}(x)P(x)\,\mathrm{d}x \qquad \text{(b)} \\
&= \int_I xP_n'(x)P_{n-1}(x)P(x)\,\mathrm{d}x - 2\alpha a\int_I (x+a)^{-1}P_n(x)P_{n-1}(x)P(x)\,\mathrm{d}x \\
&\quad + 2\beta b\int_I (x-b)^{-1}P_n(x)P_{n-1}(x)P(x)\,\mathrm{d}x \\
&= J_n - i_n^{(n-1)} + h_n^{(n-1)},
\end{aligned}$$

where

$$\begin{aligned}
J_n &= \int_I xP_n'(x)P_{n-1}(x)P(x)\,\mathrm{d}x \\
&= \int_I \big[nP_n(x) - r_1^{(n)}q_n^{-1}P_{n-1}(x) + \pi_{n-2}\big]P_{n-1}(x)P(x)\,\mathrm{d}x \\
&= -r_1^{(n)}q_n^{-1}, \qquad \text{(c)}
\end{aligned}$$

$$i_n^{(n-1)} = 2\alpha a\int_I (x+a)^{-1}P_n(x)P_{n-1}(x)P(x)\,\mathrm{d}x, \qquad \text{(d)}$$

$$h_n^{(n-1)} = 2\beta b\int_I (x-b)^{-1}P_n(x)P_{n-1}(x)P(x)\,\mathrm{d}x. \qquad \text{(e)}$$

From (b) in virtue of (c) – (e) we get

$$q_n = J_n + h_n^{(n-1)} - i_n^{(n-1)} = -r_1^{(n)}q_n^{-1} + 2\delta_n.$$

From there (28) follows. □

Lemma 10. *For $n \to +\infty$*

$$q_n = n + O(1). \qquad (29)$$

P r o o f. (29) follows from (28), (17), (18), (21) and (22). □

LEMMA 11. *Let $P_n(x)$ be a polynomial of the degree n and s_k be the sum of kth powers of its zeros. Then for a natural number r*

$$x^r P_n'(x) = \sum_{k=0}^{r-1} s_k x^{r-k-1} P_n(x) + \pi_{n-1} \,. \tag{30}$$

Proof. See [3; p. 352]. □

LEMMA 12. *Let $P_n(x)$ be a polynomial of the degree n and s_k be the sum of kth powers of its zeros. Then for a natural number r*

$$x^r P_n''(x) = P_n(x) \sum_{k=0}^{r-2} \sigma_k x^{r-k-2} + \pi_{n-1} \,, \tag{31}$$

where

$$\sigma_k = \sum_{m=0}^{k} s_m s_{k-m} - (k+1) s_k \,. \tag{32}$$

Proof. (31) follows from (30). □

3. Differential equations for polynomials $P_n(x)$

THEOREM 1. *Let us denote*

$$\varepsilon = \begin{cases} \alpha \\ \beta \end{cases} \qquad \delta = \begin{cases} \beta \\ \alpha \end{cases} \qquad \Longrightarrow \quad \varepsilon + \delta = \alpha + \beta \,, \tag{33}$$

$$e = \begin{cases} a \\ -b \end{cases} \qquad f = \begin{cases} -b \\ a \end{cases} \qquad \Longrightarrow \quad e + f = a - b \,. \tag{34}$$

Let $n = 0, 1, 2, \dots$. Then

$$\begin{aligned} P^{-1}(x) \frac{\mathrm{d}}{\mathrm{d}x} \big[(x+e) x P_n'(x) P(x) \big] - 2\delta f (e-f)(x+f)^{-1} P_n'(x) \\ + \big[n(x+e-1) + \sigma_n \big] P_n(x) \\ = \sum_{k=0}^{n-1} \chi_k P_k(x) + q_n P_{n-1}(x) \,, \end{aligned} \tag{35}$$

where σ_n is defined in (22),

$$\chi_k = -2\delta (e-f) f^{-1} P_k(0) P_n(0) P(0) \tag{36}$$

and

$$xP_n''(x) + \left[2\delta - f(x+f)^{-1}\right]P_n'(x) + \left[n + \frac{\sigma_n - n}{x+e}\right]P_n(x) = \frac{1}{x+e}R_n(x)\,, \quad (37)$$

where $R_n(x) = \pi_{n-1}$.

P r o o f.

a) Let

$$B_n(x) = P^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[x(x+e)P_n'(x)P(x)\right], \tag{a}$$

i.e. according to (4)

$$B_n(x) = P^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left\{x(x+e)P_n'(x)\left[(x+e)^{2\varepsilon}(x+f)^{2\delta}\right]\mathrm{e}^{-x}\right\}.$$

Then from the previous relation we get

$$\begin{aligned} B_n(x) &= x(x+e)P_n''(x) + \big[-x(x+e) + (x+e) + x \\ &\quad + x(x+e)2\varepsilon(x+e)^{-1} + 2\delta x(x+e)(x+f)^{-1}\big]P_n'(x) \\ &= x(x+e)P_n''(x) \\ &\quad + \left[-x^2 + (2+2\varepsilon-e)x + 2\delta x(x+e)(x+f)^{-1} + e\right]P_n'(x)\,. \end{aligned} \tag{b}$$

At first we take the expression $x(x+e)(x+f)^{-1}$ which can be written in the form

$$\begin{aligned} x(x+e)(x+f)^{-1} &= \left[(x+f)(x+e) - f(x+e)\right](x+f)^{-1} \\ &= x+e-f(x+e)(x+f)^{-1} \\ &= x+e-f\left[(x+f)+(e-f)\right](x+f)^{-1} \\ &= x+e-f-f(e-f)(x+f)^{-1}\,. \end{aligned} \tag{c}$$

Substituting (c) into (b) we get

$$\begin{aligned} B_n(x) &= x(x+e)P_n''(x) + \big\{-x^2 + (2+2\varepsilon-e)x \\ &\quad + 2\delta\left[x+e-f-f(e-f)(x+f)^{-1}\right] + e\big\}P_n'(x) \\ &= x(x+e)P_n''(x) + \big\{-x^2 + (2+2\varepsilon+2\delta-e)x \\ &\quad + e + 2\delta(e-f) + 2\delta f(e-f)(x+f)^{-1}\big\}P_n'(x)\,. \end{aligned} \tag{d}$$

Because according to (30) and (31) there is

$$x(x+e)P_n''(x) = n(n-1)P_n(x) + \pi_{n-1}\,, \tag{e}$$

$$-x^2P_n'(x) = \left(-nx + r_1^{(n)}\right)P_n(x) + \pi_{n-1}\,, \tag{f}$$

$$xP_n'(x) = nP_n(x) + \pi_{n-1}\,, \tag{g}$$

then from (d) under notation

$$C_n(x) = 2\delta f(e-f)(x+f)^{-1}P'_n(x) \tag{h}$$

we have

$$B_n(x) - C_n(x) + \left[-n(n-1) + nx - r_1^{(n)} - (2 + 2\varepsilon + 2\delta - e)n\right]P_n(x) = \pi_{n-i}\,.$$

From there by using (21) (because $2\alpha + 2\beta = 2\varepsilon + 2\delta$)

$$B_n(x) - C_n(x) + \left[n(x+e) - n + \sigma_n\right]P_n(x) = \pi_{n-i} = A_n(x)\,. \tag{i}$$

According to (12) and (13)

$$A_n(x) = \sum_{k=0}^{n-1} \chi_k P_k(x)\,, \tag{j}$$

where

$$\chi_k = \int_I A_n(x)P_k(x)P(x)\,\mathrm{d}x\,, \tag{k}$$

so by using (i)

$$\chi_k = \int_I [B_n(x) - C_n(x)]P_k(x)P(x)\,\mathrm{d}x + \int_I [nx + en - n + \sigma_n]P_n(x)P_k(x)P(x)\,\mathrm{d}x\,. \tag{l}$$

Using (a) for every k

$$\begin{aligned}
&\int_I B_n(x)P_k(x)P(x)\,\mathrm{d}x \\
&= \int_I P_k(x)\,\mathrm{d}\left[x(x+e)P'_n(x)P(x)\right] \\
&= \left[P_k(x)x(x+e)P'_n(x)P(x)\right]_0^\infty - \int_I P'_k(x)P'_n(x)x(x+e)P(x)\,\mathrm{d}x \\
&= \int_I P_n(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[x(x+e)P'_k(x)P(x)\right]\,\mathrm{d}x \\
&= \int_I P_n(x)B_k(x)P(x)\,\mathrm{d}x\,,
\end{aligned}$$

so

$$\int_I B_n(x)P_k(x)P(x)\,\mathrm{d}x = \int_I P_n(x)B_k(x)P(x)\,\mathrm{d}x\,. \tag{m}$$

Because according to (i)

$$A_k(x) = B_k(x) - C_k(x) + kxP_k(x) + \pi_k(x) = \pi_{k-1}\,,$$

from there

$$B_k(x) = C_k(x) - kxP_k(x) + \pi_k(x)\,. \tag{n}$$

Then from (m) using (n)

$$\int_I B_n(x)P_k(x)\,\mathrm{d}x = \int_I P_n(x)C_k(x)P(x)\,\mathrm{d}x - \int_I P_n(x)\big[kxP_k(x) + \pi_k\big]P(x)\,\mathrm{d}x\,. \tag{o}$$

For $k < n-1$ from (l) by using (o) and (5)

$$\chi_k = \int_I \big[P_n(x)C_k(x) - C_n(x)P_k(x)\big]P(x)\,\mathrm{d}x\,. \tag{p}$$

For $k = n-1$ from (l) using (o), (5) and (28a)

$$\chi_{n-1} = \int_I P_n(x)C_{n-1}(x)P(x)\,\mathrm{d}x - (n-1)q_n + q_n n - \int_I P_{n-1}(x)C_n(x)P(x)\,\mathrm{d}x\,,$$

and thus

$$\chi_{n-1} = \int_I \big[P_n(x)C_{n-1}(x) - C_n(x)P_{n-1}(x)\big]P(x)\,\mathrm{d}x + q_n\,. \tag{r}$$

Let us denote in (h)

$$K = 2\delta(e-f)f\,. \tag{s}$$

Then from (p) using (h) after integration by parts we have

$$\begin{aligned}
\chi_k &= K\int_I (x+f)^{-1}\big[P_k'(x)P_n(x) - P_n'(x)P_k(x)\big]P(x)\,\mathrm{d}x\\
&= K\int_I (x+f)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{P_k(x)}{P_n(x)}\right]P_n^2(x)P(x)\,\mathrm{d}x\\
&= -Kf^{-1}P_k(0)P_n(0)P(0) + K\int_I (x+f)^{-2}P_k(x)P_n(x)P(x)\,\mathrm{d}x\\
&\qquad + K\int_I (x+f)^{-1}P_k(x)P_n(x)P(x)\,\mathrm{d}x\\
&\qquad + K\int_I (x+f)^{-1}\big[2\varepsilon(x+a)^{-1} + 2\delta(x-b)^{-1}\big]P_k(x)P_n(x)P(x)\,\mathrm{d}x\\
&\qquad - 2K\int_I (x+f)^{-1}P_k(x)P_n'(x)P(x)\,\mathrm{d}x\,.
\end{aligned} \tag{t}$$

The integrals on the right-hand side of (t) equal to zero according to (5) besides the last one.

Substituting the last nonzero integral into (j) and using (10) we have

$$\sum_{k=0}^{n-1} P_k(x) \int_I (t+f)^{-1} P_k(t) P_n'(t) P(t)\ \mathrm{d}t = \int_I (t+f)^{-1} P_n'(t) P_{n-1}(x,t) P(t)\ \mathrm{d}t = 0\,,$$

because $P_{n-1}(x,t) = \pi_{n-1}$ at t and $P_{n-1}(x,t)(t+f)^{-1} = \pi_{n-2}$.

Hence for $k < n-1$ there is

$$\chi_k = -K f^{-1} P_k(0) P_n(0) P(0)\,, \qquad f > 0\,.$$

Substituting (24) into (35) and dividing by positive expression $(x+e)$, we get (37). □

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Received March 12, 1996
Revised October 10, 1996

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