Stanislav Jendroľ; Ernest Jucovič

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GENERALIZATION OF A THEOREM BY V. EBERHARD

STANISLAV JENDROĽ—ERNEST JUCOVIČ

1. Introduction

Let a cell-complex M decompose the closed connected orientable 2-manifold T_g of genus g. From Euler's formula $v(M)-e(M)+f(M) = 2(1-g)$ ($v(M)$ or $e(M)$ or $f(M)$ denote the number of 0- or 1- or 2-cells of M, respectively). Then there follows

$$\sum_{k\geq 3}(6-k)p_k(M)+2\sum_{k\geq 3}(3-k)v_k(M)=12(1-g), \tag{1}$$

where $p_k(M)$ or $v_k(M)$ denotes the number of 2-cells of M with k 0-cells on the boundary (*k-gons*, *k*-gonal faces) or the number of 0-cells of M belonging to the boundary of k 2-cells (*k-valent vertices*), respectively. Obviously there holds

$$\sum_{k\geq 3}k\cdot p_k(M)=\sum_{k\geq 3}k\cdot v_k(M)=2e(M).$$

However, there are pairs of sequences $p=(p_k|k\geq 3)$, $v=(v_k|k\geq 3)$ of non-negative integers satisfying the condition

$$\sum_{k\geq 3}(6-k)p_k+2\sum_{k\geq 3}(3-k)v_k=12(1-g) \tag{2}$$

with some g and $\sum_{k\geq 3}k\cdot p_k=\sum_{k\geq 3}k\cdot v_k\equiv 0 \pmod 2$ for which there exists no cell-decomposition M of T_g with $p_k(M)=p_k$, $v_k(M)=v_k$ for all k. The equality (2) does not restrict in any way the numbers p_6, v_3. This has led to the following question: Given sequences of non-negative integers $p=(p_k|k\geq 3,\ k\neq 6)$, $v=(v_k|k>3)$ satisfying (2) with some g, do there exist non-negative integers p_6 and v_3 and a cell-decomposition M of T_g such that $p_k(M)=p_k$, and $v_k(M)=v_k$ for all k? (If so, M is called a realization of (p, v) on T_g, the pair of sequences (p, v) is called *realizable* on T_g.)

For $g=0$ and $v=(0, 0, 0, \dots)$ an answer was given already in 1891 by

Eberhard [2]. B. Grünbaum renewed the interest in Eberhard's Theorem, gave a clear proof in [4] and some ramifications and analogues of it in [5], [6], and posed the above problem for all orientable 2-manifolds. (For another proof of Eberhard's theorem see Jendroľ [8].) For $g=0$ and $v \neq (0, 0, 0, \dots)$ a solution is contained in Jendroľ—Jucovič [9], for $g=1$ and $v=(0, 0, 0, \dots)$ in Jendroľ—Jucovič [10]. (For analogues of and relatives to Eberhard's Theorem see also Barnette—Jucovič—Trenkler [1], Fisher [3], Grünbaum—Zaks [7], Jucovič [11], Jucovič—Trenkler [12, 13], Trenkler [14], Zaks [15, 16, 17].)

The present paper presents a solution of the above problem for all closed orientable 2-manifolds. It is contained in the

Theorem. *A pair of sequences of non-negative integers* $p=(p_k \mid k \geq 3,\ k \neq 6)$, $v=(v_k \mid k > 3)$ *is realizable on the closed orientable 2-manifold* T_g *of genus* g *if and only if*

a) $\sum_{k \geq 3}(6-k)p_k + 2\sum_{k \geq 3}(3-k)v_k = 12(1-g)$, and

b) *for* $g=0$ *the following conditions do not hold simultaneously:* $\Sigma p_k = 0$ *for* k *odd and* $\Sigma v_k = 1$ *for* $k \not\equiv 0 \pmod 3$, *and*

c) *for* $g=1$ *it differs from the pair* $p=(0, 0, 1, 1, 0, \dots)$, $v=(0, 0, 0, \dots)$.

The proof of the Theorem is rather extensive. It is contained in the next three sections. Sections 2 and 3 have a preparatory character.

In Section 2 a general outline of the construction realizing a given pair of sequences on a manifold with prescribed genus is given, and fundamental cell--aggregates allowing to create faces and vertices required are presented. Furthermore, in this Section a procedure for increasing the genus of a decomposed 2-manifold and a certain transformation of cell-decompositions of 2-manifolds are described.

In Section 3 the methods for inserting faces and vertices with "more" edges into the fundamental cell-aggregates described in Section 2 are shown in details.

The actual proof of Theorem is contained in Section 4. The individual steps of construction described in Sections 2 and 3 are here aligned to show procedures for realizing the prescribed pairs of sequences on the prescribed manifolds. Different cases of pairs of sequences and in particular different possibilities of numbers of faces with ≤ 5 edges must be taken into consideration.

2. Outline of the proof, fundamental constructive elements

2.1. The general procedure of constructing the cellcomplex realizing a pair of sequences p, v prescribed on a 2-manifold required is as follows: First a certain planar or toroidal map is constructed containing some of the required i-gons, $i \geq 7$, or j-valent vertices, $j \geqslant 4$, and certain cell-aggregates *("configurations")* in it are employed for constructing all the required j-valent vertices and i-gons, $j \geqslant 4$, $i \geqslant 7$. These configurations form a set C which is self-reproducing in the following sense: We can increase the valency of a vertex or the number of edges of a face belonging to any member of C in the map and to get in the map obtained again a member of C. This makes it possible to proceed with the construction, i.e. to create other vertices and faces required. In the course of creating the faces and vertices mentioned quadrangles arise as well, and they are grouped into triples (called *plugs* in the sequel) (Fig. 5a). The plugs in the concluding stage of the construction are arranged in two ways: g or $g-1$ pairs of them — depending on whether the starting map is planar or toroidal — are joined by handles to get manifolds of the required genus, and the quadrangles in the remaining plugs are transformed into the k-gons required, $3 \leqslant k \leqslant 5$.

2.2. The face-aggregate in Fig. 1a (or its mirror image) or 2a or 3a, called configuration A_m or B_m or C_m (conf A_m, conf B_m, conf C_m in the sequel) consists of an m-gon, $m \geqslant 6$, and two hexagons and one or two or three quadrangles, respectively. (We will omit the letter m in the symbols if it is not relevant, and write simply conf A etc.)

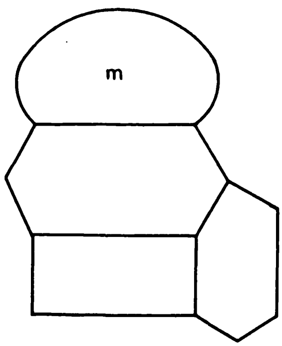

Fig. 1a

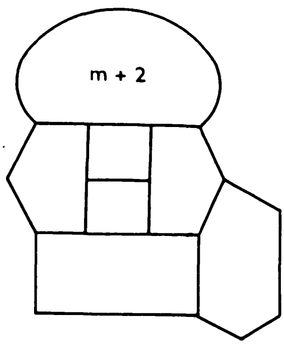

Fig. 1b

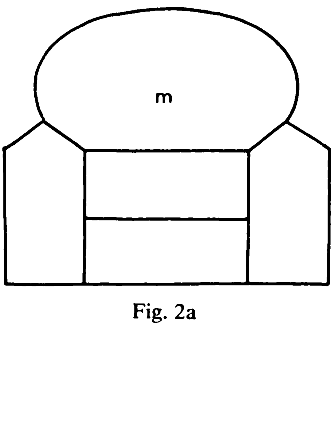

Fig. 2a

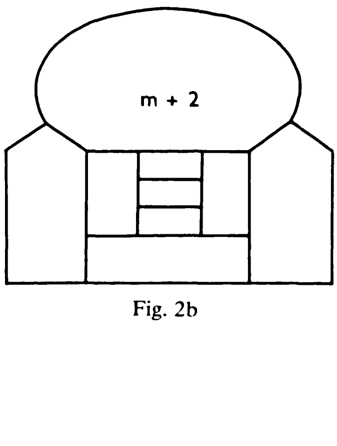

Fig. 2b

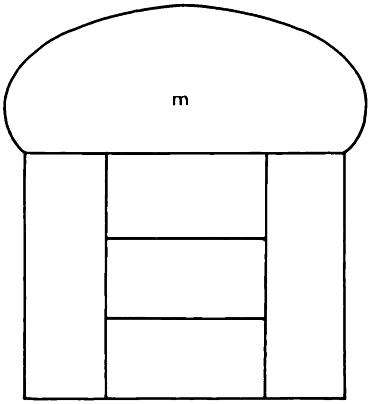

Fig. 3a

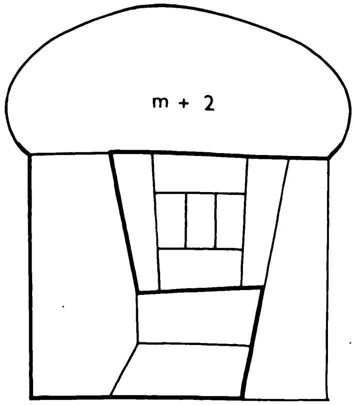

Fig. 3b

Increasing the number of edges of the m-gon in conf A_m is performed in such a way that new edges are inserted into its "middle" hexagon and two of its edges are broken into three; each is marked in Fig. 1b. We get a conf B_{m+2} and a conf B_6 (considering the "lower" hexagon) having some faces in common. The conf B_{m+2} obtained works in the subsequent construction if the number of edges of the $(m+2)$-gon is to be increased. If not, the conf B_6 is used for creating other required cells. — Analogously from conf B_m conf C_{m+2} and conf C_6 in Fig. 2b are constructed. Conf A_{m+2} and conf A_6 are obtained equally from conf C_m — Fig. 3b. The triple of quadrangles we get (see also Fig. 5a) is the plug; its precise role will be described later.

Having an m-gon in conf A_m and desiring to form an i-gon, $i \geq m+6$ from it, we construct succesively a conf B_{m+2}, conf C_{m+4} and conf A_{m+6}, and so on. This transition from conf A_m to conf A_{m+6} increasing the number of edges of a polygon in conf A by six and forming one plug will be called an *A-step* in the sequel. Analogously, increasing the number of edges by six in conf B or conf C and forming a plug is called a *B-step* or a *C-step*, respectively.

For constructing vertices of the required valencies ≥ 4 the face-aggregate in Fig. 4a (or its mirror image) with the designated vertex X of valency x, called conf V_x, is employed. To increase the valency of the designated vertex in conf V_x by three, edges are inserted as drawn in Fig. 4b. The result is a conf V_{x+3} but a plug occurs in the map, as well. This increasing of the valency of a vertex and creating a plug will be called a *V-step* in the sequel.

Some remarks should be added here: α) All the conf A, conf B, conf C appearing in the map in the course of the construction to be employed for creating the required faces and vertices have all their vertices trivalent. β) The plugs appearing in a map are used not only for forming the handles and i-gons, $3 \leq i \leq 5$,

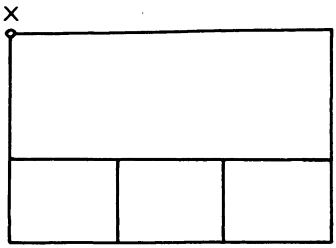

Fig. 4a

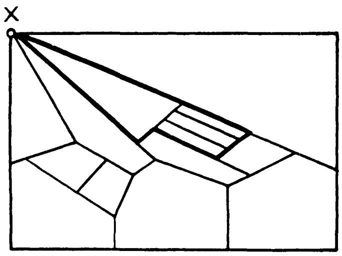

Fig. 4b

as will be described below (see Section 2.3) but from a plug a conf C_6 can be formed. This transformation is shown in Figs. 5a, 5b, and will be employed in realizing some pairs of sequences p, v. (See Sections 3.5b, 3.8b, 4.1.)

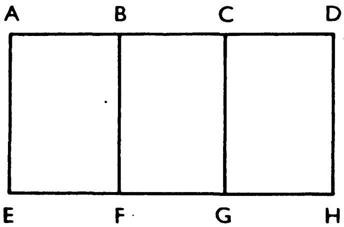

Fig. 5a

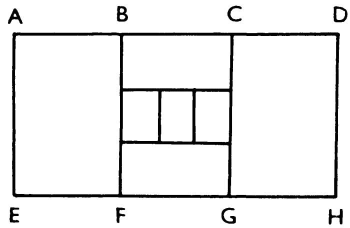

Fig. 5b

2.3. Now we describe the construction of a handle employing two plugs. Let us have two disjoint plugs P and P' with vertices A, B, C, D, E, F, G, H or $A', B', C', D', E', F', G', H'$, respectively, in a cell-decomposition R of a 2-manifold T_g of genus g. A cylindrical map Q (Fig. 6a) containing six hexagons and four quadrangles is added to R (Fig. 6b). Equally marked vertices in Q and in the plugs P, P' are identified. Only the edges BF, CG, $B'F'$, $C'G'$ are removed. Adding a handle to T_g, i.e. creating T_{g+1} has been performed. In the cell-complex on T_g this has caused the replacing of six quadrangles (i.e. the deleting of P, P') by hexagons only (i.e. decreasing the number $\sum_{k\geqslant 3}(6-k)p_k(R)+2\sum_{k\geqslant 3}(3-k)v_k(R)$ by 12); none of the i-gons, $i\geqslant 7$, and j-valent vertices, $j>3$ have changed their types.

2.4. A very useful transformation of a map M into M', called the *replacing of edges by hexagons,* will now be described. (See Grünbaum [4, p. 263].) In this transformation every edge of M is replaced by a hexagon so that an adjacent pair of faces k-gon K — l-gon L in M is replaced by the pair k-gon K' — l-gon L' in M' which are separated by a hexagon. The vertices of the faces K' and L' become trivalent while no b-valent vertex of M changes its place, and is incident with b hexagons in M' (see Fig. 7 where the initial map M is drawn in dashed lines).

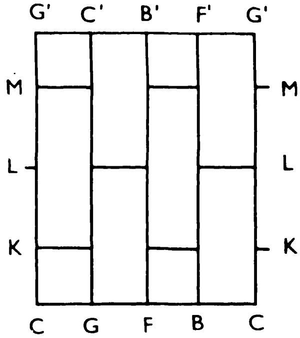

Fig. 6a

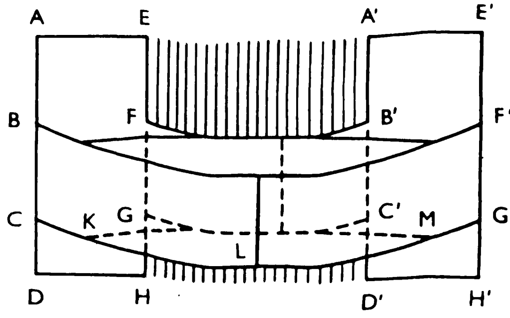

Fig. 6b

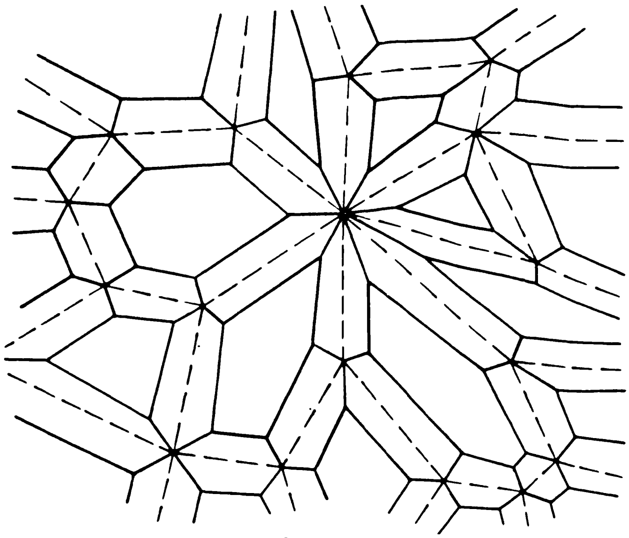
Fig. 7

3. Procedures of construction of i-gons, $i \geqslant 7$, and j-valent vertices, $j \geqslant 4$

As already mentioned, the required i-gons, $i \geqslant 7$, and j-valent vertices, $j \geqslant 4$, will be formed from conf A_6, conf B_6, and conf C_6, if not contained in the initial map. In this Section these procedures are described. $3r$-valent vertices and even-gonal faces are constructed one by one, $(3r+1)$-, $(3r+2)$-valent vertices and odd-gonal faces are constructed in pairs. (In chapter 3, r, s, t, w are positive integers.) Of course the pairs of sequences (p, v) can be such that an odd number of vertices with valencies $\not\equiv 0 \pmod 3$ is needed; this fact is taken into consideration in the choice of the fundamental (planar or toroidal) map (see Section 4). In Section 3.9 we settle the case when an odd number of odd-gonal faces (with $\geqslant 7$ vertices) is required.

The following is an important rule to be observed in the successive forming of the vertices and faces mentioned: If in some step of the construction a conf A_6 or conf B_6 and a conf C_6 appears in the map, for constructing other faces and vertices required, this conf A_6 or conf B_6, respectively, is employed, and from the conf C_6 the triple of quadrangles is employed as a plug. It should be remarked that in any step of the construction at most one of the conf A_6 or conf B_6 appears in the map. If neither conf A_6 nor conf B_6 but some conf C_6 appear, only one from among the conf C_6 is employed for creating other vertices and faces required, and from the remaining conf C_6 only the plugs are used. — If no conf A_6, conf B_6 appears in the map, from one plug a conf C_6 is formed as described in Section 2.2 (remark β)) to continue the construction.

3.1. A $3r$*-valent vertex,* $r \geqslant 2$, is constructed

a) in conf A_6 by inserting edges as drawn in Fig. 8a. We get a 6-valent vertex in conf V, and a conf A_6. If $r > 2$, we use $(r-2)$ times the V-step to get the r-valent vertex and $(r-1)$ plugs. Other vertices and faces required are constructed in using the conf A_6 mentioned.

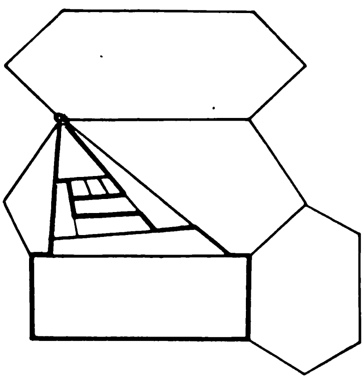

Fig. 8a

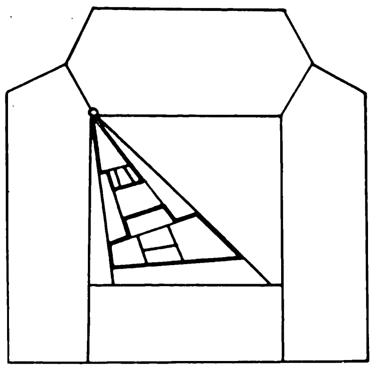

Fig. 8b

b) in conf B_6: Edges are added as drawn in Fig. 8b. Again a 6-valent vertex in conf V and a conf B_6 are obtained. Further we proceed as in the preceding case a).

c) in conf C_6: Note that conf C_6 contains a conf V_3 (and two hexagons which are retained in the map). After employing the V-step $(r-1)$ times we get, besides the r-valent vertex required, r plugs from which one can be changed into a conf C_6 so that the construction may proceed.

3.2. *A pair* $(3r+1)$*-valent and a* $(3s+1)$*-valent vertex,* $r, s \geqslant 1$, are equally inserted in all the conf A_6, conf B_6, conf C_6 employing the circumstance that all these configurations contain an adjacent pair of faces hexagonquadrangle with all the vertices trivalent. (In conf A_6 the "middle" hexagon is used, in the other cases the "upper" one.)

If $r = s = 1$ edges are inserted into the hexagon as drawn in Fig. 9a, two 4-valent vertices appear. If the initial configuration was conf A_6, we get also conf C_6, if it was conf B_6, we get also conf A_6 (not shown in the drawings) and a plug, if it was conf C_6, then conf B_6 and a plug appear. (Cf. Fig. 9a and Figs. 1a, 2a; 3a.)

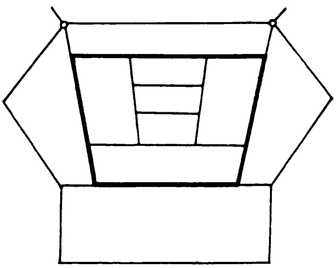

Fig. 9a

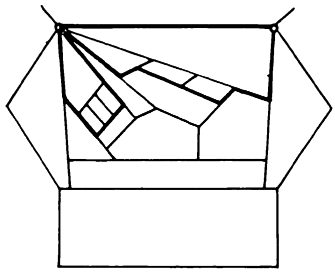

Fig. 9b

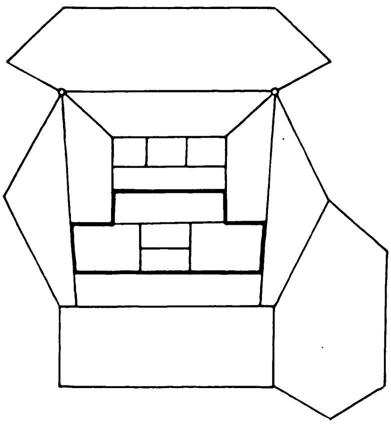

Fig. 10a

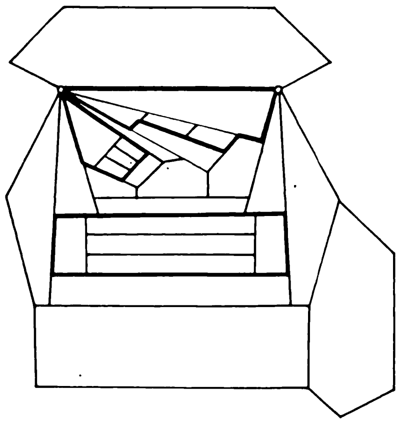

Fig. 10b

For $r \geqslant 2$ or $s \geqslant 2$ edges are inserted into the hexagon as drawn in Fig. 9b. We get a conf V_7 and a conf V_4. The V-step is employed $(r-2)$ or $(s-2)$ times on the first vertex and $(s-1)$ or $(r-1)$ times on the second one. We get, besides the required vertices, some plugs and at most one from among conf A_6, conf B_6 to be used for creating other faces and vertices required, according to the rule mentioned at the beginning of Section 3.

3.3. *The insertion of a* $(3r+2)$- *and a* $(3s+2)$-*valent vertex,* $r, s \geqslant 1$

a) into conf A_6 is performed as follows. If $r=s=1$, edges are added as in Fig. 10a; we get two 5-valent vertices, a plug and a conf B_6. For $r \geqslant 2$ or $s \geqslant 2$ the insertion is drawn in Fig. 10b; we get a conf V_8, a conf V_5 and conf B_6. The V-step is performed the appropriate number of times on the conf V's; the required vertices and $(r+s-1)$ plugs are obtained.

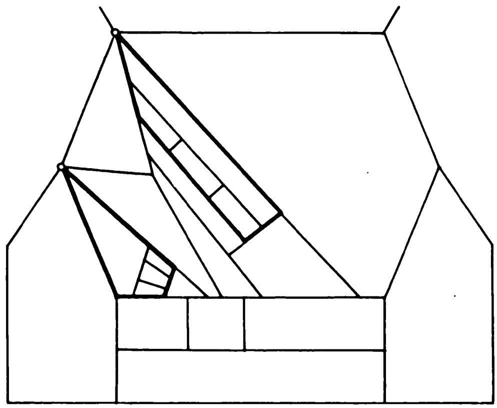

Fig. 11

b) into conf B_6 is performed as drawn in Fig. 11. We get two conf V_5. The V-step is performed with the first $(r-1)$ times and with the second one $(s-1)$ times. We get, besides the required vertices some plugs.

c) into conf C_6 is performed analogically as into conf B_6 because this last one is a part of a conf C_6. We get, besides the required vertices, a conf A_6 and plugs. (Cf. Figs. 11 and 3a).

3.4. *Even-gonal faces*

A $6t$-gon, $t \geqslant 2$, is obtained from conf A_6 or conf B_6 or conf C_6 so that $(t-1)$ times the A-step or the B-step or the C-step, respectively, is performed in it. We get, besides the polygon required, some plugs and a conf A_6 or conf B_6 or conf C_6, respectively.

To construct a $(6t+2)$- or $(6t+4)$-gon first an octagon or decagon from the hexagon in the appropriate configuration is constructed and then the A-step or the B-step or the C-step is employed. (See Section 2.2.)

3.5. *The procedure of inserting a* $(6t+1)$-*gon and a* $(6w+1)$-*gon,* $t, w \geqslant 1$,

a) into conf A_6: If $t=1$ or $w=1$, the faces are changed as drawn in Fig. 12a. We get two 7-gons, one of them in conf B_7. With this configuration the B-step is performed $(t-1)$ or $(w-1)$-times — depending on whether $w=1$ or $t=1$.

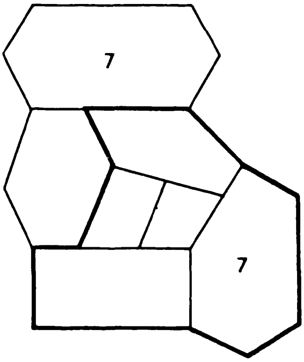

Fig. 12a

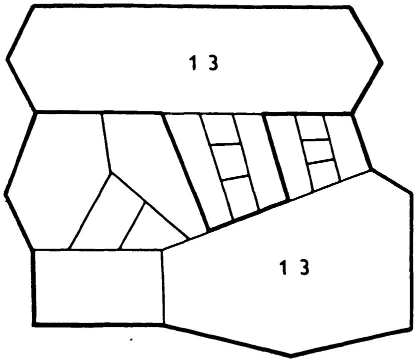

Fig. 12b

If $t \geqslant 2$, $w \geqslant 2$, the conf A_6 considered is arranged as in Fig. 12b. We get two conf C_{13}, a conf B_6 and plugs. The C-step is used $(t-2)$ and $(w-2)$ times to get the required polygons, and conf B_6 is used to construct the remaining faces and vertices.

b) into conf B_6: If $t=1$ or $w=1$, the configuration is changed as drawn in Fig. 13a. One of the two 7-gons obtained is contained in conf C_7. The C-step is performed $(t-1)$ or $(w-1)$ times; plugs and a conf C_6 result. It should be noted here that if $t=w=1$, a conf C_6 is formed from the plug in Fig. 13a (see Section 2.3, Fig. 5a, b) and is used for creating other faces and vertices required.

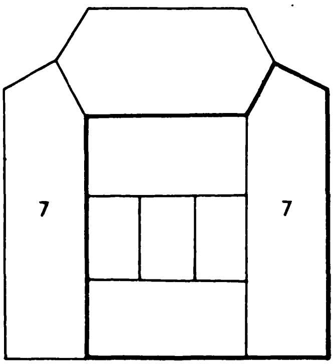

Fig. 13a

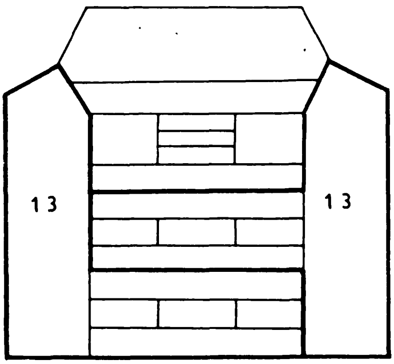

Fig. 13b

If $t \geqslant 2$ and $w \geqslant 2$, new edges are added as in Fig. 13b. We get two conf C_{13} and a conf C_6. The construction continues as in the foregoing cases.

c) into conf C_6: If it is subdivided as drawn in Fig. 14, we get two 7-gons, one in conf A_7 and the second in conf C_7. After performing the A-step $(t-1)$ times, the

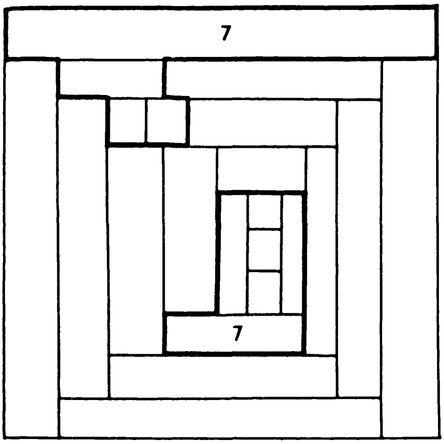

Fig. 14

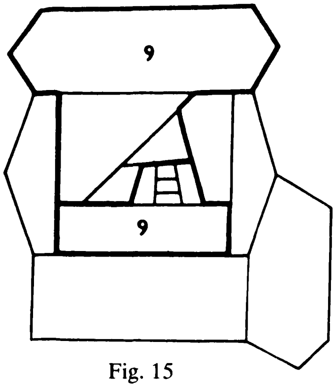

Fig. 15

C-step $(w-1)$ times the required two faces occur in the map, and besides plugs as well as a conf A_6.

3.6. *The insertion of the pair* $(6t+1)$*-gon — * $(6w+3)$*-gon and of the pair* $(6t+1)$*-gon — * $(6w+5)$*-gon*

a) into conf A_6. If $t=1$, we start again with the face-aggregate in Fig. 12a containing two 7-gons. From that belonging to conf B_7 the desired $(6w+3)$-gon or $(6w+5)$-gon is created by a successive increase of the number of its edges by two. If $t \geqslant 2$, the considered conf A_6 is changed as drawn in Fig. 15. There we have a conf A_9 and a conf C_9. As described above, the conf A_9 is changed into conf C_{13}, and the C-step is performed then $(t-2)$ times. The C-step with the conf C_9 mentioned is performed $(w-1)$ times, too. Thus the $(6t+1)$-gon and the $(6w+3)$-gon required, and plugs and two conf C_6 are constructed. In the same way we proceed in constructing the pair $(6t+1)$-gon — $(6w+5)$-gon. However, conf A_6 instead of one conf C_6 appears.

b) into conf B_6: If $t=1$, we start again with the face-aggregate in Fig. 13a. From the 7-gon belonging to the conf C_7 the required $(6w+3)$-gon or $(6w+5)$-gon is constructed as described in some previous cases.

If $t \geqslant 2$, the conf B_6 considered is arranged as in Fig. 16; there we have a conf B_9 and a conf C_9. First, the conf B_9 is changed into conf A_{13} and the A-step is performed $(t-2)$ times. Then, with conf C_9 $(w-1)$ times C-step is performed. We get, besides the required $(6t+1)$- and $(6w+3)$-gon, plugs and a conf A_6.

To construct a $(6t+1)$-gon and a $(6w+5)$-gon we must as follows change the preceding procedure (because if proceeding equally we would not be able in all cases — after creating these two polygons — to proceed the construction): The $(6w+5)$-gon is constructed from conf B_9 in Fig. 16. First, an 11-gon is formed and then the C-step is performed $(w-1)$ times. The $(6t+1)$-gon, $t \geqslant 2$ is formed from the conf C_9; this is changed into conf B_{13} and then the B-step is performed $(t-2)$ times. A conf B_6 appears.

c) into conf C_6: This is arranged as shown in Fig. 14, where we have a conf C_7 and a conf A_7. With conf C_7 the C-step is performed $(t-1)$ times. The conf A_7 is changed into conf B_9 (or conf C_{11}) and then the B-step (C-step) is performed $(w-1)$ times to get the required $(6w+3)$-gon $(6w+5)$-gon. A conf B_6 (conf C_6) appears in the map as well.

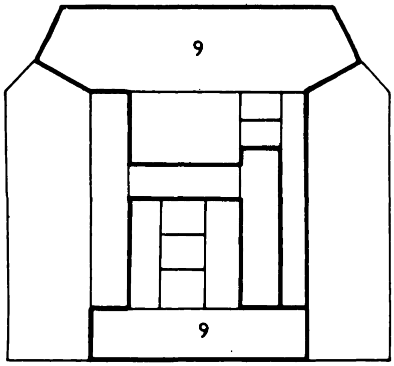

Fig. 16

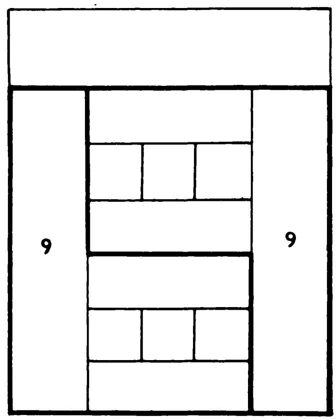

Fig. 17

3.7. *The construction of the pair* $(6t+3)$-*gon*, $(6w+3)$-*gon*, $(t, w \geqslant 1)$, *and of the pair* $(6t+3)$-*gon*, $(6w+5)$-*gon* begins with the arranged conf A_9 or conf B_9 or conf C_9 in Fig. 15 or 16 or 17, respectively. In all the cases one 9-gon occurs in a conf C_9; from it the $(6t+3)$-gon is constructed by C-steps. The second 9-gon appears in a conf A_9 or conf B_9 or conf C_9; this 9-gon is changed by A-steps or B-steps or C-steps into the $(6w+3)$-gon. The forming of the $(6w+5)$-gon from the $(6w+3)$-gon proceeds as described in Section 2.2.

3.8. In the *construction of the pair* $(6t+5)$-*gon* — $(6w+5)$-*gon*, t, $w \geqslant 1$ the procedure from Section 3.7 must be slightly changed.

a) The insertion into conf A_6: First, by procedures already described from this conf A_6, a conf C_{6t+4} is formed. Next new edges are added in this configuration as drawn in Fig. 18, to get the desired $(6t+5)$-gon, and an 11-gon in conf C_{11} from which by applying the C-step $(w-1)$ times the $(6w+5)$-gon arises. From one of the arisen plugs a conf C_6 is created (see Section 2.2 remark β) and Figs. 5a, 5b) to be employed for forming other faces and vertices required.

b) The insertion into conf B_6: New edges are added as drawn in Fig. 16 (cf. Fig. 2a) to get conf B_9 and a conf C_9. From them conf C_{11} and conf A_{11} are formed (see Section 2). To get the polygons required the C-step is performed $(t-1)$ times and the A-step $(w-1)$ times. The conf A_6 is used for forming further faces and vertices.

c) The insertion into conf C_6: It is changed first as drawn in Fig. 14. A conf A_7 and a conf C_7 occur there. From them conf C_{11} and conf B_{11} are formed as described in Section 2. Now the B-step is performed $(t-1)$ times and the C-step $(w-1)$ times, from which the required polygons, plugs and a conf B_6 result. This last is employed for continuing the construction.

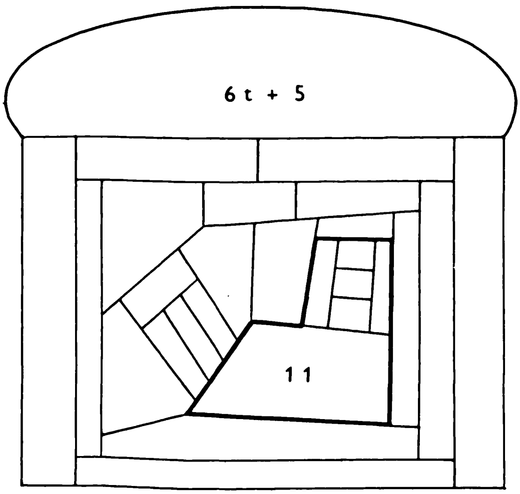

Fig. 18

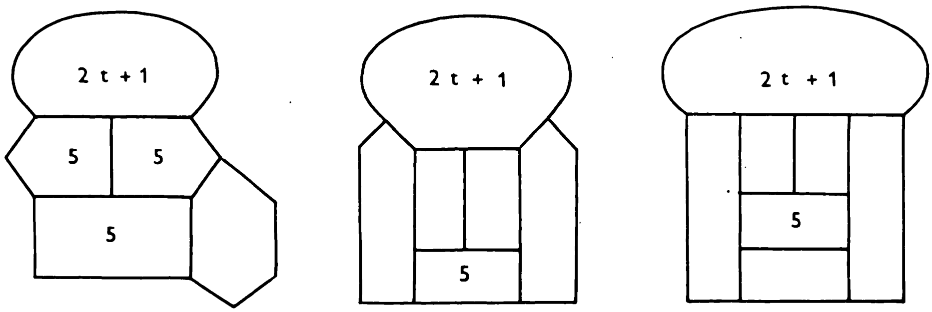

Fig. 19a Fig. 19b Fig. 19c

3.9. If *an odd number of odd-gonal faces* with $\geqslant 7$ edges is required, the last of them, a $(2t+1)$-gon, $t \geqslant 3$, can be inserted into conf A_6 or conf B_6 or conf C_6 as follows: A $2t$-gon is inserted into this configuration by the procedures already described. It is then contained in conf A_{2t} or conf B_{2t} or conf C_{2t}. Fig. 19a or 19b or 19c, respectively, shows how these configurations are changed by adding edges to get a $(2t+1)$-gon, $t \geqslant 3$. — The face-aggregates in Fig. 19 will be called *concluding configurations* in the sequel. Let us remark that this last odd-gonal face is constructed alone, if not stated otherwise, *after* constructing all i-valent vertices, $i \geqslant 4$, and remaining j-gons, $j \geqslant 7$.

4. Proof of Theorem

We will omit the proof of Theorem for $g = 0$ and the proof of non-realizability of the pair of sequences in c) because they are contained in [9, 10]. The realizability of all the remaining pairs of sequences on all manifolds of genus $g \geqslant 1$ will be demonstrated by effective constructions of the appropriate cell-complexes. To cover all pairs of sequences and all manifolds, various procedures of construction and distinguishing between many cases and subcases are needed. The procedure differ, depending on the pairs of sequences to be realized, in the initial maps and in the concluding stages when the k-gons, $k \leqslant 5$ required should be formed from the plugs and from conf A or conf B or one of the concluding configurations in Fig. 19. The i-gons, $i \geqslant 7$, and j-valent, $j \geqslant 4$, vertices are constructed by procedures described in Section 3.

4.1. $3p_3 + 2p_4 + p_5 = 0$:
For $g = 1$ the pair ($p_i = 0$ for $i \neq 6$), ($v_i = 0$ for $i \geqslant 4$) satisfies (2) only and its realization is the well-known hexagonal decomposition of the torus. Therefore decompositions of surfaces of genus $\geqslant 2$ appear in the sequel only.

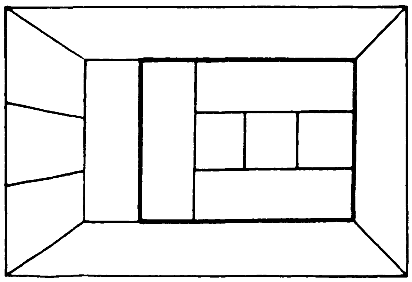

Fig. 20

a) If $\sum_{k \geqslant 1} v_{3k+1} \equiv \sum_{k \geqslant 1} v_{3k+2} \equiv 0 \pmod 2$, the construction starts with the planar map in Fig. 20 containing conf C_6, a plug and hexagons. We construct — beginning with the conf C_6 — all the vertices of valencies $\equiv 0 \pmod 3$ (see Section 3.1), $\left[\frac{\sum_{k \geqslant 1} v_{3k+1}}{2}\right]$

pairs of vertices of valencies $\equiv 1 \pmod 3$ (see Section 3.2), $\left[\frac{\sum_{k\geq 1} v_{3k+2}}{2}\right]$ pairs of vertices of valencies $\equiv 2 \pmod 3$ (see Section 3.3), all the even-gonal faces (see Section 3.4) and $\left[\frac{\sum_{k\geq 3} p_{2k+1}}{2}\right]$ pairs of odd-gonal faces (see Sections 3.5—3.8) required. (Notice that from (2) there follow $\sum_{k\geq 3} kp_k \equiv 0 \pmod 2$ and the evenness of the number of odd-gonal faces required.) After doing so we get a map M on the sphere with $p_i(M) = p_i$ for all $i \geq 7$, $v_j(M) = v_j$ for all $j \geq 4$, $p_3(M) = p_5(M) = 0$. From (1) then there follows $2p_4(M) = 12g$, and as the quadrangles are grouped in plugs, we have exactly $2g$ of them. By the procedure described in Section 2.3 they are joined in pairs to form a realization required of the pair of sequences (p, v) on T_g.

b) Let $\sum_{k\geq 1} v_{3k+1} \equiv 1 \pmod 2$, $\sum_{k\geq 1} v_{3k+2} \equiv 0 \pmod 2$. The construction starts with the toroidal maps in Fig. 21 (equally marked vertices should be identified); if $v_4 \neq 0$, it begins with the map in Fig. 21a containing a 4-valent vertex and a conf A_6; if $v_4 = 0$, it begins with the map in Fig. 21b containing a 7-valent vertex in conf V_7 and conf A_6. Applying an appropriate number of V-steps in the second case and using conf A_6 in both cases for constructing all the remaining vertices and faces required, we get a toroidal map containing all j-valent vertices, $j \geq 4$, and i-gons, $i \geq 7$, hexagons and $2(g-1)$ plugs. The plugs are joined in pairs by handles as in case a).

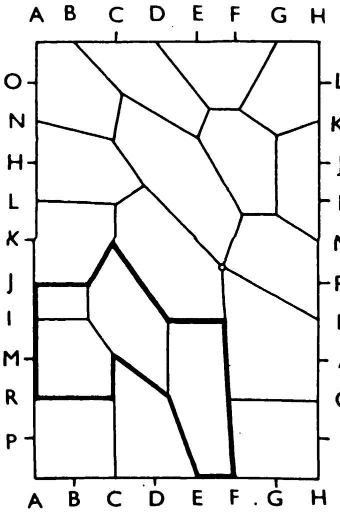

Fig. 21a

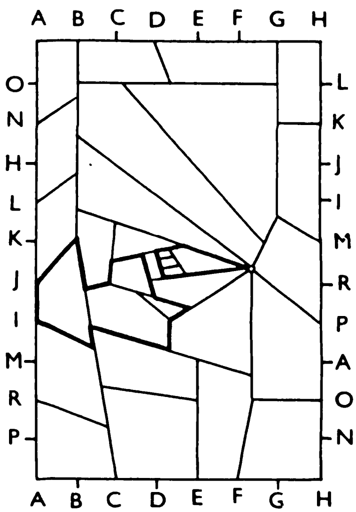

Fig. 21b

c) If $\sum_{k\geqslant 1} v_{3k+1}\equiv 0 \pmod 2$, $\sum_{k\geqslant 1} v_{3k+2}\equiv 1 \pmod 2$, the starting toroidal map is drawn in Fig.22a or Fig. 22b, provided $v_5\neq 0$ or $v_5=0$, respectively. In both cases we have at our disposal a conf B_6 for constructing all the i-gons, $i\geqslant 7$, and the remaining j – valent vertices, $j\geqslant 4$. The valency of the 8-valent vertex in conf V_8 in Fig. 22b is increased by the V-step as needed. Handle forming as in case b) can follow.

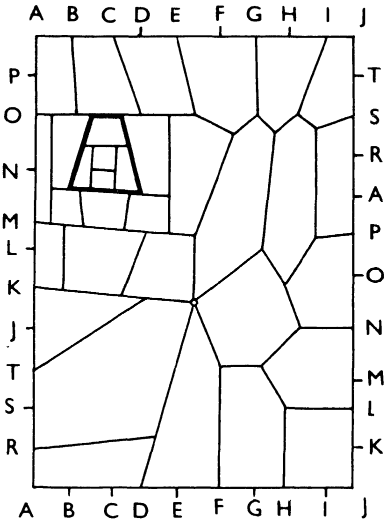

Fig. 22a

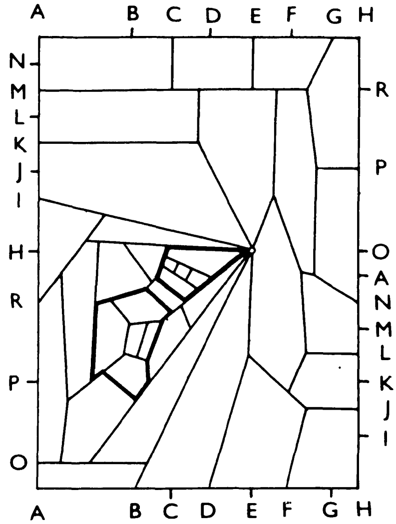

Fig. 22b

d) If $\sum_{k\geqslant 1} v_{3k+1}\equiv \sum_{k\geqslant 1} v_{3k+2}\equiv 1 \pmod 2$, the construction begins with the toroidal map in Fig. 23a or 23b depending on whether $v_5\neq 0$ or $v_5=0$, respectively. In the first case the valency of the 4-valent vertex and in the second valencies of the 4-valent and 8-valent vertices in conf V are increased by the V-step. One of the plugs appearing in conf V after creating the first two vertices required is changed according to the procedure described in remark β) in Section 2.2 (Fig. 5) into conf C_6. This conf C_6 is employed for constructing all the remaining i-valent vertices, $i\geqslant 4$, and j-gons, $j\geqslant 7$ as described in Section 3. Handle forming concludes the construction as in the preceding cases.

4.2. $3p_3+2p_4+p_5=1$, i.e. $p_3=p_4=0$, $p_5=1$.

If we proceeded in this case as in Section 4.1, we would get a concluding configuration as in Fig. 19c and $2g-1$ plugs in case $\sum_{k\geqslant 1} v_{3k+1}\equiv \sum_{k\geqslant 1} v_{3k+2}\equiv 0$ (mod 2), and $2g-3$ plugs in the remaining cases, which does not suffice for forming a T_g. We are unable to change the triple of quadrangles in Fig. 19c so as to form a plug. Therefore the construction must be changed so as to get the unique pentagon in the map, but the quadrangles in constructing the last k-gon, $k \geqslant 7$ are to be grouped in a plug. We use the fact that (2) requires the number of odd-gonal faces to be even, and because $p_5 = 1$, $\sum_{k \geqslant 3} p_{2k+1} \equiv 1 \pmod 2$ holds.

a) $\sum_{k \geqslant 1} v_{3k+1} \equiv \sum_{k \geqslant 1} v_{3k+2} \equiv 0 \pmod 2$.

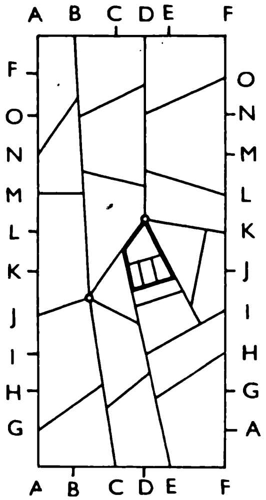

Fig. 23a

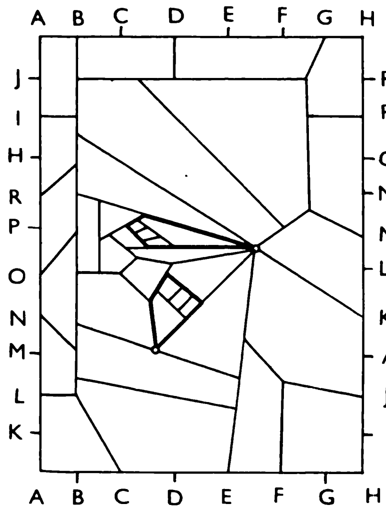

Fig. 23b

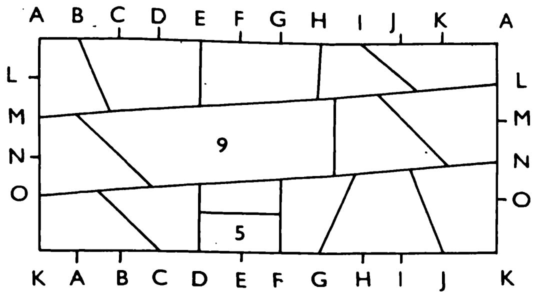

Fig. 24

If $\Sigma p_k \neq 0$ for odd $k \geqslant 9$, we start with the trivalent toroidal map in Fig. 24. It contains a 9-gon, a quadrangle, a pentagon and hexagons. After replacing edges by hexagons (see Section 2.4) we get the 9-gon in a conf A_9, and a conf A_6. This conf A_6 is employed to construct all the remaining i-gons, $i \geqslant 7$, and j-valent vertices, $j \geqslant 4$, required by procedures already described — if the 9-gon in conf A_9

mentioned is needed. The pentagon is retained in the map. The quadrangles are grouped into $(2g-2)$ plugs, so that we can use them in forming $(g-1)$ handles.

If $\Sigma p_k = 0$ for odd $k \geq 9$, then $p_7 \neq 0$.

For $p_7 \geq 3$ we start with the planar map in Fig. 25 containing three 7-gons, two plugs and a conf A_6 which is employed for constructing all the remaining i-gons, $i \geq 7$, and j-valent vertices, $j \geq 4$.

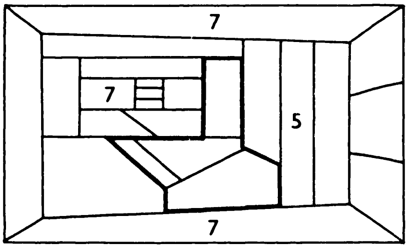

Fig. 25

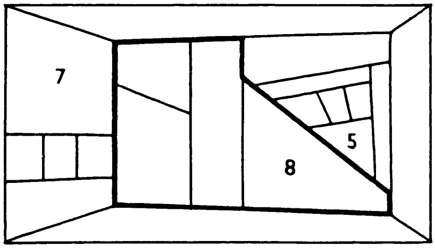

Fig. 26

maps containing one or two vertices of valencies $\equiv 0 \pmod 3$ or $\equiv 1 \pmod 3$ or $\equiv 2 \pmod 3$ in suitable configurations are drawn in Fig. 27a or 27b or 27c, respectively. The construction proceeds as before.

b) For $\sum_{k \geq 1} v_{3k+1} \equiv 1 \pmod 2$, $\sum_{k \geq 1} v_{3k+2} \equiv 0 \pmod 2$ or $\sum_{k \geq 1} v_{3k+1} \equiv 0 \pmod 2$, $\sum_{k \geq 1} v_{3k+2} \equiv 1 \pmod 2$ or $\sum_{k \geq 1} v_{3k+1} \equiv \sum_{k \geq 1} v_{3k+2} \equiv 1 \pmod 2$ the starting planar maps are

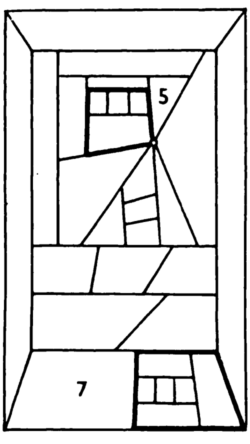

Fig. 27a

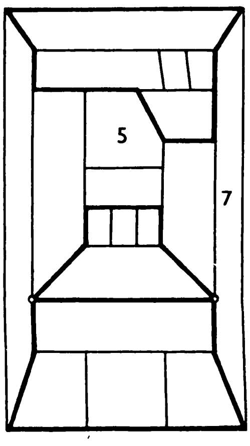

Fig. 27b

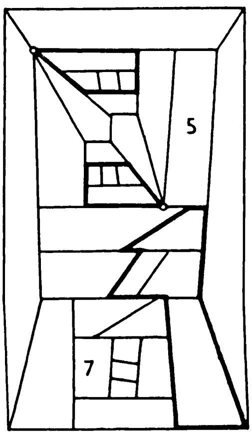

Fig. 27c

For $p_7=1$ and $\sum_{k\geq 4} p_{2k}\neq 0$ the starting planar map containing conf A_8 to be used for creating other faces and vertices required is drawn in Fig. 26.

For $p_7=1$, $\sum_{k\geq 8} p_k=0$ and $\sum_{k\geq 2} v_{3k}\neq 0$ or $\Sigma\, v_{3k+1}\geq 2$ or $\Sigma\, v_{3k+2}\geq 2$ the starting planar

drawn in Fig. 28a or 28b or 28c, respectively. The maps contain a pentagon, a conf A_7 and conf V_4 or a conf B_7 and conf V_5 or a conf C_7, conf V_4 and conf V_5, respectively. Thus from the configurations we have at our disposal in the starting maps all the i-gonal faces, $i\geq 7$, and j-valent vertices, $j\geq 4$, are created as described in Section 3. Besides the pentagon required, $2g$ plugs (and hexagons) appear in the map.

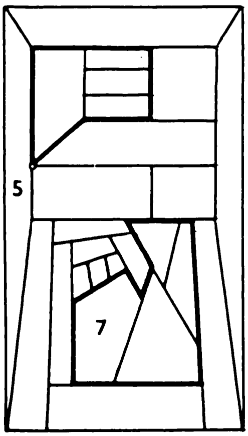

Fig. 28a

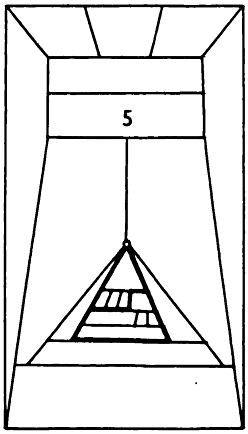

Fig. 28b

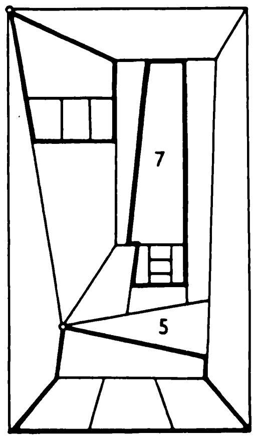

Fig. 28c

The plugs are joined to form the g handles. The construction is finished in this case, too.

4.3. $3p_3+2p_4+p_5\geq 2$.

The construction proceeds in three steps: 1. A decomposition M' of T_g with $v_j(M')=v_j$ for all $j\geq 4$, $p_i(M')=p_i$ for all $i\geq 7$ is formed. 2. M' is changed into such a decomposition M'' that $v_j(M'')=v_j$ for all $j\geq 4$, $p_i(M'')=p_i$ for all $i\geq 7$ and $i=3$ holds. 3. Creating the decomposition required.

The first step

is performed in the same way as described in Section 4.1. The starting map is that in Fig. 20 or 21 or 22 or 23 depending on whether $\sum_{k\geq 1} v_{3k+1}\equiv\sum_{k\geq 1} v_{3k+2}\equiv 0 \pmod 2$ or $\sum_{k\geq 1} v_{3k+1}\equiv 1 \pmod 2$ and $\sum_{k\geq 1} v_{3k+2}\equiv 0 \pmod 2$ or $\sum_{k\geq 1} v_{3k+1}\equiv 0 \pmod 2$ and $\sum_{k\geq 1} v_{3k+2}\equiv 1 \pmod 2$ or $\sum_{k\geq 1} v_{3k+1}\equiv\sum_{k\geq 1} v_{3k+2}\equiv 1 \pmod 2$, respectively. All remaining

j-valent vertices, $j \geq 4$, $\sum_{k \geq 4} p_{2k}$ even-gonal faces and $\left[\frac{\sum_{k \geq 3} p_{2k+1}}{2}\right]$ pairs of odd-gonal faces are constructed as in Section 4.1. While in Section 4.1 always an even number of i-gons with odd $i \geq 7$ is required, here an *odd* number of such faces (and some triangles and/or pentagons) may be needed. If so, the last of such faces are constructed as described in Section 3.9. Another difference between the situation in Section 4.1 and this case is that after joining the g or $g-1$ pairs of plugs by handles we get here on T_g a map M' with $p_i(M') = p_i$ for $i \geq 7$, $v_j(M') = v_j$ for $j \geq 4$ and we have $p_4(M') = p_4'$, $p_5(M') = p_5' \leq 3$, $p_3(M') = 0$ and $2p_4(M') + p_5(M') \geq 2$ (while the maps in Section 4.1 do not contain i-gons, $i \leq 5$). From $2p_4' + p_5' = 12(1-g) + \sum_{i \geq 7}(i-6)p_i + 2\sum_{i \geq 4}(i-3)v_i$ and (2) it follows that $2p_4' + p_5' = 3p_3 + 2p_4 + p_5$. The procedures of construction of M' impose that the p_4' quadrangles and p_5' pentagons are either in plugs or in conf A or in conf B or in the concluding configurations in Fig. 19a, b, c.

The second step:
M' is transformed into M'' such that

$$\begin{aligned}
&p_3'' = p_3(M'') = p_3, \\
&p_4'' = p_4(M'') = p_4 + \left[\frac{p_5}{2}\right], \\
&p_5'' = p_5(M'') = p_5 - 2\left[\frac{p_5}{2}\right] \quad (=0 \text{ or } 1), \\
&\qquad p_i(M'') = p_i \quad \text{for all } i \geq 7, \\
&\qquad v_j(M'') = v_j \quad \text{for all } j \geq 4 \text{ will hold.}
\end{aligned} \tag{3}$$

From the procedures of construction employed in the first step and (2), (3) it follows that

$$u = 2p_4' + p_5' = 3p_3'' + 2p_4'' + p_5'' \equiv i \pmod 6 \tag{4}$$

implies the map M' to admit

$\left[\frac{u}{6}\right]$ plugs in case $i = 0$,

$\left[\frac{u}{6}\right] - 1$ plugs and the configuration in Fig. 19c if $i = 1$,

$\left[\frac{u}{6}\right]$ plugs and conf A_6 in case $i = 2$,

$\left[\frac{u}{6}\right]$ plugs and the configuration in Fig. 19a if $i=3$,

$\left[\frac{u}{6}\right]$ plugs and conf B_6 in case $i=4$,

$\left[\frac{u}{6}\right]$ plugs and the configuration in Fig. 19b if $i=5$.

The following table contains all the residue classes mod 2 or mod 3, to which the numbers p_3'' or p_4'', respectively, can belong and the precise values of p_5'' if satisfying (3), (4):

$$
\begin{aligned}
i=0(\alpha)p_3''&\equiv 0 \pmod 2,\ p_4''\equiv 0 \pmod 3,\ p_5''=0 \text{ or}\\
(\alpha\alpha)p_3''&\equiv 1 \pmod 2,\ p_4''\equiv 1 \pmod 3,\ p_5''=1;\\
i=1(\beta)p_3''&\equiv 0 \pmod 2,\ p_4''\equiv 0 \pmod 3,\ p_5''=1 \text{ or}\\
(\beta\beta)p_3''&\equiv 1 \pmod 2,\ p_4''\equiv 2 \pmod 3,\ p_5''=0;\\
i=2(\gamma)p_3''&\equiv 0 \pmod 2,\ p_4''\equiv 1 \pmod 3,\ p_5''=0 \text{ or}\\
(\gamma\gamma)p_3''&\equiv 1 \pmod 2,\ p_4''\equiv 2 \pmod 3,\ p_5''=1;\\
i=3(\delta)p_3''&\equiv 1 \pmod 2,\ p_4''\equiv 0 \pmod 3,\ p_5''=0 \text{ or}\\
(\delta\delta)p_3''&\equiv 0 \pmod 2,\ p_4''\equiv 1 \pmod 3,\ p_5''=1;\\
i=4(\varepsilon)p_3''&\equiv 0 \pmod 2,\ p_4''\equiv 2 \pmod 3,\ p_5''=0 \text{ or}\\
(\varepsilon\varepsilon)p_3''&\equiv 1 \pmod 2,\ p_4''\equiv 0 \pmod 3,\ p_5''=1;\\
i=5(\varphi)p_3''&\equiv 0 \pmod 2,\ p_4''\equiv 2 \pmod 3,\ p_5''=1 \text{ or}\\
(\varphi\varphi)p_3''&\equiv 1 \pmod 2,\ p_4''\equiv 1 \pmod 3,\ p_5''=0.
\end{aligned}
$$

(In all the cases the data concerning p_3'', p_4'' are imposed by the possible values 0 or 1 of p_5''.)

The transformation of M' into M'' proceeds as follows (it suffices to deal with the k-gons, $k\leqslant 5$):

For $i\neq 1$

from $\left[\frac{p_3''}{2}\right]$ plugs $2\left[\frac{p_3''}{2}\right]$ triangles are constructed as shown in Fig. 29a, in $\left[\frac{p_4''}{3}\right]$ plugs we have $3\left[\frac{p_4''}{3}\right]$ quadrangles and in the case of

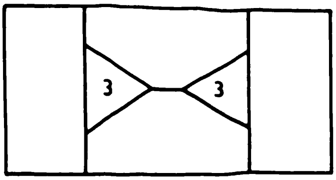

Fig. 29a

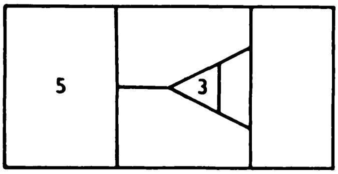

Fig. 29b

$i=0$ we either have finished the construction (case (α) in the Table above) or one more triangle and quadrangle and pentagon are needed ($\alpha\alpha$) — they are obtained in changing the last plug as drawn in Fig. 29b;

$i=2$ (γ) the last quadrangle needed is in conf A_6;

($\gamma\gamma$) one quadrangle needed is in conf A_6, one more quadrangle, triangle and pentagon needed are constructed from the last plug as shown in Fig. 29b;

$i=3$ Fig. 30a or 30b shows the change of the configuration in Fig. 19a so as to get a triangle (δ) or a quadrangle and pentagon ($\delta\delta$), respectively;

$i=4$ (ε) the two quadrangles are in conf B_6;

($\varepsilon\varepsilon$) Fig. 31a shows the change of conf B_6 into a triangle and pentagon needed;

$i=5$ (φ) the two quadrangles and one pentagon are contained in the concluding configuration in Fig. 19b;

($\varphi\varphi$) Fig. 31b shows the change of the face-aggregate in Fig. 19b into one triangle and a quadrangle.

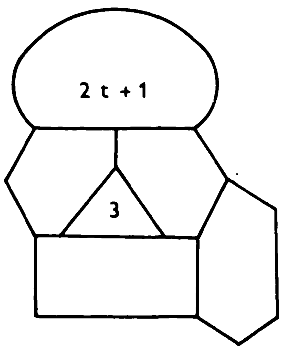

Fig. 30a

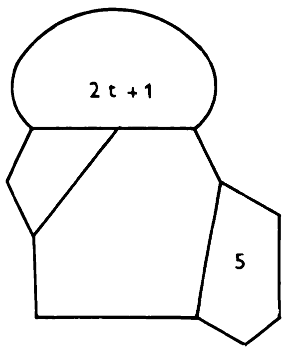

Fig. 30b

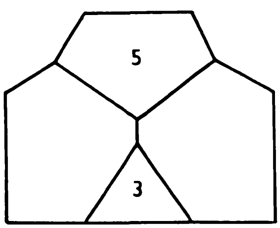

Fig. 31a

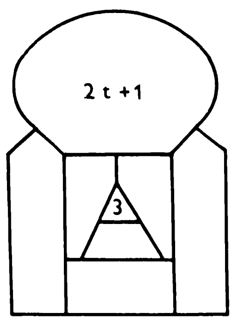

Fig. 31b

For $i=1$;

(β) If $p_5''=1$, $p_4''=0$, $p_3''\equiv 0 \pmod 2$, from the plugs $p_3''-2$ triangles are formed (Fig. 29a). The last two triangles and the pentagon needed are formed from the concluding configuration in Fig. 19c as shown in Fig. 32a. If $p_4''\neq 0$ from $\frac{p_3''}{2}=\frac{p_3}{2}$ plugs are obtained all p_3'' triangles, in the remaining plugs we have $p_4''-3$ quadrangles. The last three quadrangles and the pentagon are in the concluding configuration in Fig. 19c.

($\beta\beta$) From the plugs $p_3''-1$ triangles and $p_4''-2$ quadrangles are formed. We get the last two quadrangles and the triangle required in changing the concluding configuration in Fig. 19c as drawn in Fig. 32b.

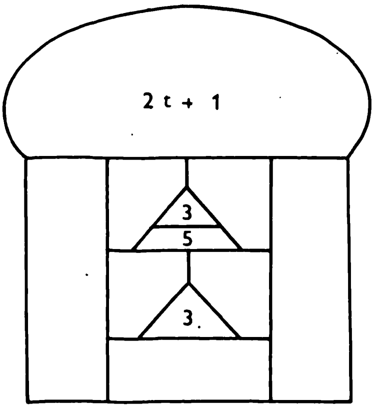

Fig. 32a

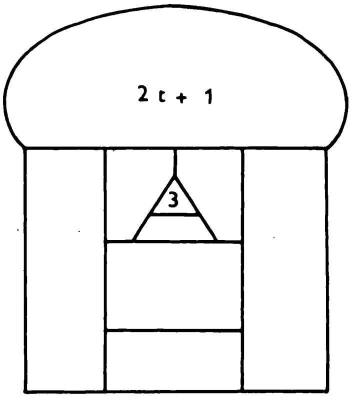

Fig. 32b

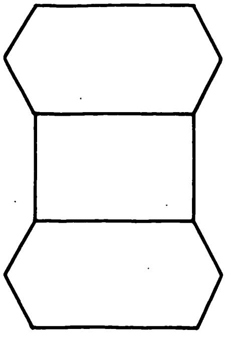
Fig. 33a

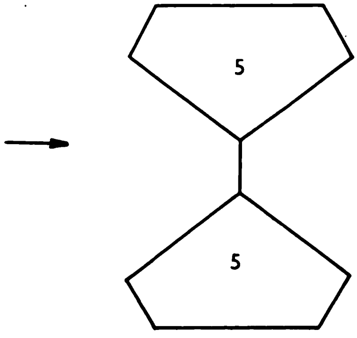

Fig. 33b

The third step

consists of changing $\left[\frac{p_5}{2}\right]$ quadrangles into pentagons. To do this, first with the map M'' the transformation of replacing edges by hexagons (see Section 2.4) is performed. In the map M_0 obtained, every quadrangle is adjacent to two hexagons as drawn in Fig. 33a; to two different quadrangles disjoint pairs of hexagons can be associated. Now $\left[\frac{p_5}{2}\right]$ such quadrangles are replaced by single edges as drawn in Fig. 33b. The resulting map is M with $p_i(M)=p_i$ for all $i\neq 6$, $v_j(M)=v_j$ for all $j>3$. This completes the proof of the Theorem.

Remark

In the proof of the Theorem those parts pertinent to the sphere (T_0) and the torus (T_1) were omitted which deal with the non-realizability of the pairs of sequences in cases b), c) and with the existence of realizations of the remaining pairs on the sphere. These parts are contained in [2, 4, 8, 9, 10]. The constructive part of the proof in Section 4 could be altered so as to cover the sphere, too. Nothing is necessary to be added for the pairs (p, v) satisfying one of the conditions (a) $\sum_{k\geq 1} v_{3k+1}\equiv\sum_{k\geq 1} v_{3k+2}\equiv 0 \pmod 2$ or (b) $p_5>0$, $\Sigma p_i>0$ for odd $i>6$. For the remaining pairs new starting planar maps must be chosen. We do not present here a complete proof because in its present form the proof is already rather complicated. Possibly a different approach is needed to reach a simpler proof.

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II. katedra matematiky
Prírodovedeckej fakulty UPJŠ
Komenského 14
041 54 Košice

ОБОБЩЕНИЕ ОДНОЙ ТЕОРЕМЫ В. ЭБЕРГАРДА

Станислав Ендроль–Эрнест Юцович

Резюме

Пусть $p_k(M)(v_k(M))$ обозначает число k-угольных граней (вершин k-той степени) клеточного комплекса M. В статье следующим образом обобщена известная теорема Эбергарда [2, 4] о выпуклых многогранниках с регулярным графом третьей степени.

Если $p=(p_i | i\geqslant 3,\ i\neq 6)$, $v=(v_i | i\geqslant 4)$ последовательности неотрицательных целых чисел, то существуют числа p_6, v_3 и такой комплекс M разделяющий связную ориентируемую поверхность рода g, что $p_k(M)=p_k$, $v_k(M)=v_k$ для всех k тогда и только тогда, когда

а) $\sum_{k\geqslant 3}(6-k)p_k+2\sum_{k\geqslant 3}(3-k)v_k=12(1-g)$.

б) для $g=0$ следующие условия одновременно не выполнены: $\Sigma p_k=0$ для нечетных k и $\Sigma v_k=1$ для $k\not\equiv 0 \pmod 3$,

в) для $g=1$, p, v разные от пары $p=(0, 0, 1, 1, 0, \ldots)$, $v=(0, 0, 0, \ldots)$.