

Martin Kochol

Latin $(n \times n \times (n - 2))$ -parallelepipeds not completing to a Latin cube

Mathematica Slovaca, Vol. 39 (1989), No. 2, 121--125

Persistent URL: <http://dml.cz/dmlcz/128702>

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1989

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

LATIN $(n \times n \times (n - 2))$ -PARALLELEPIPEDS NOT COMPLETING TO A LATIN CUBE

MARTIN KOCHOL

A latin $(n \times n \times d)$ -parallelepiped is a three dimensional array $A = [a_{i,j,k}]$, where $1 \leq i, j \leq n, 1 \leq k \leq d, a_{i,j,k} \in \{1, \dots, n\}$ and $a_{i,j,k} \neq a_{r,s,t}$ whenever exactly two of the following equalities hold: $i = r, j = s, k = t$ (See also [3]). In the case $d = n, A$ is called a latin cube of order n . A latin cube $B = [b_{i,j,k}]$ of order n is called an extension of a latin $(n \times n \times d)$ -parallelepiped $A = [a_{i,j,k}]$ if $b_{i,j,k} = a_{i,j,k}$ for all $1 \leq i, j \leq n, 1 \leq k \leq d$.

The following problem (see also [3], [4]) was mentioned during the Sixth Hungarian Colloquium on Combinatorics, Eger 1981: Given a latin $(n \times n \times d)$ -parallelepiped A , does there exist a latin cube of order n , which is an extension of A ? An analogous problem for latin rectangles was answered in the affirmative by Hall in [2]. On the contrary, P. Horák [3] constructed for all $n = 2^k, k \geq 3$, a latin $(n \times n \times (n - 2))$ -parallelepiped that cannot be extended to a latin cube of order n . This result was extended by H.-L. Fu [1] for $n = 6$ and $n \geq 12$. In this paper, using completely distinct methods, we construct such parallelepipeds for all $n \geq 5$. Moreover, this is the best possible result, because using the computer we have verified that every latin $(4 \times 4 \times 2)$ -parallelepiped can be extended to a latin cube of order 4, and for $n \leq 3$ it is obviously impossible to construct such a parallelepiped. It is easy to see that every $(n \times n \times 1)$ -parallelepiped can be

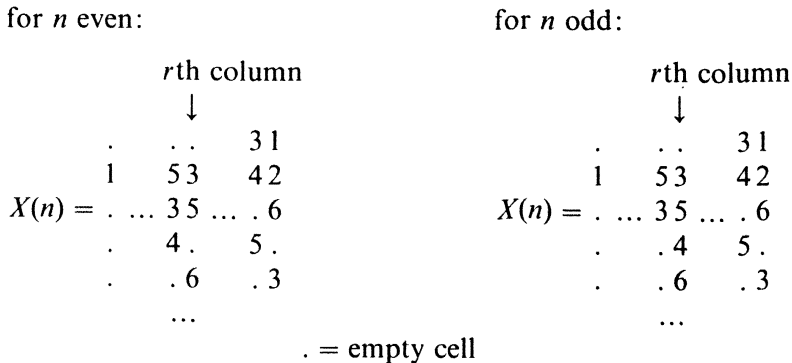


Fig. 1.

extended to a latin cube of order n , but we know nothing about $(n \times n \times d)$ -parallelepipeds if $2 \leq d \leq n - 3$.

A three-dimensional array can be described by its partial arrays. For this purpose we need some more notations. Let $A = [a_{i,j,k}]$ be a latin $(n \times n \times d)$ -parallelepiped ($1 \leq i, j \leq n, 1 \leq k \leq d$). Let us denote by $A_i^{(n)} = [a_{i,j,k}]$ ($1 \leq j \leq n, 1 \leq k \leq d$) the two-dimensional array where i is fixed, $1 \leq i \leq n$. Obviously $A_i^{(n)}$ is a latin rectangle. Similarly we can denote $A_j^{(j)} = [a_{i,j,k}]$ ($1 \leq i \leq n, 1 \leq k \leq d$) and $A_k^{(k)} = [a_{i,j,k}]$ ($1 \leq i, j \leq n$). For every $1 \leq i, j \leq n$ let us denote $A_{i,j}^{(i,j)} = [a_{i,j,k}]$ ($1 \leq k \leq d$). Similarly we can define $A_{i,k}^{(i,k)} = [a_{i,j,k}]$ ($1 \leq j \leq n$), and $A_{j,k}^{(j,k)} = [a_{i,j,k}]$ ($1 \leq i \leq n$). An incomplete latin square of order n is an $n \times n$ array such that the entries are integers from $\{1, \dots, n\}$, no integer occurs in any row or column more than once and some cells may be empty.

$$n = 8, r = 4, s = 3$$

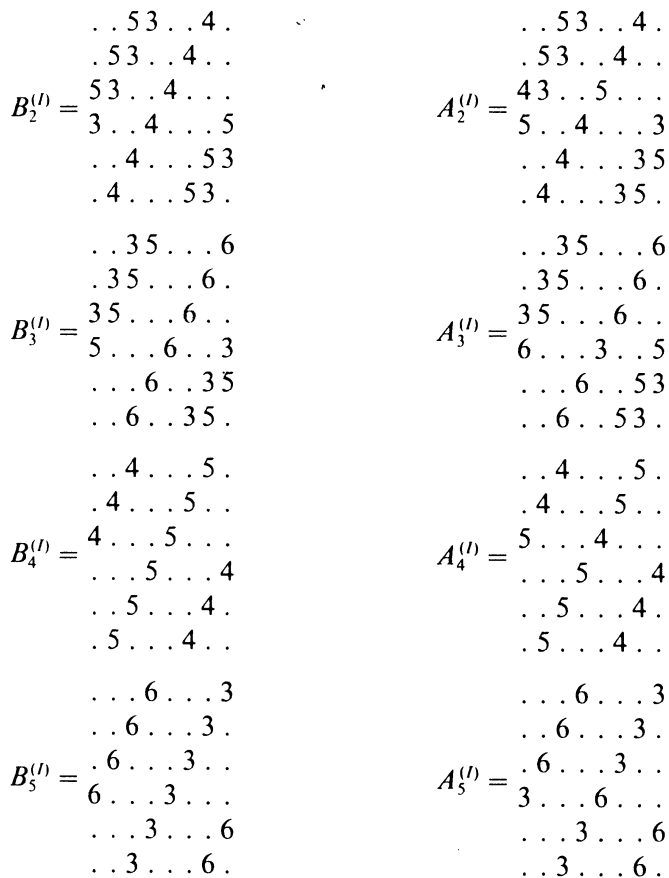


Fig. 2.

Take $n \geq 7$, let $r = \lfloor n/2 \rfloor$, $s = \lfloor (n-1)/2 \rfloor$. Take an incomplete latin square $X(n)$ to an incomplete latin square $Y(n)$ which has nonempty just 25 cells and these are exactly the intersections of the 1st, ..., 5th rows and the 1st, $(r-1)$ th, r th, $(n-1)$ th, n th columns in such a way that the number $N(i)$ of the times that the symbol i appears in the matrix $Y(n)$ is at least $5 + 5 - n$, for every $i \in \{1, \dots, n\}$. Then by Ryser [5] $Y(n)$ (means also $X(n)$) can be extended to a latin square $Z(n)$ of order n . Let us denote by Z_1, Z_2, \dots, Z_n the columns of $Z(n)$.

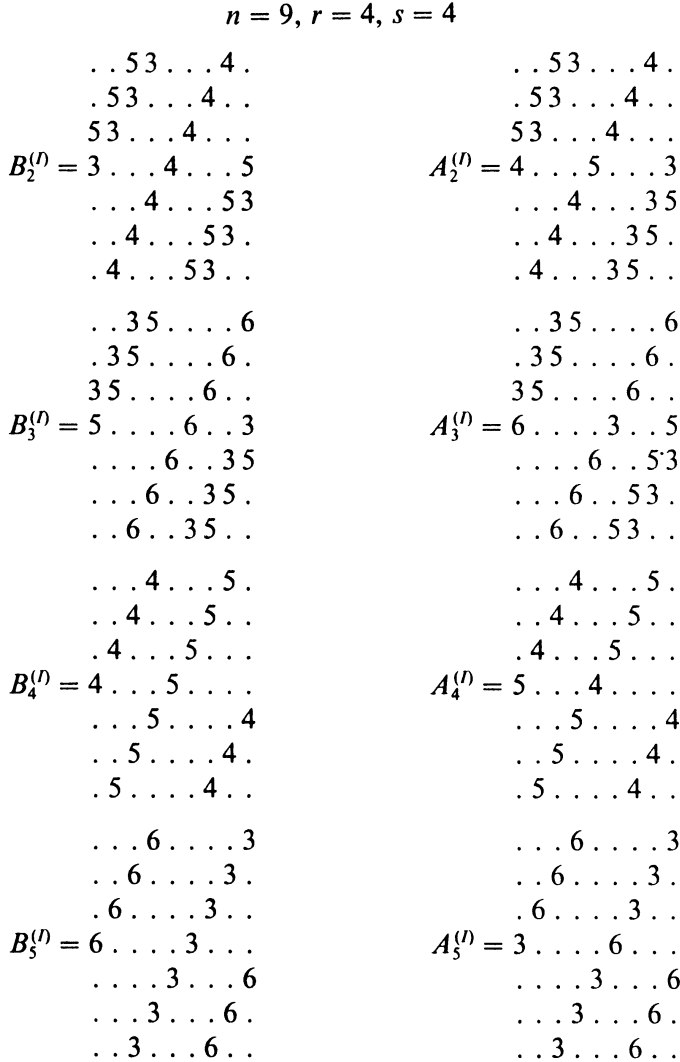


Fig. 3.

Consider the latin $(n \times n \times (n - 2))$ -parallelepiped $B = [b_{i,j,k}]$ such that for $1 \leq j \leq n$, $1 \leq k \leq n - 2$ there holds $B_{j,k}^{(j,k)} = Z_{j+k-1}$, where the indices are taken mod n .

By means of B we will construct $A = [a_{i,j,k}]$ as follows: Primarily we interchange 3 and 5 in the r th, $(r + 1)$ th, ..., n th rows of $B_2^{(n)}$, $B_3^{(n)}$, then we interchange 5 and 4 in the s th row of $B_2^{(n)}$, $B_4^{(n)}$ and 6 and 3 in the r th row of $B_3^{(n)}$, $B_5^{(n)}$. See all these members in Figs. 2., 3. for $n = 8, 9$ respectively ($a_{i,j,k}$, $b_{i,j,k}$ are in the k th row and the j th column of $A_i^{(n)}$, $B_i^{(n)}$ respectively). All the other members of A are the same as in B . It is easy to see that A is also a latin parallelepiped.

Denote by $M_{i,j}(A)$ the subset of the numbers 1, 2, ..., n which do not occur in the set $A_{i,j}^{(i,j)}$. As $a_{1,n,1} = 1$, $a_{1,n-1,1} = 3$, then it is easy to see that $M_{1,1}(A) = \{1, 3\}$, and $M_{1,2}(A) = \{1, x\}$, where $x \in \{1, \dots, n\}$. From $A_2^{(n)}$ and $Y(n)$ we can see that $M_{2,1}(A) = \{3, 2\}$, $M_{2,2}(A) = \{1, 2\}$. Let $C = [c_{i,j,k}]$ be a latin cube of order n , which is an extension of B . Provided $c_{1,2,n} = 1$ (if $c_{1,2,n-1} = 1$, the proof is similar) then $c_{1,1,n} = 3$, $c_{2,2,n} = 2$, and $c_{2,1,n} = 2$ or 3 — a contradiction.

3 5 4 2 1	5 4 3 1 2	4 1 2 5 3
5 4 3 1 2	4 5 1 2 3	3 2 5 4 1
$A_1^{(K)} = 4 2 1 5 3$	$A_2^{(K)} = 2 1 5 3 4$	$A_3^{(K)} = 1 3 4 2 5$
1 3 2 4 5	3 2 4 5 1	2 5 1 3 4
2 1 5 3 4	1 3 2 4 5	5 4 3 1 2

Fig. 4.

Then B cannot be extended to a latin cube of order n .

For $n = 6$ we replace in $X(6)$ 4 by 6. We can extend this $X(6)$ to a latin square $Z(6)$ of order 6. The rest of the proof is the same as that for n even, replacing 4 by 6. For $n = 5$ take A depicted in Fig. 4. Now $M_{1,1}(A) = \{1, 2\}$, $M_{1,2}(A) = \{2, 3\}$, $M_{2,1}(A) = \{1, 2\}$, $M_{2,2}(A) = \{1, 3\}$ and A cannot be extended to a latin cube of order n . We can conclude:

Theorem: *There exist a latin $(n \times n \times (n - 2))$ -parallelepiped that cannot be extended to a latin cube of order n if and only if $n \geq 5$.*

REFERENCES

- [1] FU H.-L.: On latin $(n \times n \times (n - 2))$ -parallelepipeds. Tamkang J. of Mathematics 17, 1986, 107—111.
- [2] HALL, M. Jr.: An existence theorem for latin squares. Bull. Amer. Math. Soc. 51, 1945, 387—388.
- [3] HORÁK, P.: Latin parallelepipeds and cubes. J. Combinatorial Theory Ser. A 33, 1982, 213—214.

- [4] HORÁK, P.: Solution of four problems from Eger, 1981, I. In: Graphs and Other Combinatorial Topics, Proc. of the 3rd Czechoslovak Symposium on Graph Theory, Teubner-Texte zur Mathematik, band 59, Leipzig, 1983, 115—117.
- [5] RYSER, H. J.: A combinatorial theorem with an application to latin rectangles. Proc. Amer. Math. Soc. 2, 1951, 550—552.

Received November 6, 1987

*Matematický ústav SAV,
Obrancov mieru 49,
814 73 Bratislava*

ЛАТИНСКИЕ $(n \times n \times (n - 2))$ -ПАРАЛЛЕЛЕПИПЕДЫ, НЕ ДОПОЛНИМЫЕ
В ЛАТИНСКИЙ КУБ

Martin Kochol

Резюме

В работе показана конструкция латинских $(n \times n \times (n - 2))$ -параллелепипедов, которые не возможно дополнить в латинский куб порядка n , для всех $n \geq 5$.