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LATIN $(n \times n \times (n-2))$-PARALLELEPIPEDS NOT COMPLETING TO A LATIN CUBE

MARTIN KOCHOL

A latin $(n \times n \times d)$-parallelepiped is a three dimensional array $A = [a_{i,j,k}]$, where $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant d$, $a_{i,j,k} \in \{1, \ldots, n\}$ and $a_{i,j,k} \neq a_{r,s,t}$ whenever exactly two of the following equalities hold: $i = r, j = s, k = t$ (See also [3]). In the case $d = n$, A is called a latin cube of order n. A latin cube $B = [b_{i,j,k}]$ of order n is called an extension of a latin $(n \times n \times d)$-parallelepiped $A = [a_{i,j,k}]$ if $b_{i,j,k} = a_{i,j,k}$ for all $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant d$.

The following problem (see also [3], [4]) was mentioned during the Sixth Hungarian Colloquium on Combinatorics, Eger 1981: Given a latin $(n \times n \times d)$-parallelepiped A, does there exist a latin cube of order n, which is an extension of A? An analogous problem for latin rectangles was answered in the affirmative by Hall in [2]. On the contrary, P. Horák [3] constructed for all $n = 2^k$, $k \geqslant 3$, a latin $(n \times n \times (n-2))$-parallelepiped that cannot be extended to a latin cube of order n. This result was extended by H.-L. Fu [1] for $n = 6$ and $n \geqslant 12$. In this paper, using completely distinct methods, we construct such parallelepipeds for all $n \geqslant 5$. Moreover, this is the best possible result, because using the computer we have verified that every latin $(4 \times 4 \times 2)$-parallelepiped can be extended to a latin cube of order 4, and for $n \leqslant 3$ it is obviously impossible to construct such a parallelepiped. It is easy to see that every $(n \times n \times 1)$-parallelepiped can be

for n even:

		rth column ↓	
	.	. .	3 1
	1	5 3	4 2
$X(n) =$	. ...	3 5 ...	. 6
	.	4 .	5 .
	.	. 6	. 3
		...	

for n odd:

		rth column ↓	
	.	. .	3 1
	1	5 3	4 2
$X(n) =$	. ...	3 5 ...	. 6
	.	. 4	5 .
	.	. 6	. 3
		...	

. = empty cell

Fig. 1.

extended to a latin cube of order n, but we know nothing about $(n \times n \times d)$-parallelepipeds if $2 \leqslant d \leqslant n-3$.

A three-dimensional array can be described by its partial arrays. For this purpose we need some more notations. Let $A = [a_{i,j,k}]$ be a latin $(n \times n \times d)$-parallelepiped $(1 \leqslant i, j \leqslant n,\ 1 \leqslant k \leqslant d)$. Let us denote by $A_i^{(I)} = [a_{i,j,k}]$ $(1 \leqslant j \leqslant n,\ 1 \leqslant k \leqslant d)$ the two-dimensional array where i is fixed, $1 \leqslant i \leqslant n$. Obviously $A_i^{(I)}$ is a latin rectangle. Similarly we can denote $A_j^{(J)} = [a_{i,j,k}]$ $(1 \leqslant i \leqslant n,\ 1 \leqslant k \leqslant d)$ and $A_k^{(K)} = [a_{i,j,k}]$ $(1 \leqslant i, j \leqslant n)$. For every $1 \leqslant i, j \leqslant n$ let us denote $A_{i,j}^{(I,J)} = [a_{i,j,k}]$ $(1 \leqslant k \leqslant d)$. Similarly we can define $A_{i,k}^{(I,K)} = [a_{i,j,k}]$ $(1 \leqslant j \leqslant n)$, and $A_{j,k}^{(J,K)} = [a_{i,j,k}]$ $(1 \leqslant i \leqslant n)$. An incomplete latin square of order n is an $n \times n$ array such that the entries are integers from $\{1, \ldots, n\}$, no integer occurs in any row or column more than once and some cells may be empty.

$$n = 8,\ r = 4,\ s = 3$$

$$B_2^{(I)} = \begin{array}{cccccccc} . & . & 5 & 3 & . & . & 4 & . \\ . & 5 & 3 & . & . & 4 & . & . \\ 5 & 3 & . & . & 4 & . & . & . \\ 3 & . & . & 4 & . & . & . & 5 \\ . & . & 4 & . & . & . & 5 & 3 \\ . & 4 & . & . & . & 5 & 3 & . \end{array} \qquad A_2^{(I)} = \begin{array}{cccccccc} . & . & 5 & 3 & . & . & 4 & . \\ . & 5 & 3 & . & . & 4 & . & . \\ 4 & 3 & . & . & 5 & . & . & . \\ 5 & . & . & 4 & . & . & . & 3 \\ . & . & 4 & . & . & . & 3 & 5 \\ . & 4 & . & . & . & 3 & 5 & . \end{array}$$

$$B_3^{(I)} = \begin{array}{cccccccc} . & . & 3 & 5 & . & . & . & 6 \\ . & 3 & 5 & . & . & . & 6 & . \\ 3 & 5 & . & . & . & 6 & . & . \\ 5 & . & . & . & 6 & . & . & 3 \\ . & . & . & 6 & . & . & 3 & 5 \\ . & . & 6 & . & . & 3 & 5 & . \end{array} \qquad A_3^{(I)} = \begin{array}{cccccccc} . & . & 3 & 5 & . & . & . & 6 \\ . & 3 & 5 & . & . & . & 6 & . \\ 3 & 5 & . & . & . & 6 & . & . \\ 6 & . & . & . & 3 & . & . & 5 \\ . & . & . & 6 & . & . & 5 & 3 \\ . & . & 6 & . & . & 5 & 3 & . \end{array}$$

$$B_4^{(I)} = \begin{array}{cccccccc} . & . & 4 & . & . & . & 5 & . \\ . & 4 & . & . & . & 5 & . & . \\ 4 & . & . & . & 5 & . & . & . \\ . & . & . & 5 & . & . & . & 4 \\ . & . & 5 & . & . & . & 4 & . \\ . & 5 & . & . & . & 4 & . & . \end{array} \qquad A_4^{(I)} = \begin{array}{cccccccc} . & . & 4 & . & . & . & 5 & . \\ . & 4 & . & . & . & 5 & . & . \\ 5 & . & . & . & 4 & . & . & . \\ . & . & . & 5 & . & . & . & 4 \\ . & . & 5 & . & . & . & 4 & . \\ . & 5 & . & . & . & 4 & . & . \end{array}$$

$$B_5^{(I)} = \begin{array}{cccccccc} . & . & . & 6 & . & . & . & 3 \\ . & . & 6 & . & . & . & 3 & . \\ . & 6 & . & . & . & 3 & . & . \\ 6 & . & . & . & 3 & . & . & . \\ . & . & . & 3 & . & . & . & 6 \\ . & . & 3 & . & . & . & 6 & . \end{array} \qquad A_5^{(I)} = \begin{array}{cccccccc} . & . & . & 6 & . & . & . & 3 \\ . & . & 6 & . & . & . & 3 & . \\ . & 6 & . & . & . & 3 & . & . \\ 3 & . & . & . & 6 & . & . & . \\ . & . & . & 3 & . & . & . & 6 \\ . & . & 3 & . & . & . & 6 & . \end{array}$$

Fig. 2.

Take $n \geqslant 7$, let $r = \lfloor n/2 \rfloor$, $s = \lfloor (n-1)/2 \rfloor$. Take an incomplete latin square $X(n)$ to an incomplete latin square $Y(n)$ which has nonempty just 25 cells and these are exactly the intersections of the 1st, ..., 5th rows and the 1st, $(r-1)$th, rth, $(n-1)$th, nth columns in such a way that the number $N(i)$ of the times that the symbol i appears in the matrix $Y(n)$ is at least $5 + 5 - n$, for every $i \in \{1, ..., n\}$. Then by Ryser [5] $Y(n)$ (means also $X(n)$) can be extended to a latin square $Z(n)$ of order n. Let us denote by $Z_1, Z_2, ..., Z_n$ the columns of $Z(n)$.

$$n = 9,\ r = 4,\ s = 4$$

$$B_2^{(I)} = \begin{array}{ccccccccc} \cdot&\cdot&5&3&\cdot&\cdot&\cdot&4&\cdot\\ \cdot&5&3&\cdot&\cdot&\cdot&4&\cdot&\cdot\\ 5&3&\cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot\\ 3&\cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot&5\\ \cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot&5&3\\ \cdot&\cdot&4&\cdot&\cdot&\cdot&5&3&\cdot\\ \cdot&4&\cdot&\cdot&\cdot&5&3&\cdot&\cdot \end{array} \qquad A_2^{(I)} = \begin{array}{ccccccccc} \cdot&\cdot&5&3&\cdot&\cdot&\cdot&4&\cdot\\ \cdot&5&3&\cdot&\cdot&\cdot&4&\cdot&\cdot\\ 5&3&\cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot\\ 4&\cdot&\cdot&\cdot&5&\cdot&\cdot&\cdot&3\\ \cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot&3&5\\ \cdot&\cdot&4&\cdot&\cdot&\cdot&3&5&\cdot\\ \cdot&4&\cdot&\cdot&\cdot&3&5&\cdot&\cdot \end{array}$$

$$B_3^{(I)} = \begin{array}{ccccccccc} \cdot&\cdot&3&5&\cdot&\cdot&\cdot&\cdot&6\\ \cdot&3&5&\cdot&\cdot&\cdot&\cdot&6&\cdot\\ 3&5&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot\\ 5&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&3\\ \cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&3&5\\ \cdot&\cdot&\cdot&6&\cdot&\cdot&3&5&\cdot\\ \cdot&\cdot&6&\cdot&\cdot&3&5&\cdot&\cdot \end{array} \qquad A_3^{(I)} = \begin{array}{ccccccccc} \cdot&\cdot&3&5&\cdot&\cdot&\cdot&\cdot&6\\ \cdot&3&5&\cdot&\cdot&\cdot&\cdot&6&\cdot\\ 3&5&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot\\ 6&\cdot&\cdot&\cdot&\cdot&3&\cdot&\cdot&5\\ \cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&5&3\\ \cdot&\cdot&\cdot&6&\cdot&\cdot&5&3&\cdot\\ \cdot&\cdot&6&\cdot&\cdot&5&3&\cdot&\cdot \end{array}$$

$$B_4^{(I)} = \begin{array}{ccccccccc} \cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot&5&\cdot\\ \cdot&\cdot&4&\cdot&\cdot&\cdot&5&\cdot&\cdot\\ \cdot&4&\cdot&\cdot&\cdot&5&\cdot&\cdot&\cdot\\ 4&\cdot&\cdot&\cdot&5&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&5&\cdot&\cdot&\cdot&\cdot&4\\ \cdot&\cdot&5&\cdot&\cdot&\cdot&\cdot&4&\cdot\\ \cdot&5&\cdot&\cdot&\cdot&\cdot&4&\cdot&\cdot \end{array} \qquad A_4^{(I)} = \begin{array}{ccccccccc} \cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot&5&\cdot\\ \cdot&\cdot&4&\cdot&\cdot&\cdot&5&\cdot&\cdot\\ \cdot&4&\cdot&\cdot&\cdot&5&\cdot&\cdot&\cdot\\ 5&\cdot&\cdot&\cdot&4&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&5&\cdot&\cdot&\cdot&\cdot&4\\ \cdot&\cdot&5&\cdot&\cdot&\cdot&\cdot&4&\cdot\\ \cdot&5&\cdot&\cdot&\cdot&\cdot&4&\cdot&\cdot \end{array}$$

$$B_5^{(I)} = \begin{array}{ccccccccc} \cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot&\cdot&3\\ \cdot&\cdot&6&\cdot&\cdot&\cdot&\cdot&3&\cdot\\ \cdot&6&\cdot&\cdot&\cdot&\cdot&3&\cdot&\cdot\\ 6&\cdot&\cdot&\cdot&\cdot&3&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&3&\cdot&\cdot&\cdot&6\\ \cdot&\cdot&\cdot&3&\cdot&\cdot&\cdot&6&\cdot\\ \cdot&\cdot&3&\cdot&\cdot&\cdot&6&\cdot&\cdot \end{array} \qquad A_5^{(I)} = \begin{array}{ccccccccc} \cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot&\cdot&3\\ \cdot&\cdot&6&\cdot&\cdot&\cdot&\cdot&3&\cdot\\ \cdot&6&\cdot&\cdot&\cdot&\cdot&3&\cdot&\cdot\\ 3&\cdot&\cdot&\cdot&\cdot&6&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&3&\cdot&\cdot&\cdot&6\\ \cdot&\cdot&\cdot&3&\cdot&\cdot&\cdot&6&\cdot\\ \cdot&\cdot&3&\cdot&\cdot&\cdot&6&\cdot&\cdot \end{array}$$

Fig. 3.

Consider the latin $(n \times n \times (n-2))$-parallelepiped $B = [b_{i,j,k}]$ such that for $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n-2$ there holds $B_{j,k}^{(J,K)} = Z_{j+k-1}$, where the indices are taken mod n.

By means of B we will construct $A = [a_{i,j,k}]$ as follows: Primarily we interchange 3 and 5 in the rth, $(r+1)$th, ..., nth rows of $B_2^{(I)}$, $B_3^{(I)}$, then we interchange 5 and 4 in the sth row of $B_2^{(I)}$, $B_4^{(I)}$ and 6 and 3 in the rth row of $B_3^{(I)}$, $B_5^{(I)}$. See all these members in Figs. 2., 3. for $n = 8, 9$ respectively ($a_{i,j,k}$, $b_{i,j,k}$ are in the kth row and the jth column of $A_i^{(I)}$, $B_i^{(I)}$ respectively). All the other members of A are the same as in B. It is easy to see that A is also a latin parallelepiped.

Denote by $M_{i,j}(A)$ the subset of the numbers 1, 2, ..., n which do not occur in the set $A_{i,j}^{(I,J)}$. As $a_{1,n,1} = 1$, $a_{1,n-1,1} = 3$, then it is easy to see that $M_{1,1}(A) = \{1, 3\}$, and $M_{1,2}(A) = \{1, x\}$, where $x \in \{1, ..., n\}$. From $A_2^{(I)}$ and $Y(n)$ we can see that $M_{2,1}(A) = \{3, 2\}$, $M_{2,2}(A) = \{1, 2\}$. Let $C = [c_{i,j,k}]$ be a latin cube of order n, which is an extension of B. Provided $c_{1,2,n} = 1$ (if $c_{1,2,n-1} = 1$, the proof is similar) then $c_{1,1,n} = 3$, $c_{2,2,n} = 2$, and $c_{2,1,n} = 2$ or 3 — a contradiction.

$$A_1^{(K)} = \begin{matrix} 35421 \\ 54312 \\ 42153 \\ 13245 \\ 21534 \end{matrix} \qquad A_2^{(K)} = \begin{matrix} 54312 \\ 45123 \\ 21534 \\ 32451 \\ 13245 \end{matrix} \qquad A_3^{(K)} = \begin{matrix} 41253 \\ 32541 \\ 13425 \\ 25134 \\ 54312 \end{matrix}$$

Fig. 4.

Then B cannot be extended to a latin cube of order n.

For $n = 6$ we replace in $X(6)$ 4 by 6. We can extend this $X(6)$ to a latin square $Z(6)$ of order 6. The rest of the proof is the same as that for n even, replacing 4 by 6. For $n = 5$ take A depicted in Fig. 4. Now $M_{1,1}(A) = \{1, 2\}$, $M_{1,2}(A) = \{2, 3\}$, $M_{2,1}(A) = \{1, 2\}$, $M_{2,2}(A) = \{1, 3\}$ and A cannot be extended to a latin cube of order n. We can conclude:

Theorem: *There exist a latin $(n \times n \times (n-2))$-parallelepiped that cannot be extended to a latin cube of order n if and only if $n \geqslant 5$.*

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ЛАТИНСКИЕ $(n \times n \times (n-2))$-ПАРАЛЛЕЛЕПИПЕДЫ, НЕ ДОПОЛНИМЫЕ В ЛАТИНСКИЙ КУБ

Martin Kochol

Резюме

В работе показана конструкция латинских $(n \times n \times (n-2))$-параллелепипедов, которые не возможно дополнить в латинский куб порядка n, для всех $n \geqslant 5$.