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Mathematica Slovaca, Vol. 26 (1976), No. 4, 299--307

Persistent URL: http://dml.cz/dmlcz/128672

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ON THE DISTRIBUTION OF ZEROS OF EXPONENTIAL POLYNOMIALS

A. J. van der POORTEN

1. Introduction

We employ a generalisation of the method described in P. Turán's book *Eine neue Methode in der Analysis und deren Anwendungen* [3] and find a lower bound for

$$\max_{1 \leq \mu \leq m} |G^{(\mu-1)}(z_0)|$$

Where $G(z)$ is a function of the shape

$$G(z) = \sum_{k=1}^{m} a_k z^{\alpha_k} .$$

As an illustration of the technique of generalisation by particularisation [5] we show that this result implies a similar lower bound for functions of the shape

$$F(z) = \sum_{k=1}^{m} \sum_{s=1}^{n(k)} a_{ks} (\ln z)^{s-1} z^{\alpha_k}$$

Jensen's Theorem now permits the estimation of an upper bound for the number of zeros of $F(z)$ in discs in the complex plane. By writing $z = e^u$ we obtain upper bounds for the number of zeros of exponential polynomials in certain regions of the complex plane. We show that:

Theorem. *Let* $E(z) = \sum_{k=1}^{m} \sum_{p=1}^{n(k)} a_{kp} z^{p-1} e^{\alpha_k z}$ *be an exponential polynomial and let*

$$\sigma = \sum_{k=1}^{m} n(k) \qquad \Delta = \max_{j,k} |\alpha_j - \alpha_k|$$

Then in any disc of radius R *in the complex plane* $E(z)$ *has less than*

$$3(\sigma - 1) + 6R\Delta$$

zeros.

The problem of analysing the distribution of zeros of exponential polynomials is of considerable importance in many fields of pure and applied mathematics. The results mentioned therefore have some intrinsic interest. Our principal purpose however is to illustrate a useful technique and to show that methods previously only applied to exponential polynomials or power sums are applicable in somewhat more general contexts. For this reason we do not directly consider exponential polynomials (see [4]) though to do so would have produced results similar to those of this paper.

2. We consider a function $G(z)$ of the form

$$G(z)=\sum_{k=1}^{m}\sum_{p=1}^{n(k)} f_{kp} z^{\alpha_{kp}} .$$

where the quantities f_{kp} are described below, and the α_{kp} are complex numbers. Further let σ be the sum $n(1)+\ldots+n(m)$ of the non-negative integers $n(k)$. Finally let z_0 be distinct from zero. Denote by Δ the $\sigma\times\sigma$ determinant

$$\Delta=\left|\left\{\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\lambda-1} z^{\alpha_{kp}}\right\}_{z=z_0}\right|_{kp,\lambda}$$

where the notation implies that rows are indexed by the pairs (k,p), $1\leqslant k\leqslant m$; $1\leqslant p\leqslant n(k)$, arranged lexicographically, whilst columns are indexed by λ, $1\leqslant\lambda\leqslant\sigma$. Further, denote by $\Delta_{\lambda,kp}$ the cofactor of the (kp,λ) element of Δ.

Then we have immediately,

$$\sum_{\lambda=1}^{\sigma} G^{(\lambda-1)}(z_0)\,\Delta_{\lambda,kp}=f_{kp}\Delta ,$$

whence, for any $z_1\neq 0$

$$\sum_{\lambda=1}^{\sigma} G^{(\lambda-1)}(Z_0)\sum_{k=1}^{m}\sum_{p=1}^{n(k)}\Delta_{\lambda,kp} z_1^{\alpha_{kp}}=\sum_{k=1}^{m}\sum_{p=1}^{n(k)} f_{kp} z_1^{\alpha_{kp}}\Delta=G(z_1)\Delta ,$$

and thus

$$\sum_{\lambda=1}^{\sigma} z_0^{\lambda-1}\frac{(\sigma-\lambda)!}{(\sigma-1)!}G^{(\lambda-1)}(z_0)\,.\,z_0^{-(\lambda-1)}\frac{(\sigma-1)!}{(\sigma-\lambda)!}\sum_{k=1}^{m}\sum_{p=1}^{n(k)}\Delta_{\lambda,kp} z_1^{\alpha_{kp}}=G(z_1)\Delta .$$

Taking absolute values, we obtain immediately:

Lemma 1. *There is an integer* μ, $1\leqslant\mu\leqslant\sigma$, *such that*

$$\left|z_0^{\mu-1}\frac{(\sigma-\mu)!}{(\sigma-1)!}G^{(\mu-1)}(z_0)\right|\geqslant\left|G(z_1)\right|\left\{\sum_{\lambda=1}^{\sigma}\left|\frac{(\sigma-1)!}{(\sigma-\lambda)!}\right| z_0{}^{(\lambda-1)}\sum_{k=1}^{m}\sum_{p=1}^{n(k)}\frac{\Delta_{\lambda,kp}}{\Delta} z_1^{\alpha_{kp}}\right|\right\}^{-1} .$$

3. We adopt the notation, that for non-negative integers n

$$x^{\underline{n}}=x(x-1)\ldots(x-n+1) ,\qquad x^{\underline{0}}=1 .$$

Then we see that the determinant Δ may be expressed in the form

$$\Delta = |\alpha_{kp}^{!(\lambda-1)} z_0^{\alpha_{kp}-(\lambda-1)}|_{kp,\lambda}$$

whence it follows that the cofactors $\Delta_{\lambda,kp}$ satisfy:

$$z_0^{\alpha_{hq}} \sum_{\lambda=1}^{\sigma} \alpha_{hq}^{!(\lambda-1)} . \Delta_{\lambda,kp} . z_0^{-(\lambda-1)} = \begin{cases} 0 & (h,q) \neq (k,p) \\ \Delta & (h,q) = (k,p) \end{cases}$$

Hence we obtain by summing over (k,p)

$$(1) \qquad \sum_{\lambda=1}^{\sigma} z_0^{-(\lambda-1)} \sum_{k=1}^{m} \sum_{p=1}^{n(k)} \frac{\Delta_{\lambda,kp}}{\Delta} z_1^{\alpha_{kp}} \alpha_{hq}^{!(\lambda-1)} = (z_1/z_0)^{\alpha_{hq}}$$

Now let

$$P(z) = \sum_{\lambda=1}^{\sigma} c_\lambda z^{!(\lambda-1)}$$

be the unique polynomial of degree $\sigma-1$ defined by the σ conditions

$$P(\alpha_{hq}) = (z_1/z_0)^{\alpha_{hq}}, \quad \ldots 1 \leqslant h \leqslant m;\ 1 \leqslant q \leqslant n(h)$$

Then by uniqueness, the equations (1) imply

$$(2) \qquad z_0^{-(\lambda-1)} \sum_{k=1}^{m} \sum_{p=1}^{n(k)} \frac{\Delta_{\lambda,kp}}{\Delta} . z_1^{\alpha_{kp}} = c_\lambda; \qquad 1 \leqslant \lambda \leqslant \sigma .$$

If we write $P(z)$ as an interpolation series

$$(3) \qquad P(z) = \sum_{k=1}^{m} \sum_{p=1}^{n(k)} b_{kp} \prod_{(h,q)<(k,p)} (z - \alpha_{hq}),$$

then the interpolation coefficients b_{lp} are given by the contour integrals

$$b_{kp} = \frac{1}{2\pi i} \oint_\Gamma (z_1/z_0)^\zeta \prod_{(h,q)\leqslant(k,p)} (\zeta - \alpha_{hq})^{-1} d\zeta$$

where the closed contour Γ may be taken as a circle about the origin of sufficiently large radius to contain all the points x_{kp}.

We assert that, with the notation as above:

Lemma 2.

$$S = \sum_{\lambda=1}^{\sigma} \frac{(\sigma-1)!}{(\sigma-\lambda)!} |c_\lambda| \leqslant \sum_{k=1}^{m} \sum_{p=1}^{n(k)} |b_{kp}| \prod_{(h,q)<(k,p)} (\sigma - 1 + |\alpha_{hq}|) .$$

To see this, observe that our claim is trivial when all the c_λ are real and positive, since then the left hand side is simply $P(\sigma-1)$, and by (3) the right hand side is assuredly as large as $P(\sigma-1)$. Since the c_λ are expressible as a polynomial combination of the b_{kp} and the $(-\alpha_{kp})$ with positive coefficients (apply the difference operator, which sends $f(z)$ to $f(z+1)-f(z)$, $\lambda-1$ times to $P(z)$ in order to obtain $(\lambda-1)!c_\lambda$), the assertion remains, *a fortiori*, true in general.

Recalling (2) we note that Lemma 1 has now become: There is an integer μ, $1 \leqslant \mu \leqslant \sigma$, such that

$$|G^{(\mu-1)}(z_0)| \geqslant |z_0|^{(\mu-1)} \frac{(\sigma-1)!}{(\sigma-\mu)!} S^{-1} |G(z_1)| . \tag{4}$$

with Lemma 2 providing a convenient lower bound for S^{-1}. Writing

$$A = \max_{(k,s)} |\alpha_{ks}|$$

we obtain

$$|b_{kp}| \leqslant \left| \frac{1}{2\pi i} \int_\Gamma |z_1/z_0|^{\pm\zeta} \prod_{(h,q)\leqslant(k,p)} (\zeta - A)^{-1} \mathrm{d}\zeta \right|$$

with the $+$ or $-$ sign being selected according as $|z_1| \geqslant |z_0|$ or $|z_1| < |z_0|$. Hence by Lemma 2,

$$S \leqslant \sum_{l=1}^{\sigma} \left| \frac{1}{2\pi i} \oint_l |z_1/z_0|^{\pm\zeta} (S-A)^{-l} \mathrm{d}\zeta \right| . (\sigma - 1 + A)^{l+1} =$$

$$= \sum_{l=1}^{\sigma} \frac{(\pm \log |z_1/z_0|)^{l-1}}{(l-1)!} |z_1/z_0|^{\pm A} (\sigma - 1 + A)^{l+1} .$$

Then if $|z_1/z_0|$ is close to 1, we obtain the tidy estimate

$$S \leqslant \exp \{ \pm (\log |z_1/z_0|)(\sigma - 1 + 2A) \} = |z_1/z_0|^{\pm(\sigma-1+2A)} . \tag{5}$$

A more precise estimate may be produced as follows: we write $x = \pm \log |z_1/z_0| (\sigma - 1 + 2A) = k(\sigma - 1)$ and assume $k \geqslant 1$. Then

$$\sum_{l=1}^{\sigma} \frac{x^{l-1}}{(l-1)!} = \frac{1}{2\pi i} \int_{|z|=R} \left(1 - \left(\frac{x}{z} \right)^\sigma \right) \mathrm{e}^z \frac{\mathrm{d}z}{z-x} = \frac{1}{2\pi i} \int_{|z|=R} \left(\sum_{l=1}^{\sigma} x^{\sigma-l} z^{l-1} \right) \mathrm{e}^z \frac{\mathrm{d}z}{z^\sigma}$$

Hence taking $R = x/k = \sigma - 1$ we obtain

$$\left| \sum_{l=1}^{\sigma} \frac{x^{l-1}}{(l-1)!} \right| \leqslant \left(\sum_{l=1}^{\sigma} k^{l-1} \right) \mathrm{e}^{x/k} \leqslant (k/k-1) k^{\sigma-1} \mathrm{e}^{\sigma-1} = (k/k-1) \mathrm{e}^{(1+\log k)(\sigma-1)}$$

whence finally

$$S \leqslant |z_1/z_0| \ldots (k/k-1)\, \mathrm{e}^{(1+\log k)(\sigma-1)} \tag{6}$$

4. We employ a subterfuge to finally obtain the result we require. Thus we

choose the coefficients f_{kp} of $G(z)$ as functions of the α_{kp} so that, if the a_{kp} are constants, not all zero,

$$\sum_{p=1}^{n(k)} f_{kp} z^{\alpha_{kp}} = \sum_{p=1}^{n(k)} (p-1)!\,\frac{a_{kp}}{2\pi i}\oint_{\Gamma} z^{\zeta}\prod_{q=1}^{p}(\gamma-\alpha_{kq})^{-1}\mathrm{d}\zeta$$

where the closed contour Γ contains all the α_{kp}. If we now identify the α_{kp} according to, say

$$\alpha_{kp}=\alpha_{k1}=\alpha_k\ ,\qquad 1\leqslant k\leqslant m;\quad 1\leqslant p\leqslant n(k)\ ,$$

then the function $G(z)$ becomes

$$F(z)\doteq\sum_{k=1}^{m}\sum_{p=1}^{n(k)} a_{kp}\,(\log z)^{p-1}z^{\alpha_{kp}}\ .$$

Hence, by (4) we obtain the principal auxiliary result

Lemma 3. *There is an integer* μ, $1\leqslant\mu\leqslant\sigma$, *such that*

$$|F^{(\mu-1)}(z_0)|\geqslant|z_0|^{-(\mu-1)}\frac{(\sigma-1)!}{(\sigma-\mu)!}\,S^{-1}|F(z_1)|$$

where upper bounds for S are given by (5) *or* (6).

This result is an analogue of the Main Theirems of P. Turán [2, 3, 5]. From it we immediately derive a further auxiliary result by Cauchy's inequality and the maximum modulus principle.

5. Thus let $0<R_1, R_2<|z_0|$. By the maximum modulus principle the maximum of $|F(z)|$ on the disc $|z-z_0|\leqslant R_1$ occurs at some point z_1 on the circle $|z-z_0|=R_1$. We observe that $|z_1|\geqslant|z\ |-R_1$. Hence by Lemma 3 we obtain

$$\max_{|z-z_0|\leqslant R_1}|F(z)|=|F(z_1)|\leqslant\max_{1\leqslant\mu\leqslant\sigma}\frac{(\sigma-\mu)!}{(\sigma-1)!}\,|z_0|^{\mu-1}|F^{(\mu-1)}(z_0)|\,.\,S \tag{7}$$

where S is given by (5) or (6) with $|z_1/z_0|^{\pm1}$ replaced by $|z_0|/|z_0|-R_1$. Further, by Cauchy's inequality and the maximum modulus principle there is a point z_2 on the circle $|z-z_0|=R_2$ so that for $|\leqslant\mu\leqslant\sigma$

$$|F^{(\mu-1)}(z_0)|=(\mu-1)!\,R_2^{-(\mu-1)}\max_{|z\ \ z_0|\leqslant R_2}|F(z)|=(\mu-1)!\,R_2^{-(\mu\ \ 1)}|F(z_2)| \tag{8}$$

Thus we obtain by combining (7) and (8) that:

Lemma 4. *If* $0<R_1, R_2<|z_0|$ *then there is a point* z_2 *on the circle* $|z-z_0|=R_2$ *so that*

$$\max_{|\ \ z_0\ \leqslant R_1}|F(z)/F(z_2)|\leqslant S\,.\max_{1<\mu\leqslant\sigma}\left\{(|z_0|/R_2)^{\mu\ \ 1}\Big/\binom{\sigma-1}{\mu-1}\right\}=$$

$$= S(|z_0/R_2)^{\sigma-1}$$

An application of Jensen's Theorem now suffices to give us our final results. Denote by $N(w, R, F)$ the number of zeros of the function $F(z)$ in the disc $\{z: |z-w|<R\}$. Then if F is holomorphic on the disc $\{z: |z-w|\leq SR\}$ and $F(w)\neq 0$, Jensen's Theorem asserts:

$$\int_0^{sR}\frac{N(w, r, F)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi}\log|F(w+sR\,e^{i\theta})|\,d\theta - \log|F(w)|\,.$$

Then, as in [2; p 54—5] we see that if $s>1$

$$\int_0^{sR}\frac{N(w, r, F)}{r}\,dr = \sum_{k=0}^{\infty}\int_{Rs^{-k}}^{sRs^{-k}}\cdot\frac{N(w, r, F)}{r}\,dr$$

$$\geq \sum_{k=0}^{\infty} N(w, Rs^{-k}, F)(\log sRs^{-k} - \log Rs^{-k})$$

$$= \log s\cdot\sum_{k=0}^{\infty} N(w, Rs^{-k}, F)\,.$$

Hence it follows from Jensen's Theorem that:

Lemma 5.

$$\sum_{k=0}^{\infty} N(w, Rs^{-k}, F) \leq \frac{1}{\log s}\max_{|z-w|\leq sR}\log|F(z)/F(w)|\,.$$

We now apply Lemma 4 and Lemma 5 with the following parameters: Let u, s, t be positive numbers with $s>1$, $t<1$ and $u>s+t$. We select R such that $|z_0|=uR$ and R_1, R_2 such that $R_1=(s+t)R$, $R_2=tR$. Then according to (5) and (6) the quantity s is such that

(9) $$\log S \leq (\sigma-1+2A)\log(u/u-s-t)$$

or

(10) $$\log S \leq A\log(u/u-s-t) + \log(k/k-1) + (1+\log k)(\sigma-1)$$

with k given by:

$$(\sigma-1)k = (\sigma-1+A)\log(u/u-s-t)\,.$$

Moreover since the disc $\{z: |z-z_0|\leq R(1-t)\}$ is contained in $\{z: |z-z_2|\leq R\}$ we see that

$$N(z_0, R(1-t), F) \leq N(z_2, R, F) \leq \sum_{k=0}^{\infty} N(z_2, Rs^{-k}, F)\,.$$

Hence by Lemma 4 and 5 and the estimate (9) for S:

$$N(z_0, R(1-t), F) \leq \frac{1}{\log s} \max_{|z-z_2| \leq sR} \log|F(z)/F(z_2)|$$

(II) $$\leq \frac{1}{\log s} \max_{|z-z_0| \leq (s+t)R} \log|F(z)/F(z_2)|$$

$$\leq \frac{1}{\log s} \{(\sigma-1+2A)\log(u/u-s-t)+(\sigma-1)\log(u/t)\}$$

Selecting, say, $u=20$, $s=5{,}3$, $t=0{,}2$ we obtain

(12) $$N(z_0, R(1-t), F) z (0.2)(\sigma-1+2A)+(2.8)(\sigma-1)$$

(more precisely, the constants are respectively 0,/93... and 2,762...). This result, which does not appear to have been observed previously, is perhaps of some incidental interest in its own right. For example, the the independence of the result on the value of R is at first sight startling. It is not difficult to see however why this should be so and we will observe a similar phenomenon on considering exponential polynomials.

6. By replacing z by e^z the function $F(z)$ becomes the exponential polynomial

$$E(z)=\sum_{k=1}^{m}\sum_{p=1}^{n(k)} a_{kp} z^{p-1} e^{\alpha_k z}$$

and the estimate (12) asserts that $E(z)$ has at most $(0.386)A+(2.955)(\sigma-1)$ zeros in the region defined by $\{Z: |e^z - z_0| \leq (0.8)R\}$ where $|z_0|=20R$. But it is known that because the estimate is independent of the coefficients a_{kp} and only roughly dependent on the exponents α_k and the degrees $n(k)$ and independent of R, therefore it is valid for any convenient value of R and will hold for any affine transform of the region then obtained; for details see [4]. Selecting $z_0=1$ so $R=0.05$, the region becomes $\{Z: |e^z-1| \leq 0.04\}$. But this region contains the circle $\{Z: |z| \leq 0.039\}$. So, when $r=0.039$ it is the case that in the disc $\{z: |z| \leq r\}$ the function $E(z)$ has at most

(13) $$10Ar+3(\sigma-1)...$$

zeros. We assert that this result is valid for all r and indeed for discs with any centre, as a consequence of the invariance mentioned above; one can see this by observing that the transformation $z \mapsto zl$, $\alpha_k \mapsto \alpha_k/l$ (all k) leaves the estimate (13) invariant and affects only the coefficients of $E(z)$ and similarly that the transformation $z \mapsto z-\beta$ affects only the coefficients of $E(z)$. Finally, if $\Delta = \max_{j,k} |\alpha_j - \alpha_k|$ then there exists $\alpha \in C$ such that $\max_k |\alpha_{k-\alpha}| = \Delta/\sqrt{3}$; observing that $E(z)$ and

$e\ {}^{\alpha}E(z)$ have the same zeros, it is clear that ın (13) we may replace A by $\Delta/\sqrt{3}$. Hence we have shown that the exponential polynomial $E(z)$ has at most

$$(14) \qquad 6\Delta R + 3(\sigma - 1)$$

zeros in any disc of radius R in the complex plane, which is the Theorem.

7. There is of course no suggestion that the estimate (14) is optimal; indeed ın asymptotic cases even our method provides better results. For example, suppose n is large and in (11) we select the parameters as $s = 2^n\left(1+\frac{1}{n}\right)^n$, $t = us^{(1+_n)}$ and $u = s/1 - s^{1/n}$. Then (12) becomes

$$N(z_0, R(1-t), F) \leq \frac{1}{n}(\sigma - 1 + 2A) + \left(1+\frac{1}{n}\right)\cdot(\sigma - 1)$$

and as the right-hand side of this ˙neauality must be an ınteger we obtaın

$$N(z_0, R(1-t), F) \leq \sigma - 1 \quad \text{ıf} \quad 2(A + \sigma - 1) < n$$

It follows that if n satisfies this inequality then in any disc of rad us les than n e $^{1}2$ ^{n} the exponential polynomial $E(z)$ has at most $\sigma - 1$ zeros. Thıs r sult is best possible in the sense that the exponential polynomial may have a zero of order $\sigma - 1$ at any given point.

Conversely it can be shown that for large R our estimates imply that in dıscs of radıus R the number of zeros of the exponential polynomial $E(z)$ is of order e $R\Delta$. This result is non-optimal. Indeed, by other methods [1] it is known that the correct result is, order less than $R\Delta$.

I acknowledge the assıstance of Mr. P. O'Sullivan in performing some of the calculations for this paper.

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R ceıved June 3, 1974

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О РАСПРЕДЕЛЕНИИ НУЛЕЙ ПОКАЗАТЕЛЬНЫХ МНОГОЧЛЕНОВ

А. Й. ван дер Поортен

Резюме

В работе обобщены результаты П. Турана нижней оценки максимума для производной обобщенного многочлена. Даны тоже оценки числа нулей показательного многочлена.