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ON SOME PROBLEMS IN THE OSCILLATION THEORY OF SELF-ADJOINT LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The known results concerning the oscillation properties of second order equations are extended to self-adjoint equations of the even order. Some open problems associated with this extension are formulated.

1. Introduction

Consider a self-adjoint linear differential equation of the second order

$$-(p(x)y')' + q(x)y = 0, \tag{1}$$

where $p(x) \in C^1$, $q(x) \in C^0$, $p(x) > 0$ on $I = (a, b)$, $-\infty \leqslant a < b \leqslant \infty$. The following statements concerning the oscillation behaviour of this equation are well known, see [3], [12].

i) Let u_1, u_2 be (linearly independent) solutions of (1) for which $p(x)(u_1(x)u_2'(x) - u_1'(x)u_2(x)) = 1$. Equation (1) is oscillatory at b (for terminology see Sec. 2) if and only if

$$\int_c^b \frac{\mathrm{d}x}{p(x)(u_1^2(x) + (u_2^2(x))} = \infty, \; c \in I.$$

ii) Let u_1, u_2 be the same as in i). Equation (1) is disconjugate on I if and only if

$$J(u_1, u_2, I) = \int_a^b \frac{\mathrm{d}x}{p(x)(u)_1^2(x) + u_2^2(x))} \leqslant \pi .$$

Particularly, (1) is 1-special on I if $J(u_1, u_2, I) = \pi$ and 1-general on I if $J(u_1, u_2, I) < \pi$ for every pair of solutions u_1, u_2 for which $p(u_1u_2' - u_1'u_2) = 1$.

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iii) Let equation (1) be conjugate on an interval $I_0 \subseteq I$. There exists a function $\tilde{q}(x) \in C^0$ such that $\tilde{q}(x) > q(x)$ on I_0 and the equation

$$-(p(x)y')' + \tilde{q}(x)y = 0$$

is conjugate on I_0.

The aim of this paper is to extend these results to selfadjoint linear differential equations of the even order

$$\sum_{k=0}^{n} (-1)^k (p_k(x)y^{(k)})^{(k)} = 0\,, \tag{2}$$

$p_j \in C^k$, $p_n > 0$ on I, and to formulate some open problems associated with this extension.

The principal method we use is the application of the results of the transformation theory of linear Hamiltonian systems to equation (2), combined with oscillation criteria for the so-called trigonometric systems.

2. Preliminary results

Two points $x_1, x_2 \in I$ are said to be conjugate relative to equation (2) if there exists a nontrivial solution of this equation for which $y^{(i)}(x_1) = 0 = y^{(i)}(x_2)$, $i = 0, \dots, n-1$. Equation (2) is said to be disconjugate on an interval $I_0 \subseteq I$ whenever there exists no pair of points of I_0 which are conjugate relative to (2), in the opposite case (2) is said to be conjugate on I_0. Equation (2) is said to be nonoscillatory at b if there exists $c \in I$ such that this equation is disconjugate on (c, b), in the opposite case (2) is said to be oscillatory at b.

Let y be a solution of (2). Set $u_1 = y$, $u_2 = y'$, $\dots$, $u_n = y^{(n-1)}$, $v_n = p_n y^{(n)}$, $v_{n-k} = -v'_{n-k+1} + p_{n-k}y^{(n-k)}$, $k = 1, \dots, n-1$. Then the pair of vectors $u = (u_1, \dots, u_n)^T$, $v = (v_1, \dots, v_n)^T$ („T" denotes the transpose) is a solution of the linear Hamiltonian system

$$\begin{aligned} u' &= A(x)u + B(x)v\,, \\ v' &= C(x)u - A^T(x)v\,. \end{aligned} \tag{3}$$

where A, B, C are $n \times n$ matrices with entries

$$\begin{aligned} A_{ij} &= \begin{cases} 1 \text{ for } j = i+1,\ i = 1, \dots, n-1\,, \\ 0 \text{ elsewhere}\,, \end{cases} \\ B &= \operatorname{diag}\{0, \dots, 0, p_n^{-1}\}\,, \\ C &= \operatorname{diag}\{p_0, \dots, p_n\}\,. \end{aligned} \tag{4}$$

We say that the solution (u, v) of (3) is generated by the solution y of (2).

Two points $x_1, x_2 \in I$ are said to be conjugate relative to a linear Hamiltonian system of the form (3) if there exists a solution (u, v) of this system such that $u(x_1) = 0 = u(x_2)$ and $u(x)$ is not identically zero between x_1 and x_2. Oscillation, nonoscillation, conjugacy and disconjugacy of (3) are defined by means of conjugate points in the same way as in the case of equation (2). It is obvious that two points are conjugate relative to (2) if and only if they are conjugate relative to (3) with the matrices A, B, C given by (4). This fact enables us to make use of the results of oscillation theory of linear Hamiltonian systems for the investigation of oscillation properties of (2).

Simultaneously with (3) consider the matrix system

$$\begin{aligned} U' &= A(x)U + B(x)V, \\ V' &= C(x)U - A^T(x)V, \end{aligned} \tag{5}$$

where U, V are $n \times n$ matrices. It is obvious that the columns of these matrices form solutions of (3). If these solutions of (3) are generated by solutions $y_1, \ldots, y_n$ of (2), we say that the matrix solution (U, V) of (5) is generated by $y_1, \ldots, y_n$. A self-conjugate solution (U_b, V_b) of (5) (i.e. $U_b^T(x)V_b(x) = V_b^T(x)U_b(x)$) is said to be principal at b if the matrix $U_b(x)$ is nonsingular near b and $\lim_{x \to b-} \left[\int_c^x U^{-1}(s)B(s)U^{T-1}(s)\,\mathrm{d}s\right]^{-1} = 0$, where $c \in I$ is sufficiently close to b. One can show that the solution (U_b, V_b) is principal at b if and only if there exists a self-conjugate solution (U, V) of (5) such that $U(x)$ is nonsingular near b, $U^T(x)V_b(x) - V^T(x)U_b(x) = E$ (the identity matrix) and $\lim_{x \to b-} U^{-1}(x)U_b(x) = 0$. The principal solution (U_a, V_a) at a is defined in the same way. System (3) is said to be identically normal on I whenever the trivial solution $(u, v) = (0, 0)$ is the only solution for which $u(x)$ vanishes on a nondegenerate subinterval of I. Two solutions (U_1, V_1), (U_2, V_2) of (5) are said to be linearly independent if every solution of (5) can be expressed in the form $(U, V) = (U_1C_1 + U_2C_2, V_1C_1 + V_2C_2)$, C_1, C_2 being constant $n \times n$ matrices. If (U_1, V_1), (U_2, V_2) are self-conjugate, one can show that they are linearly independent if and only if their "Wronskian" $U_1^T(x)V_2(x) - V_1^T(x)U_2(x)$ is nonsingular.

The following theorem describes a certain transformation of (5) and plays an important role in the further investigation.

Theorem A. [5, Th. 1]. *There exist $n \times n$ matrices of real-valued functions $H(x)$, $K(x) \in C^1(I)$, $H(x)$ being nonsingular, such that the transformation*

$$U = H(x)S, \; V = K(x)S + H^{T-1}(x)C \tag{6}$$

transforms (5) *into the so-called trigonometric system*

$$S' = Q(x)C, \; C' = -Q(x)S, \tag{7}$$

where

$$Q(x) = H^{-1}(x)B(x)H^{T-1}(x). \tag{8}$$

This theorem implies that a given pair of self-conjugate solutions (U_1, V_1), (U_2, V_2) of (5) for which $U_1^T V_2 - V_1^T U_2 = E$ can be expressed in the form

$$\begin{aligned}(U_1, V_1) &= (HS, KC + H^{T-1}S),\\ (U_2, V_2) &= (HC, -KS + H^{T-1}C),\end{aligned} \tag{9}$$

where (S, C) is a self-conjugate solution of (7) for which $S^T(x)S(x) + C^T(x)C(x) = E$ (the so-called trigonometric matrices). The matrix $H(x)$ is given by the relation

$$H(x)H^T(x) = U_1(x)U_1^T(x) + U_2(x)U_2^T(x).$$

Note that the concept of the trigonometric system was introduced by Barrett in connection with the extension of the Prüfer transformation to matrix systems. The possibility to transform (5) into the trigonometric system was proved in [5] and any $n \times n$ matrix $\mathscr{A}(x)$ for which $\mathscr{A}'(x) = Q(x)$ was called the phase matrix of (5) determined by the pair of solutions (U_1, V_1), (U_2, V_2). If $n = 1$ and $B(x) = 1$, then the phase matrix is identical with the phase function of the second order equation $u'' - C(x)u = 0$, introduced by Borůvka.

The followng statement concerns the oscillation behaviour of trigonometric systems. In a somewhat different form it can be found in [8] and under a stronger assumption ($Q(x)$ is positive definite) it is expressed in book [12].

Theorem B. *Let the matrix $Q(x)$ be nonnegative definite near b and neither $S(x)$ nor $C(x)$ can be identically singular on a nondegenerate subinterval of I for every self-conjugate solution (S, C) of (7). Then (7) is oscillatory at b if and only if*

$$\int^b \operatorname{Tr} Q(s)\, \mathrm{d}s = \infty, \tag{10}$$

where Tr( ) denotes the trace of the matrix indicated.

Now recall the extension of the Sturm comparison theorem to linear Hamiltonian systems. Let A_1, B_1, C_1 be $n \times n$ matrices of continuous, real-valued functions, B_1, C_1 being symmetric and B_1 nonnegative definite on I. Consider the system

$$\begin{aligned}U' &= A_1(x)U + B_1(x)V,\\ V' &= C_1(x)U - A_1^T(x)V.\end{aligned} \tag{11}$$

If the $2n \times 2n$ matrix

$$\begin{pmatrix} B_1(x) - B(x) & A_1(x) - A(x) \\ A_1^T(x) - A^T(x) & C(x) - C_1(x) \end{pmatrix}$$

is nonnegative definite, (11) is disconjugate on some interval $I_0 \subseteq I$ and both systems (5), (11) are identically normal on I_0, then (5) is also dinconjugate on I_0, see [4, Proposition 10]. Consequently, if the functions $\tilde{p}_j(x) \in C^j(I)$, $j = 0, \dots, n-1$, satisfy $\tilde{p}_j(x) > p_j(x)$ on I_0 and (2) is disconjugate on I_0, then the equation $(-1)^n(p_n y^{(n)})^{(n)} + \sum_{k=0}^{n-1} ((-1)^k \tilde{p}_k y^{(k)})^{(k)} = 0$ is also disconjugate on I_0.

3. Main results

Let $y_1, \dots, y_{2n}$ be linearly independent solutions of (2) and let (U_1, V_1), (U_2, V_2) be the solutions of the corresponding linear Hamiltonian system, generated by $y_1, \dots, y_n$ and $y_{n+1}, \dots, y_{2n}$, respectively. The system of solutions $y_1, \dots, y_{2n}$ is said to be normalized if (U_1, V_1), (U_2, V_2) are self-conjugate and $U_1^T(x)V_2(x) - V_1^T(x)U_2(x) = E$.

Theorem 1. *Let $y_1, \dots, y_{2n}$ be a normalized system of solutions of* (2) *and let (U_1, V_1), (U_2, V_2) be the solutions of* (5) *generated by $y_1, \dots y_n$ and $y_{n+1}, \dots, y_{2n}$, respectively. Equation* (2) *is oscillatory at b if and only if*

$$\int^b p_n^{-1}(x) e_n^T (U_1(x)U_1^T(x) + U_2(x)U_2^T(x))^{-1} e_n \, dx = \infty\,, \tag{12}$$

where $e_n = (0, \dots, 0, 1)^T \in R^n$.

Proof. By Theorem A the solutions (U_1, V_1), (U_2, V_2) can be expressed by (9), where (S, C) is a solution of (7) with the matrix Q given by (8). As both U_1, U_2 have only isolated singularities (see, e.g., [8]) and $H(x)$ is nonsingular, the same holds for S and C. Since the transformation (6) preserves oscillation behaviour, according to Theorem B it suffices to prove that (10) holds. $\operatorname{Tr} Q = \operatorname{Tr} H^{-1}BH^{T-1} = \operatorname{Tr} BH^{T-1}H^{-1} = \operatorname{Tr} p_n^{-1} \operatorname{diag}\{0, \dots, 0, 1\} (HH^T)^{-1} = p_n^{-1} \operatorname{Tr} \operatorname{diag}\{0, \dots, 0, 1\}(U_1U_2^T + U_2U_2^T)^{-1} = p_n^{-1} e_n (U_1U_2^T + U_2U_2^T)^{-1} e_n$, i. e., in view of (12) $\int^b \operatorname{Tr} Q(x)\, dx = \infty$ and (2) is oscillatory at b.

Now recall some fact concerning the classification of disconjugate differential equations and systems. Suppose that (1) is disconjugate on I. As this equation is a special case of (3), the definition of its principal solutions at a and b is the same as for (3). If the principal solutions at a and b of this equation are linearly idependent, then (1) is said to be 1-general on I. In the opposite case (i. e., if $y_a = y_b \cdot k$, k beging a nonzero real constant) (1) is said to be 1-special on I, see [3]. This classification of the second order equations can be extended to linear Hamiltonian systems in the following way (see [6]). Let (3) be disconjugate on I and let (U_a, V_a), (U_b, V_b) be the principal solutions at a and b of (5), respectively. The system (3) (or (5)) is said to be k-general on I if the rank of the matrix

$$\begin{pmatrix} U_a(x) & U_b(x) \\ V_a(x) & V_b(x) \end{pmatrix}$$

equals $n + k$, $k \in \{0, \dots, b\}$, for every $x \in I$. Analogously, equation (2) is said to be k-general on I whenever the corresponding linear Hamiltonian system has this property.

The next theorem generalizes the above given statement ii) concerning second order equations.

Theorem 2. *Let* (2) *be disconjugate and k-general on* I, $0 \leqslant k \leqslant n$. *If* $y_1, \dots \dots, y_{2n}$ *is a normalized system of solutions of of this equation and* (U_1, V_1), (U_2, V_2) *are the same as in Theorem* 1, *then*

$$(n-k)\pi \leqslant \int_a^b p_n^{-1}(x) e_n^T (U_1(x) U_1^T(x) + U_2(x) U_2^T(x))^{-1} e_n \, \mathrm{d}x \leqslant n\pi. \quad (13)$$

Proof. Let $\mathscr{A}(x)$ be a phase matrix of the linear Hamiltonian system corresponding to (2) determined by (U_1, V_1), (U_2, V_2). By Theorem A there exist $n \times n$ matrices $H(x), K(x) \in C^1(I)$ such that (U_1, V_1), (U_2, V_2) can be expressed by (9), where (S, C) is a self-conjugate solution of (7) with $Q = \mathscr{A}' = H^{-1} B H^{T-1}$, satisfying

$$S^T(x) S(x) + C^T(x) C(x) = E. \quad (14)$$

Denote by (S_a, C_a) the principal solution of (7) at a, for which (14) holds. Note that such a solution always exists, since the principal solution at a is the limit for $t_1 \to a-$ of the solutions $(S(x; t_1, t_2), C(x; t_1, t_2))$ given by the boundary condition $S(t_1, t_1, t_2) = 0$, $S(t_2, t_1, t_2) = E$, $t_1, t_2 \in I$, see [4, Chap. II], and $S^T S + C^T C = K$, K being a constant $n \times n$ matrix, for every solution of (7). Self-conjugacy of (S_a, C_a) and (14) imply

$$S_a(x) S_a^T(x) + C_a(x) C_a^T(x) = E. \quad (15)$$

Denote by $X = C_a + iS_a$, $G = XX^T = C_a C_a^T - S_a S_a^T + 2iS_a C_a^T$, $i^2 = -1$. One can verify directly that $X' = iQX$ and $G' = i(QG + GQ)$. The Jacobi formula yields $\det G(x) = \det^2 X(x) = \det^2 X(x_0).\ \exp\left\{2i \int_{x_0}^x \operatorname{Tr} Q(s) \, \mathrm{d}s\right\}$. Passing to the limit for $x_0 \to a-$ in the last expression, we have

$$\det G(x) = \exp\left\{2i \int_a^x \operatorname{Tr} Q(s) \, \mathrm{d}s\right\}. \quad (16)$$

Indeed, we have $\lim_{x \to a-} C_a^{-1}(x) S_a(x) = 0$, hence $\lim_{x \to a-} S_a(x) = 0$ (because by (14) $\| C_a^{-1}(x) \| \geqslant 1$, where $\| \ \|$ denotes the spectral matrix norm) and thus by (15)

$\lim_{x \to a-} X(x) = \lim_{x \to a-} C_a(x) = E$. The matrix $G(x)$ is unitary (i.e., $G^* = G^{-1}$, where "*" denotes the conjugate transpose of the matrix indicated), hence its eigenvalues lie on the unit circle in the complex plane. Denote these eigenvalues by $\exp\{i\alpha_j(x)\}$, $j = 1, \dots, n$, where $\alpha_j(x)$ are such that $\lim_{x \to a-} \alpha_j(x) = 0$. Then $\det G(x) = \prod_{j=1}^{n} \exp\{i\alpha_j(x)\} = \exp\left\{i \sum_{j=1}^{n} \alpha_j(x)\right\}$ and according to (16)

$$\sum_{j=1}^{n} \alpha_j(x) = 2 \int_a^x \operatorname{Tr} Q(s)\, ds. \tag{17}$$

Now we need the following statements whose proofs can be found, e.g., in [1, Chap. X].

i) If $\det S(x)$, $\det C(x)$ do not vanish on any nondegenerate subinterval of I, then $\alpha_j(x)$ are increasing functions.

ii) Det $S_a(x_0) = 0$ for some $x_0 \in I$ if and only if the nuber 1 is an eigenvalue of $G(x_0)$. The multiplicity of x_0 as the zero of $\det S_a(x_0)$ equals the dimension of the linear space generated by the eigenvectors of $G(x_0)$ corresponding to the eigenvalue 1.

Let (S_b, C_b) be the principal solution of (7) at b for which (14) holds. Similarly as above $\lim_{x \to b+} S_b(x) = 0$. Since equation (2) is k-general on I and the transformation (6) preserves this property, the system (7) is also k-general on I. It implies that there exist linearly indepenedent vectors $c_1, \dots, c_{n-k}$ such that $\lim_{x \to b+} S_a(x)c_j = 0, j = 1, \dots, n-k$. Hence, according to i) and ii), at least $n-k$ eigenvalues of $G(x)$ must make the full run around the unit circle, i.e., $\lim_{x \to b-} \left(\sum_{j=1}^{n} \alpha_j(x)\right) \geqslant 2(n-k)\pi$. On the other hand, disconjugacy of (7) implies that $\det S_a(x) \neq 0$ on I (see, e.g., [14]), i.e., $\lim_{x \to b-} \left(\sum_{j=1}^{n} \alpha_j(x)\right) \leqslant 2n\pi$. This, together with (17), completes the proof.

Remark. Consider the fourth order equation

$$(p(x)y'')'' = 0, \tag{18}$$

where $p(x) \in C^2(R), p(x) > 0$. After some computation one can verify that (18) is 2-general on R if and only if at last one of the integrals $\int_0^\infty x^2 p^{-1}(x)\, dx$, $\int_{-\infty}^0 x^2 p^{-1}(x)\, dx$ is convergent, it is 1-general if $\int_0^\infty x^2 p^{-1}(x)\, dx = \int_{-\infty}^0 x^2 p^{-1}(x)\, dx =$

$= \infty$ and at least one of the integrals $\int_0^\infty p^{-1}(x)\,dx$, $\int_{-\infty}^0 p^{-1}(x)\,dx$ is convergent, and it is 0-general if $\int_0^\infty p^{-1}(x)\,dx = \infty = \int_{-\infty}^0 p^{-1}(x)\,dx$, see [7]. Using this fact and Theorem 2 we can get estimates for certain improper integrals involving solutions of (18). Particularly, one can directly verify that $y_1 = 1$, $y_2 = x$, $y_3 = \int_0^x (x-t)p^{-1}(t)\,dt$, $y_4 = \int_0^x t(x-t)p^{-1}(t)\,dt$ form the normalized system of solutions of (18). In this case (13) reduces to the inequality

$$(2-k)\pi \leqslant \int_{-\infty}^{\infty} \frac{p^{-1}\sum_{i=1}^4 y_i^2}{\left(\sum_{i=1}^4 y_i^2\right)\left(\sum_{i=1}^4 y_i'^2\right) - \left(\sum_{i=1}^4 y_i y_i'\right)^2}\,dx \leqslant 2\pi$$

$0 \leqslant k \leqslant 2$, and if, e.g., $p(x) = 1$ we have the formula

$$\int_{-\infty}^{\infty} \frac{1 + x^2 + x^2/4 + x^6/36}{(1 + x^2 + x^4/4 + x^6/36)(1 + x^2 + x^4/4) - (x + x^3/2 + x^5/12)^2}\,dx = 2\pi$$

(which can be also derived by the residuum theorem, but the computations are rather complicated). Similar formulae can be also obtained for the integrals involving solutions of the higher order equations.

4. Open problems

Let us turn our attention to the statement iii) of Section 1. This statement can be proved as follows. Let $\alpha(x)$ be a phase function of (1), i.e., there exists a real-valued function $h(x)$ such that $p = \alpha'^{-1}h^2$, $q = -(\alpha'^{-1}h')' - h^2\alpha'$. Since (1) is supposed to be conjugate on $I_0 = (c, d)$, we have $|\alpha(d) - \alpha(c)| > \pi$. Let k be a real number for which $\pi/|\alpha(d) - \alpha(c)| < k < 1$ and set $\alpha_1 = k\alpha$, $h_1 = k^{1/2}h$, $\tilde p = \alpha_1'^{-1}h_1^2$, $\tilde q = -h_1(\alpha_1'^{-1}h_1) - h_1^2\alpha_1'$. Then $\tilde p = p$, α_1 is the phase function of $-(py')' + \tilde q y = 0$ and $\tilde q = h(\alpha'^{-1}h')' - k^2\alpha' h^2 > q$. Since $|\alpha_1(d) - \alpha_1(c)| = k|\alpha(d) - \alpha(c)| > \pi$, the last equation is also conjugate on I_0.

Follow this idea in the case of higher order equations. Let $\mathscr{A}(x)$ be a phase matrix of the linear Hamiltonian system corresponding to (2), i.e., there exist $n \times n$ matrices $H(x)$, $K(x)$ such that transformation (6) transforms this system into (7) with $Q(x) = \mathscr{A}'(x)$. Substituing (6) into (5) we have

$$B = HQH^T$$
$$C = (K' - H^{T-1}Q + A^TK)H^{-1}$$
$$A = H^{-1}(H' - BK).$$

Suppose that (2) is conjugate on I_0, i.e., it is the corresponding trigonometric system (7). Let $k \in (0, 1)$ be sufficiently close to 1 (it will be specified later). To prove that system (7) with $kQ(x)$ instead of $Q(x)$ is conjugate on I_0, we use the following statement.

Lemma. [4, Chap. II]. *Let $Q(x)$ be a nonnegative definite on I_0 such that* (7) *is identically normal on I_0. This system is conjugate on I_0 if and only if there exists a pair of vector-valued functions $y(x)$, $z(x)$ which are piecewise of the class C^1 and C, respectively, such that $y' = Q(x)z$,* supp $y \subset I_0$ *and* $\int_c^d (z^T(x)Q(x)z(x) - y^T(x)Q(x)y(x))\,\mathrm{d}x < 0$.

Let (y, z) be the pair of functions from Lemma and let $k \in (0, 1)$ be such that $\int_c^d z^TAz\,\mathrm{d}x - k^2 \int_c^d y^TQy\,\mathrm{d}x < 0$. Set $z_1 = z$, $y_1 = ky$, $Q_1 = kQ$. Then $y_1' = Q_1z_2$, supp $y_1 \subset I_0$ and $\int_c^d (z_1^TQ_1z_1 - y_1^TQ_1y_1)\,\mathrm{d}x = k\left[\int_c^d z^TQz\,\mathrm{d}x - k^2\int_c^d y^TQy\,\mathrm{d}x\right] < 0$. Consequently, system (7) with kQ instead of Q is conjugate on I_0.

Now, let $\mathscr{A}_1 = k\mathscr{A}$, $H_1 = k^{-1/2}H$, $K_1 = k^{-1/2}K$ and denote by $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ the matrices in the system of the form (5), which is transformed by (6), with H_1, K_1, into (7) with kQ. Then by (19) $\tilde{B} = H_1Q_1H_1^T = B$, $\tilde{A} = H_1^{-1}(H_1 - BK_1) = A$, $\tilde{C} = K_1'H_1 - H_1^{T-1}Q_1H_1^{-1} + A^TK_1H_1^{-1} = K'H^{-1} - k^2H^{T-1}QH^{-1} + A^TKH^{-1} = C + (1 - k^2)H^{T-1}QH^{-1}$, i.e., the matrix $\tilde{C} - C = (1 - k^2)H^{T-1}QH^{-1}$ is nonnegative definite and has rank 1 for every $x \in I_0$. From the last equality we see that the matrix $\tilde{C}(x)$ is not generally in the diagonal form. A linear Hamiltonian system with A, B given by (4) and C nondiagonal corresponds to a (formally) more general self-adjoint equation

$$(-1)^n(p_n(x)y^{(n)})^{(n)} + \sum_{k=0}^{n-1}(-1)^k\left[\sum_{j=0}^{n-1} p_{kj}(x)y^{(j)}\right]^{(k)} = 0, \tag{20}$$

where $p_{kj} = p_{jk} = C_{kj}$. Consequently, following the "second order" method we have obtained only a partial analogy of the statement iii), which is summarized in then next theorem.

Theorem 3. *Let* (2) *be conjugate on $I_0 = (c, d) \subseteq I$. There exist functions $p_{kj}(x)$, $k, j = 0, \ldots, n-1$, such that $p_{kj}(x) = p_{jk}(x)$, the matrix $(p_{kj}(x))_{k,j=0}^{n-1}$ –*

$-\operatorname{diag}\{p_0(x), \dots, p_{n-1}(x)\}$ *is nonnegative definite, has rank* 1 *for every* $x \in I_0$ *and equation* (20) *is also conjugate on* I_0.

The open question is whether or not the full analogy of iii) holds. This leads to the following conjecture.

Conjecture. *Suppose that* (2) *is conjugate on* $I_0 \subseteq I$. *There exist functions* $\tilde{p}_j(x) \in C^j(I_0)$ *such that* $\tilde{p}_j(x) > p_j(x)$ *on* I_0, $j = 0, \dots, n-1$, *and the equation* $\sum_{k=0}^{n} (-1)^k (\tilde{p}(x) y^{(k)})^{(k)} = 0$, $\tilde{p}_n(x) = p_n(x)$, *is also conjugate on* I_0.

The preceding investigation leads to a more general problem, which can be introduced in the following way: Consider a trigonometric system (7) and ask when the matrix $\int_c^x Q(s)\,\mathrm{d}x$, $c \in I$, is a phase matrix of some equation (2), i.e., when system (7) results (using trnaformation (6)) from a linear Hamiltonian system corresponding to (2). Resolving this problem, we would have a useful tool for the construction of higher order self-adjoint equations whose solutions have the described properties. Using this technique for the second order equations, Neuman [10], [11] solved several open problems in the qualitative theory of these equations.

Now investigate the inverse problem — to find a phase matrix of a given self-adjoint linear differential equation when the solutions of this equation are known. This problem is trivial for second order equations (the phase function $\alpha(x)$ is defined as a continuous function satisfying the equality $\operatorname{tg} \alpha(x) = y_1(x)$ $y_2(x)$, where y_1, y_2 are linearly independent solutions of the equation under consideration). In the higher order case one needs to solve certain first order linear differential systems with an antisymetric matrix, see [5]. In general, such systems can be solved explicitly only for 2×2 antisymetric matrices (which correspond to fourth order equations). In order to study the oscillation properties of self-adjoint equations via the properties of their phase matrices, it would be useful to know at least the phase matrices of some equation with relatively simple solutions, like $y^{(2n)} = 0$, $y^{(2n)} \pm y = 0$, etc.

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