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THREE SOLUTIONS OF A QUASILINEAR ELLIPTIC PROBLEM NEAR RESONANCE

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ABSTRACT. In this note, we show the existence of three solutions of the problem

$$-\Delta_p u - \lambda_1 |u|^{p-2}u + \varepsilon |u|^{p-2}u = f(x,u) + h(x) \qquad \text{in} \quad W_0^{1,p}(\Omega),$$

where $p \geq 2$ and $\varepsilon > 0$ is a small parameter. The result is suggested by a theorem of J. Mawhin and K. Schmitt. Our proof is based in a variational setting and uses elementary critical point theorems.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this note, we are concerned with the existence of three solutions of the nonlinear elliptic problem

$$-\Delta_p u - \lambda_1 |u|^{p-2}u + \varepsilon |u|^{p-2}u = f(x,u) + h \qquad \text{in} \quad W_0^{1,p}(\Omega), \tag{1}$$

where $p \geq 2$, $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the so called "p-Laplacian", $\varepsilon > 0$ is a small parameter, and $\lambda_1 > 0$ is the first eigenvalue of the problem

$$-\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

We recall that the first eigenvalue of (2) can be characterized by

$$\lambda_1 = \inf\left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x\,;\ u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega |u|^p \,\mathrm{d}x = 1 \right\}, \tag{3}$$

and is simple and isolated. Moreover, its corresponding eigenfunction φ_1 can be chosen to be positive. (cf., e.g., [2]).

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We study the problem (1) from a variational point of view. In fact, supposing that f has subcritical growth, it is well-known that the solutions of (1) are precisely the critical points of the C^1 functional

$$J_\varepsilon(u) = \frac{1}{p} \int_\Omega (|\nabla u|^p \, dx - \lambda_1 |u|^p) \, dx + \frac{\varepsilon}{p} \int_\Omega |u|^p \, dx - \int_\Omega \big(F(\cdot\,, u) + hu\big) \, dx\,,$$

defined in $W_0^{1,p}(\Omega)$, where $F(x,u) = \int_0^u f(x,t) \, dt$.

In [7], J. Mawhin and K. Schmitt proved the existence of at least three solutions of the two-point boundary value problem

$$-u'' - u + \varepsilon u = f(x,u) + h\,, \qquad u(0) = u(\pi) = 0\,,$$

for $\varepsilon > 0$ small enough and h orthogonal to $\sin x$ by assuming f bounded and satisfying the sign condition $uf(x,u) \geq 0$. Later, various papers related to their result have appeared. We mention for example [3], [4] and [6]. Notice that in all these papers, techniques from bifurcation and degree theory are used.

On the other hand, in [9], one of the authors studied a related problem for a fourth order equation using a variational argument; he also proved the existence of at least three solutions for $\varepsilon > 0$ small enough. Here, following [9], we assume:

(H_1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists $\theta > 1/p$ such that

$$\theta u f(x,u) - F(x,u) \to -\infty \qquad \text{as} \quad |u| \to \infty$$

uniformly in $x \in \Omega$.

(H_2) There exists $R > 0$ such that

$$uf(x,u) > 0 \quad \forall x \in \Omega\,, \ |u| \geq R\,.$$

Remarks.

(a) Note that (H_1) and (H_2) allow f to be unbounded, but with

$$-C_1 \leq F(x,u) \leq C_2 |u|^\sigma + C_3 \quad \forall x \in \overline{\Omega} \ \forall u \in \mathbb{R}\,, \tag{4}$$

where C_1, C_2, C_3 are positive constants, and $\sigma = \frac{1}{\theta} < p$. Consequently, for some $C > 0$, the following growth condition holds.

$$|f(x,u)| \leq C(1 + |u|)^{\sigma - 1} \quad \forall x \in \overline{\Omega} \text{ and } \forall u \in \mathbb{R}\,. \tag{5}$$

(b) If $a(x)$ is some continuous, positive function in $\overline{\Omega}$, and $\alpha \in (1,p)$, then $a(x)|u|^{\alpha-2}u$ satisfies (H_1) and (H_2).

(c) The existence of three solutions of (1) with $p = 2$, $N \geq 1$, and f unbounded was noticed in [4; Remark 2], but under the assumption of the Landesmann-Lazer condition,

$$\int_\Omega \Big[\limsup_{u\to-\infty} f(x,u)\Big]\varphi_1(x)\,\mathrm{d}x < \int_\Omega h(x)\varphi_1(x)\,\mathrm{d}x < \int_\Omega \Big[\liminf_{u\to+\infty} f(x,u)\Big]\varphi_1(x)\,\mathrm{d}x\,.$$

Since $f(u) = |u|^{p-2}u(1+|u|^p)^{-1}$ satisfies (H_1)–(H_2), our hypotheses do not imply the Landeman-Lazer condition, however, we need the sign condition (H_2) and $\int_\Omega h(x)\varphi_1(x)\,\mathrm{d}x = 0$.

THEOREM. *Suppose that $p \geq 2$ and conditions (H_1) and (H_2) are satisfied. Then for every $h \in L^{p'}(\Omega)$ with $\int_\Omega h(x)\varphi_1(x)\,\mathrm{d}x = 0$, problem (1) has at least three solutions if $\varepsilon > 0$ is small enough.*

Before going to the proof of the theorem, let us fix some notations. We use the norm $\|u\|_{W_0^{1,p}} = \Big(\int_\Omega |\nabla u|^p\,\mathrm{d}x\Big)^{1/p}$ in the Sobolev space $W_0^{1,p}(\Omega)$. The standard $L^p(\Omega)$ norm is denoted by $\|\cdot\|_p$. We also consider the following decomposition

$$W_0^{1,p}(\Omega) = \mathrm{Span}\{\varphi_1\} \oplus W\,,$$

where W is a closed complementary subspace of $\mathrm{Span}\{\varphi_1\}$. Then setting

$$\lambda_2 = \inf\left\{\frac{\int_\Omega |\nabla w|^p\,\mathrm{d}x}{\int_\Omega |w|^p\,\mathrm{d}x}\,;\ \ w \in W\setminus\{0\}\right\},$$

it follows from the simplicity and isolation of λ_1 that $\lambda_2 > \lambda_1$, and, by definition, for all $w \in W$,

$$\int_\Omega |w|^p\,\mathrm{d}x \leq \frac{1}{\lambda_2}\int_\Omega |\nabla w|^p\,\mathrm{d}x\,. \tag{6}$$

LEMMA 1. *For every $\varepsilon > 0$, J_ε is coercive in $W_0^{1,p}(\Omega)$. Moreover, there exists a constant $m > 0$, independent of ε, such that $\inf_W J_\varepsilon \geq -m$ $\forall\,\varepsilon > 0$.*

Proof. Choosing $0 < \varepsilon < \lambda_1$ it follows from (3) that

$$J_\varepsilon(u) \geq \frac{\varepsilon}{p\lambda_1}\|u\|^p_{W_0^{1,p}} - \int_\Omega \big(F(x,u(x)) - h(x)u(x)\big)\,\mathrm{d}x\,.$$

Using (4) and the fact that $\sigma < p$, we have that J_ε is coercive for every $\varepsilon > 0$.

Now, from the inequalities (6) and (4), we have, for all $w \in W$,

$$J_\varepsilon(w) \ge \frac{\lambda_2 - \lambda_1}{p\lambda_2} \|w\|^p_{W_0^{1,p}} - C_2\|w\|^\sigma_\sigma - C_3|\Omega| - \|h\|_{p'}\|w\|_p ,$$

and since $\sigma < p$, it follows that

$$J_\varepsilon(w) \ge k_1\|w\|^p_{W_0^{1,p}} - k_2\|w\|^\sigma_{W_0^{1,p}} - k_3\|w\|_{W_0^{1,p}} - k_4 \quad \forall \varepsilon > 0 ,$$

for some constants $k_i > 0$, $i = 1,2,3,4$, independent of $\varepsilon > 0$. Hence J_ε is coercive in W, and in particular, it is bounded from below in W. This ends the proof. □

Next we check a compactness property of J_ε. Let $\mathcal{O}$ be an open set in $W_0^{1,p}(\Omega)$. One says that J_ε satisfies the Palais-Smale condition in $\mathcal{O}$ at level $c \in \mathbb{R}$, which we write as $(PS)_{c,\mathcal{O}}$ for short, if every sequence $u_n \in \mathcal{O}$ such that $J_\varepsilon(u_n) \to c$ and $\|J'_\varepsilon(u_n)\|_* \to 0$ has a convergent subsequence in $\mathcal{O}$. When $\mathcal{O}$ is the whole space, and $(PS)_{c,\mathcal{O}}$ holds for every $c \in \mathbb{R}$, one says that J_ε satisfies the Palais-Smale condition (PS).

LEMMA 2. *For any $\varepsilon > 0$, J_ε satisfies (PS). Moreover, setting*

$$\mathcal{O}^\pm = \{u \in W_0^{1,p}(\Omega) ;\ u = \pm t\varphi_1 + w \text{ with } t > 0 \text{ and } w \in W\} ,$$

J_ε satisfies both $(PS)_{c,\mathcal{O}^+}$ and $(PS)_{c,\mathcal{O}^-}$ for every $c < -m$.

Proof. Let (u_n) be a sequence satisfying $J_\varepsilon(u_n) \to c$ and $\|J_\varepsilon(u_n)\|_* \to 0$. Since J_ε is coercive, we have necessarily that (u_n) is bounded. Then there exists a subsequence, which we still denote by (u_n), such that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$. To conclude that (u_n) has a convergent subsequence, we compute

$$\begin{aligned} J'_\varepsilon(u_n)(u_n - u) &= \langle -\Delta_p u_n, u_n - u\rangle - (\lambda_1 - \varepsilon) \int_\Omega |u_n|^{p-2}u_n(u_n - u)\, dx \\ &\quad - \int_\Omega (f(\cdot\,, u_n) + h)(u_n - u)\, dx \\ &= \delta_n \|u_n - u\|_{W_0^{1,p}} \qquad (\delta_n \to 0) . \end{aligned}$$

Now, from the growth condition (5), the Nemytskii mapping $N_f u = f(\cdot\,, u_n)$ is continuous from $L^p(\Omega)$ into $L^{p'}(\Omega)$, so that

$$\lim_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle = 0 .$$

But, as is well known, $-\Delta_p$ is of class (S_+) from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ (see, e.g., [10] or [5]), and hence $u_n \to u$ strongly.

For the second part of the lemma, let $(u_n) \subset \mathcal{O}^+$ be such that $J_\varepsilon(u_n) \to c < -m$ and $\|J_\varepsilon(u_n)\|_* \to 0$. As above, there exists $u \in W_0^{1,p}(\Omega)$ and a subsequence, still denoted by u_n, such that $u_n \to u$ strongly. So we must show that $u \in \mathcal{O}^+$. Indeed, if $u \in \partial\mathcal{O}^+ = W$, then from Lemma 1, $J_\varepsilon(u) = c \geq -m$, which contradicts the fact that $c < -m$. The proof of $(PS)_{c,\mathcal{O}^-}$ is similar. □

LEMMA 3. *If $\varepsilon > 0$ is small enough, there exist $t^- < 0 < t^+$ such that $J_\varepsilon(t^\pm\varphi_1) < -m$.*

P r o o f. The proof is similar to [9; Theorem 4], where it is assumed that $p = 2$ and $N = 1$. For the reader's convenience, we sketch it here. So, let us normalize φ_1 such that $0 \leq \varphi_1 \leq 1$ $\forall x \in \overline{\Omega}$. Given $S > R$, by (H_2), there exists $\varepsilon_S > 0$ such that $\varepsilon_S u^{p-1} < f(x,u)$ in $\overline{\Omega} \times [R,S]$. Then, if $\varphi_1(x) > R/S$, we have

$$\varepsilon_S S^{p-1}\varphi_1(x)^{p-1} < f\big(x, S\varphi_1(x)\big)\,.$$

Now, setting $A(S) = \big\{x \in \overline{\Omega};\ \varphi_1(x) > R/S\big\}$ and $B(S) = \overline{\Omega} \setminus A(S)$,

$$\begin{aligned} J_{\varepsilon_S}(S\varphi_1) &= \frac{1}{p}\int_\Omega \varepsilon_S S^p \varphi_1^p \,\mathrm{d}x - \int_\Omega F\big(x, S\varphi_1(x)\big)\,\mathrm{d}x \\ &< \int_{A(S)} \Big(\frac{1}{p} S\varphi_1 f(x, S\varphi_1) - F(x, S\varphi_1)\Big)\,\mathrm{d}x \\ &\quad + \int_{B(S)} \Big(\frac{\varepsilon_S}{p} S^p \varphi_1^p - F(x, S\varphi_1)\Big)\,\mathrm{d}x\,. \end{aligned}$$

Since the integral over $B(S)$ is bounded independently of ε_S and S, it follows from (H_1) and Fatou's lemma that $J_{\varepsilon_S}(S\varphi_1) \to -\infty$ if $S \to +\infty$. Of course, if we take $S < -R$, we should derive that $J_{\varepsilon_S}(S\varphi_1) \to -\infty$ as $S \to -\infty$, then the proof is finished by noting that $J_\varepsilon(S\varphi_1) < J_{\varepsilon_S}(S\varphi_1)$ $\forall \varepsilon \leq \varepsilon_S$. □

P r o o f o f t h e T h e o r e m . For $\varepsilon > 0$ small enough, we have from Lemmas 2 and 3 that

$$-\infty < \inf_{\mathcal{O}^\pm} J_\varepsilon < -m\,,$$

and since $(PS)_{c,\mathcal{O}^\pm}$ holds for all $c < -m$, it follows from the deformation lemma that the infima are attained, say at $u^- \in \mathcal{O}^-$ and $u^+ \in \mathcal{O}^+$. Since $\mathcal{O}^\pm$ are open in $W_0^{1,p}(\Omega)$, we have found two distinct critical points of J_ε.

Now applying the mountain pass lemma of Ambrosetti-Rabinowitz [1], the number

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\varepsilon\big(\gamma(t)\big)\,,$$

where $\Gamma = \{\gamma \in C([0,1]; W_0^{1,p}(\Omega))\,;\ \gamma(0) = u^- \text{ and } \gamma(1) = u^+\}$, is a critical value of J_ε since (PS) holds for every $\varepsilon > 0$ (Lemma 2). Noting that $\gamma([0,1]) \cap W \neq \emptyset$ $\forall \gamma \in \Gamma$, we conclude that $c \geq \inf_W J_\varepsilon \geq -m$ (Lemma 1), and once $J_\varepsilon(u^\pm) < -m$, we have found a third critical point of J_ε. This proves the theorem. □

Note added in proof. As in the case of semilinear elliptic equations studied in [8], it is easy to see that assumptions $(H1)-(H2)$ may be weakened to:

$f(x,u) = o(|u|^{p-1})$ uniformly in x as $|u| \to \infty$,
$F(x,u)$ is bounded below and

$$\lim_{|S|\to\infty} \int_\Omega F(x, S\phi(x))\, \mathrm{d}x = +\infty\,.$$

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