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OSCILLATION THEOREMS OF COMPARISON TYPE FOR NEUTRAL NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

We consider the neutral equation

$$(1)\qquad \big(x(t)+p(t)x(g_*(t))\big)^{(n)}+q(t)f\big(x(g(t))\big)=0$$

and the forced neutral equation

$$(2)\qquad \big(x(t)+p(t)x(g_*(t))\big)^{(n)}+q(t)f\big(x(g(t))\big)=e(t),$$

where n is even, e, g, g_*, p, q: $[t_0,\infty)\to\mathbb{R}=(-\infty,\infty)$, $t_0>0$ and $f\colon\mathbb{R}\to\mathbb{R}$ are continuous, $q(t)\geqslant 0$ and not identically zero on any ray of the form $[t^*,\infty)$, $t^*\geqslant t_0$, $\lim\limits_{t\to\infty} g(t)=\infty$ and $\lim\limits_{t\to\infty} g_*(t)=\infty$.

By a solution of equation (1) (or (2)) we mean a function $x\colon[T_x,\infty)\to\mathbb{R}$, $T_x\geqslant t_0$, such that $x(t)+p(t)x\big(g_*(t)\big)$ is n-times continuously differentiable and satisfies equation (1) (or (2)) for all sufficiently large $t\geqslant T_x\cdot A$ solution of equation (1) (or (2)) is said to be oscillatory if it has an infinite sequence of zeros tending to infinity; otherwise, a solution is said to be nonoscilatory. Equation (1) (or (2)) is said to be oscillatory if all its solutions are oscillatory.

Besides its theoretical interest, the study of the oscillatory behavior of solutions of neutral differential equations has some importance in many applications. Recently there has been a lot of activity in establishing sufficient conditions for the oscillation of neutral equations of type (1) and/or related equations. See, for example [4–8, 13] and the references cited therein. However, theorems on the oscillatory behavior of equations (1) and (2) (f is not a monotonic function) via comparison with that of some linear second order differential equations are in general scarce in the literature.

The purpose of this paper is to relate the oscillation problem of equations (1) and (2) to that of some linear second order equations. In Section 3 we intend to reduce the study of the oscillatory properties of equation (1) to that of linear second order equation and present four oscillation criteria for equation (1) by examining the following for cases for p and g_*: $p(t) = 0$, $\{0 \leqslant p(t) < 1, g_*(t) < t\}$, $\{p(t) > 1, g_*(t) > t\}$, $\{-1 < p(t) < 0, g_*(t) < t\}$ and in Section 4 we intend to extend the results of Section 3 to equation (2).

The results of this paper are presented in a form which is essentially new and it offer alternative means of classifying such equations with respect to oscillation.

2. Preliminaries

We denote by

$$\mathbb{R}_{t_0} = (-\infty, -t_0] \cup [t_0, \infty) \quad \text{for any } t_0 > 0,$$

and we consider the spaces:

$$C(\mathbb{R}) = \{f\colon \mathbb{R} \to \mathbb{R}\colon f \text{ is continuous and } xf(x) > 0 \text{ for } x \neq 0\}$$

and

$$C_B(\mathbb{R}_{t_0}) = \{f \in C(\mathbb{R})\colon f \text{ is of bounded variation on any interval } [a,b] \subset \mathbb{R}_{t_0}\}.$$

For our purpose, we need the following three lemmas. The first two lemmas can be found in [2], [10] and [15] while for the third one, we refer to [14].

Lemma 1. *Let u be a positive and n-times differentiable function on an interval $[t_0, \infty)$ with its n-th derivative $u^{(n)}$ nonpositive on $[t_0, \infty)$ and not identically zero on any interval of the form $[t^*, \infty)$, $t^* \geqslant t_0$. Then there exists a $t_u \geqslant t_0$ and an integer L, $0 \leqslant L \leqslant n$ with $n + L$ odd and such that for $t \geqslant t_u$*

$$L \leqslant n-1 \text{ implies } (-1)^{L+j} u^{(j)}(t) > 0, \quad (j = L, L+1, \dots, n-1),$$
$$L > 1 \text{ implies } u^{(j)}(t) > 0, \quad (j = 1, 2, \dots, L-1).$$

Lemma 2. *Let u be as in Lemma 1, and n be even. Then for any constants a and a^*, $0 < a$, $a^* < 1$ and all large t*

$$x'(t/2) \geqslant \frac{a}{(n-2)!} t^{n-2} x^{(n-1)}(t)$$

and

$$x(t) \geqslant \frac{a^*}{(n-1)!} t^{n-1} x^{(n-1)}(t).$$

Lemma 3. *Suppose $t_0 > 0$ and $f \in C(\mathbb{R})$. Then $f \in C_B(\mathbb{R}_{t_0})$ if an only if $f(x) = G(x) \cdot H(x)$ for all $x \in \mathbb{R}_{t_0}$ where $G\colon \mathbb{R}_{t_0} \to (0,\infty)$ is nondecreasing on $(-\infty, -t_0)$ and nonincreasing on (t_0, ∞) and $H\colon \mathbb{R}_{t_0} \to \mathbb{R}$ and nondecreasing on $\mathbb{R}_{t_0}$.*

We assume that there exists a differentiable function $h\colon [t_0, \infty) \to (0,\infty)$ such that

$$h(t) \leqslant \min\{t, g(t)\}, \quad h'(t) > 0 \quad \text{for } t > t_0 \text{ and } \lim_{t\to\infty} h(t) = \infty. \tag{3}$$

For $T \geqslant t_0$ and all $t \geqslant T$, we let

$$r(t) = \left(h^{n-2}(t)h(t)\right)^{-1}.$$

3. Oscillation of equation (1)

The following criterion is concerned with the oscillatory behavior of equation (1) when $p(t) = 0$.

Theorem 1. *Suppose $f \in C(\mathbb{R}_{t_0})$, $t_0 > 0$ and let G and H be a pair of continuous components of f with H being the nondecreasing one. Moreover, assume that condition (3) holds, $p(t) = 0$ and*

$$H(x) \operatorname{sgn} x \geqslant |x|^c \quad \text{for } x \neq 0 \text{ and } c \text{ is a positive constant.} \tag{4}$$

If for every constant $C \geqslant 1$, the linear equation

$$\left(r(t)y'(t)\right)' + \frac{1}{2((n-2)!)} G\left(Cg^{n-1}(t)\right) q(t) Q(t) y(t) = 0 \tag{5}$$

is oscillatory, where

$$Q(t) = \begin{cases} a_1, & \text{any positive constant} \quad \text{if } c > 1, \\ a_2, & \text{any constant}, 0 < a_2 < 1 \quad \text{if } c = 1, \\ a_3 h^{(c-1)(n-1)}(t), & a_3 \text{ is any constant}, \ 0 < a_3 < 1 \quad \text{if } 0 < c < 1, \end{cases} \tag{6}$$

then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). We may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_0 > 0$, since a parallel argument holds if $x(t) < 0$ for $t \geqslant t_0$. By Lemma 1, there exists a $t_1 \geqslant t_0$ such that

$$x'(t) > 0 \quad \text{and} \quad x^{(n-1)} > 0 \quad \text{for } t \geqslant t_1. \tag{7}$$

Since $x(t)$ is an increasing function and $x^{(n-1)}(t)$ is a decreasing function for $t \geqslant t_1$, there exist positive constants k and k_1 such that for $t \geqslant t_1$

$$x(h(t)) \geqslant k \tag{8}$$

and

$$x^{(n-1)}(t) \leqslant k_1.$$

By successive integration from t_1 to t, we conclude that there exist a $t_2 \geqslant t_1$ and a constant $k^* \geqslant 1$ such that

$$x(g(t)) \leqslant k^* g^{n-1}(t) \quad \text{and} \quad x(h(t)) \leqslant k^* h^{n-1}(t) \quad \text{for } t \geqslant t_2. \tag{9}$$

Furthermore, let us consider an arbitrary constant b with $b > 1$. Then, by applying Lemma 2, we conclude that there exists a large $t_3 \geqslant 2t_2$ such that

$$x'(h(t)/2) \geqslant \frac{h^{n-2}(t)}{b(n-2)!} x^{(n-1)}(t) \quad \text{for } t \geqslant t_3. \tag{10}$$

Next, we define the function W by

$$W(t) = -\frac{x^{(n-1)}(t)}{x(h(t)/2)} \quad \text{for } t \geqslant t_3.$$

Then for $t \geqslant t_3$, we get

$$\begin{aligned} W'(t) &= q(t)\frac{f(x(g(t)))}{x(h(t)/2)} + \frac{x^{(n-1)}(t)x'(h(t)/2)(h'(t)/2)}{x^2(h(t)/2)} \\ &= F(t)q(t) + \frac{1}{P(t)}W^2(t), \end{aligned} \tag{11}$$

where

$$F(t) = \frac{f(x(g(t)))}{x(h(t)/2)} \quad \text{and} \quad P(t) = \frac{x^{(n-1)}(t)}{x'(h(t)/2)(h'(t)/2))}. \tag{12}$$

The Ricatti equation (11) has a solution on $[t_3, \infty)$. It is well-known that this is equivalent to the nonoscillation of the linear equation

$$(13) \qquad \big(P(t)u'(t)\big)' + q(t)F(t)u(t) = 0.$$

Using (3), (9) and (10) in (12) we have

$$(14) \qquad P(t) = \frac{2x^{(n-1)}(t)}{x'(h(t)/2)(h'(t)/2)} \leqslant \frac{2b(n-2)!}{h'(t)h^{n-2}(t)} = \big(2b(n-2)!\big)r(t) \quad \text{for } t \geqslant t_3,$$

and

$$(15) \qquad \begin{aligned} F(t) = \frac{G(x(g(t)))H(x(g(t)))}{x(h(t)/2)} &\geqslant \frac{G(k^* g^{n-1}(t))x^c(h(t))}{x(h(t)/2)} \\ &\geqslant G\big(k^* g^{n-1}(t)\big)x^{c-1}\big(h(t)\big)\frac{x(h(t))}{x(h(t)/2)} \\ &\geqslant G\big(k^* g^{n-1}(t)\big)x^{c-1}\big(h(t)\big). \end{aligned}$$

Now, there are three cases to consider:

Case 1. $c > 1$. From (8) it follows that

$$x^{c-1}\big(h(t)\big) \geqslant k^{c-1} \quad \text{for } t \geqslant t_3$$

and hence (15) becomes

$$F(t) \geqslant k^{c-1}G\big(k^* g^{n-1}(t)\big) \quad \text{for } t \geqslant t_3.$$

Case 2. $c = 1$. In this case

$$F(t) \geqslant G\big(k^* g^{n-1}(t)\big) \quad \text{for } t \geqslant t_3.$$

Case 3. $0 < c < 1$. From (9), we have

$$x^{c-1}\big(h(t)\big) \geqslant \big(k^* h^{n-1}(t)\big)^{c-1} \quad \text{for } t \geqslant t_3$$

and hence (15) becomes

$$F(t) \geqslant (k^*)^{c-1}\big(h^{n-1}(t)\big)^{c-1}G\big(k^* g^{n-1}(t)\big) \quad \text{for } t \geqslant t_3.$$

Thus an application of the Picone Sturm Comparison Theorem (see [11]) to equation (12) yields the nonoscillation of the linear equation

$$\big(r(t)y'(t)\big)' + \frac{1}{2b(n-2)!}G\big(k^* g^{n-1}(t)\big)q(t)Q^*(t)y(t) = 0,$$

where

$$Q^*(t) = \begin{cases} k^{c-1} & \text{if } c > 1, \\ 1 & \text{if } c = 1, \\ k^{*c-1} h^{(c-1)(n-1)}(t) & \text{if } 0 < c < 1. \end{cases}$$

This contradicts the hypothesis that equation (5) is oscillatory. This completes the proof. □

In the following theorem, we assume that

$$(16) \qquad 0 \leqslant p(t) \leqslant p_0 < 1, \; g_*(t) < t \text{ and } g_* \text{ is strictly increasing for } t \geqslant t_0.$$

Theorem 2. *Let $f \in C(\mathbb{R}_{t_0})$, $t_0 > 0$ and let the functions G and H be defined as in Theorem 1. Moreover, suppose that conditions (3), (4) and (16) hold and for every constant $C \geqslant 1$, the linear equation*

$$(17) \qquad \big(r(t)y'(t)\big)' + \frac{(1-p_0)^c}{2((n-2)!)} G\big(Cg^{n-1}(t)\big)q(t)Q(t)y(t) = 0$$

is oscillatory, where $Q(t)$ is defined by (6). Then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$, $x\big(g_*(t)\big) > 0$ and $x\big(g(t)\big) > 0$ for $t \geqslant t_0 > 0$. Put

$$(18) \qquad z(t) = x(t) + p(t)x\big(g_*(t)\big).$$

Then $z(t) > 0$ for $t \geqslant t_0$ and equation (1) takes the form

$$(19) \qquad z^{(n)}(t) = -q(t)f\big(x(g(t))\big) \leqslant 0 \quad \text{for } t \geqslant t_0.$$

By Lemma 1, there exists a $t_1 \geqslant t_0$ such that

$$(20) \qquad z'(t) > 0 \quad \text{and} \quad z^{(n-1)}(t) > 0 \quad \text{for } t \geqslant t_1.$$

Since $z(t)$ is an increasing function and $z^{(n-1)}(t)$ is decreasing for $t \geqslant t_1$. Then there exist positive constants k and k_1 such that for all $t \geqslant t_1$

$$(21) \qquad z\big(h(t)\big) \geqslant k$$

and

$$(22) \qquad z^{(n-1)}(t) \leqslant k_1.$$

As in the proof of Theorem 1, there exist a constant $k^* \geqslant 1$ and a $t_2 \geqslant t_1$ such that

(23) $$x\big(g(t)\big) \leqslant z\big(g(t)\big) \leqslant k^* g^{n-1}(t) \quad \text{and} \quad x\big(h(t)\big) \leqslant k^* h^{n-1}(t) \quad \text{for } t \geqslant t_2.$$

Next, using (16) and (20) in (18), we obtain

$$\begin{aligned} x(t) &= z(t) - p(t)x\big(g_*(t)\big) \\ &= z(t) - p(t)\big[z\big(g_*(t)\big) - p\big(g_*(t)\big)x\big(g_* \circ g_*(t)\big)\big] \\ &\geqslant z(t) - p(t)z\big(g_*(t)\big) \\ &\geqslant (1-p_0)z(t) \quad \text{for } t \geqslant t_1. \end{aligned}$$

Thus, there exists a $t_3 \geqslant t_2$ such that

(24) $$x\big(g(t)\big) \geqslant (1-p_0)z\big(g(t)\big) \quad \text{for } t \geqslant t_3.$$

Using (23) and (24) in equation (19) we have

$$z^{(n)}(t) + (1-p_0)^c G\big(k^* g^{n-1}(t)\big)q(t)z^c\big(g(t)\big) \leqslant 0 \quad \text{for } t \geqslant t_3.$$

Therefore, as pointed out by Foster and Grimmer [1], the equation

$$z^{(n)}(t) + (1-p_0)^c G(k^* g^{n-1}(t)\big)g(t)z^c\big(g(t)\big) = 0,$$

has a positive solution. The rest of the proof proceeds as in the proof of Theorem 1. This completes the proof. □

The following criterion deals with the oscillation of equation (1) when the functions p and g_* satisfy the following conditions:

(25) $1 < p_1 \leqslant p(t) \leqslant p_2$, g_* is strictly increasing and $g_*(t) > t$ for $t \geqslant t_0$,

and

(26) there exists a positive differentiable function $h_*\colon [t_0, \infty) \to (0, \infty)$ such that $h_*(t) \leqslant \min\{t, g_*^{-1} \circ g(t)\}$, $h_*'(t) > 0$ for $t \geqslant t_0$ and $\lim\limits_{t\to\infty} h_*(t) = \infty$, where g_*^{-1} denotes the inverse function of g_*.

We let
$$p^* = \frac{p_1 - 1}{p_1 p_2} \quad \text{and} \quad r^*(t) = \big(h_*'(t)h_*^{n-2}(t)\big)^{-1}.$$

Theorem 3. *Suppose $f \in C(\mathbb{R}_{t_0})$, $t_0 > 0$, the functions G and H are defined as in Theorem 1 and let conditions (4), (25) and (26) hold. If for every $C \geqslant 1$ the linear equation*

$$(27)\qquad (r^*(t)y'(t))' + \frac{p^{*c}}{2((n-2)!)} Q_1(t) G\big(Cg^{n-1}(t)\big) q(t) y(t) = 0$$

is oscillatory, where Q_1 is the same as Q defined by (6) with h replaced by h_*, then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) and assume that $x(t) > 0$, $x\big(g_*(t)\big) > 0$ and $x\big(g(t)\big) > 0$ for $t \geqslant t_0 > 0$. As in the proof of Theorem 2, we define the function $z(t)$ by (18) and assume that there exists a $t_2 \geqslant t_1 \geqslant t_0$ such that (20)–(22) hold for $t \geqslant t_1$ and (23) holds for $t \geqslant t_2$.

Next, using (20) and (25) in (18), we have

$$\begin{aligned}
x(t) &= \frac{z(g_*^{-1}(t)) - x(g_*^{-1}(t))}{p(g_*^{-1}(t))}\\
&= \frac{z(g^{-1}(t))}{p(g_*^{-1}(t))} - \frac{1}{p(g_*^{-1}(t))}\Big(\frac{z(g_*^{-1} \circ g_*^{-1}(t)) - x(g_*^{-1} \circ g_*^{-1}(t))}{p(g_*^{-1} \circ g_*^{-1}(t))}\Big)\\
&\geqslant \frac{z(g_*^{-1}(t))}{p(g_*^{-1}(t))} - \frac{z(g_*^{-1} \circ g_*^{-1}(t))}{p(g_*^{-1}(t))p(g_*^{-1} \circ g_*^{-1}(t))}\\
&\geqslant \frac{p_1 - 1}{p_1 p_2} z\big(g_*^{-1}(t)\big) \quad \text{for } t \geqslant t_1.
\end{aligned}$$

Thus, there exists a $t_3 \geqslant t_2$ such that

$$(28)\qquad x\big(g(t)\big) \geqslant p^* z\big(g_*^{-1} \circ g(t)\big) \quad \text{for } t \geqslant t_3.$$

Using (4), (23) and (28) in equation (19), we have

$$z^{(n)}(t) + p^{*^c} G\big(k^* g^{n-1}(t)\big) q(t) z^c\big(g_*^{-1} \circ g(t)\big) \leqslant 0 \quad \text{for } t \geqslant t_3.$$

Applying the same argument as above, we led to the desired contradiction. □

The following theorem is concerned with the oscillatory behavior of equation (1) when the function $g \cdot g_*$, f and p satisfy the following conditions:

$$(29)\qquad -p_* < p(t) < 0, \text{ for some } p_*,\ 0 < p_* < 1,\ g_*(t) \text{ and } g^*(t) = g_*^{-1} \circ g(t)$$

are increasing, $g_*(t) < t$ and $g^*(t) < t$ for $t \geqslant t_0$, and

$$(30)\qquad f(x) \operatorname{sgn} x \geqslant |x|^c \quad \text{for } x \neq 0 \text{ and } c > 0.$$

Theorem 4. *Let $f \in C(\mathbb{R}_{t_0})$, $t_0 > 0$ and let conditions* (29) *and* (30) *hold. If the linear equation*

$$(r(t)y'(t)) + \frac{1}{2((n-2)!)}Q(t)q(t)y(t) = 0 \tag{31}$$

is oscillatory where the function Q is defined by (6), *and all bounded solutions of the equation*

$$w^{(n)}(t) - q(t)\left(\left|w(g^*(t))\right|^c\right)\operatorname{sgn} w(g^*(t)) = 0 \tag{32}$$

are oscillatory, then equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$, $x(g_*(t)) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_0 > 0$. We define the function $z(t)$ by (18). Then for $t \geqslant t_1 \geqslant t_0$, $z^{(n)}(t) \leqslant 0$ and $z^{(n-1)}(t)$ is of one sign. We shall show that $z^{(n-1)}(t) > 0$ for $t \geqslant t_1$. In fact, if $z^{(n-1)}(t) \leqslant 0$ for $t \geqslant t_1$, there exists a $t_2 \geqslant t_1$ so that

$$z^{(n-1)}(t) \leqslant -b < 0 \quad \text{for some } b > 0 \text{ and } t \geqslant t_2.$$

Hence

$$z(t) \to -\infty \quad \text{as } t \to \infty. \tag{33}$$

On the other hand, if $z(t) < 0$ for $t \geqslant t_2$, then we have

$$\begin{aligned} 0 < x(t) &< -p(t)x(g_*(t)) < p_*x(g_*(t)) \\ &< p_*^2 x(g_* \circ g_*(t)) < \ldots < p_*^m x(g_{*_m}(t)), \end{aligned}$$

where we define g_{*_m} as follows:

$$\begin{aligned} g_{*_1}(t) &= g_*(t), \\ g_{*_m}(t) &= g_* \circ g_{*_{m-1}}(t), \quad m > 1. \end{aligned}$$

We note that for any t, $g_{*_m}(t) < t$ and hence for each t and arbitrary m, $x(g_{*_m}(t))$ is well-defined. Since $p_*^m \to 0$ as $m \to \infty$, we conclude that $x(t) \to 0$ as $t \to \infty$. Consequently, $z(t) \to 0$ at $t \to \infty$, which contradicts (33). Therefore, we must have $z^{(n-1)}(t) > 0$ for $t \geqslant t_1$, and hence by Lemma 1, we see that $z'(t) > 0$ for $t \geqslant t_1$. Next, we consider the following two cases:

Case 1. Let $z(t) > 0$ for $t \geqslant t_1$. From (18) and (29) we have

$$x(t) > z(t) \quad \text{for } t \geqslant T \geqslant t_1. \tag{34}$$

Using (30) and (34) in equation (19), we obtain

$$z^{(n)}(t) + q(t)z^c\big(g(t)\big) \leqslant z^{(n)}(t) + q(t)\frac{f(x(g(t)))}{x^c(g(t))}x^c\big(g(t)\big) = 0,$$

for $g \geqslant T$. By a result in [1], it follows that the equation

$$y^{(n)}(t) + q(t)y^c(t) = 0,$$

has a positive solution. Now, an application of Theorem 1, yields a desired conclusion.

Case 2. Let $z(t) < 0$ for $t \geqslant t_1$. From the above proof we see that $z(t) \to 0$ as $t \to \infty$ and $z^{(n-1)}(t) > 0$ and $z'(t) > 0$ for $t \geqslant t_1$. Now, we let $v(t) = -z(t) > 0$ and hence we have

$$(-1)^i v^{(i)}(t) > 0 \quad \text{for } i = 0, 1, \ldots, n-1, \text{ and } t \geqslant t_1. \tag{35}$$

From (18) and (29), if follows that

$$x\big(g_*(t)\big) \geqslant -\frac{1}{p(t)}v(t) \geqslant v(t) \quad \text{for } t \geqslant t_1,$$

and hence there exists a $T_1 \geqslant t_1$ so that

$$x\big(g(t)\big) \geqslant v\big(g_*^{-1} \circ g(t)\big) = v\big(g^*(t)\big) \quad \text{for } t \geqslant T_1. \tag{36}$$

Using (30), (35) and (36) we get

$$\begin{aligned} v^{(n)}(t) &\geqslant q(t)\frac{f(x(g(t)))}{x^c(g(t))}x^c\big(g(t)\big) \\ &\geqslant q(t)v^c\big(g^*(t)\big) \quad \text{for } t \geqslant T_1. \end{aligned} \tag{37}$$

Integrating (37) from t to u, repeatedly n-times, letting $u \to \infty$ and using (35) we find that

$$v(t) \geqslant \int_t^\infty q(s)\frac{(t-s)^{n-1}}{(n-1)!}v^c\big(g^*(s)\big)\,\mathrm{d}s. \tag{38}$$

But by a result of Philos [16], if inequality (38) has an eventually positive solution $v(t)$, then the corresponding equation

$$w(t) = \int_t^\infty q(s)\frac{(t-s)^{n-1}}{(n-1)!}w\big(g^*(s)\big)\,\mathrm{d}s,$$

also has an eventually positive solution $w(t)$. It follows that equation (32) has the eventually positive solution $w(t)$, a contradiction. This completes the proof. □

To illustrate the results of this section, we consider the equations

$$(39)\qquad \big(x(t)+px(t-m)\big)^{(n)}+q(t)\operatorname{Sech}x(t)\big(|x(t)|^c\big)\operatorname{sgn}x(t)=0$$

and

$$(40)\qquad \big(r(t)y'(t)\big)'+\frac{B}{2((n-2)!)}q(t)\operatorname{Sech}Ct^{n-1}Q(t)y(t)=0,$$

where n is even, C, p, B and m are constants, $C\geqslant 1$, the functions $r,q\colon[t_0,\infty)\to(0,\infty)$ are continuous and the function Q is defined by (6). We consider the following:

(i) when $p=0$, we let $r(t)=t^{2-n}$, $B=1$ and $h(t)=t$,

(ii) when $0<p<1$ and $m>0$, we let $r(t)=t^{2-n}$, $B=(1-p)^c$ and $h(t)=t$,

(iii) when $p>1$ and $m<0$, we let $r(t)=(t+m)^{2-n}$, $B=(\frac{1-p}{p^2})^c$ and $h(t)=t+m$.

From Theorems 1–3, equation (39) is oscillatory if equation (40) is oscillatory provided that (i)–(iii) hold respectively.

Oscillatory behavior of equation (40) has been intensively studied in the literature. Here, we give the following most important conditions for the oscillation of equation (40):

(I) $$\liminf_{t\to\infty}\left(\int_{t_0}^{t}\frac{\mathrm{d}s}{r(s)}\right)\left(\int_{t}^{\infty}q(s)\operatorname{Sech}Cs^{n-1}Q(s)\,\mathrm{d}s\right)>\frac{2((n-2)!)}{4B},$$
(see [17, Theorem 1]).

(II) There exists a differentiable function $v\colon[t_0,\infty)\to(0,\infty)$ such that
$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{B}{2((n-2)!)}v(s)q(s)Q(s)\operatorname{Sech}Cs^{n-1}-\frac{r(s)v'^2(s)}{4v(s)}\right]\mathrm{d}s=\infty,$$
(see [3, Theorem 4]).

One can easily conclude that condition (I) (or; (II)) together with (i)–(iii) is sufficient for the oscillation of equation (39).

Also, we see that the equation

$$\left(x(t)-\frac{1}{2}x(t-2)\right)^{(n)}+q(t)\big(|x(t-4)|^c\big)\operatorname{sgn}x(t-4)=0,\quad t>4\text{ and }c>0$$

where n is even and $q\colon[t_0,\infty)\to(0,\infty)$ is continuous, is oscillatory by Theorem 4 if the equation

$$\big(t^{2-n}y'(t)\big)'+\frac{1}{2((n-2)!)}q(t)Q(t)y(t)=0,$$

is oscillatory, where Q is defined by (6) with $h(t)=t-4$, and all bounded solutions of the equation

$$u^{(n)}-q(t)\big(|u(t-2)|^c\big)\operatorname{sgn}u(t-2)=0,$$

are oscillatory.

Remark. In the results presented above, one can relate the oscillation problem of equation (1) to that of some linear equation of the form

$$(41) \qquad \big(r(t)y'(t)\big)' + q_1(t)y(t) = 0,$$

where $r, q_1\colon [t_0,\infty) \to (0,\infty)$ are continuous and $\int^\infty \frac{\mathrm{d}s}{r(s)} = \infty$. From this observation, we can proceed further in this direction and reduce the study of the oscillatory properties of equation (1) to that of linear first order delay equation of the form

$$(42) \qquad v'(t) + q_2(t)v\big(h(t)\big) = 0,$$

where the functions $q_2, h\colon [t_0,\infty) \to (0,\infty)$ are continuous, $h(t) < t$, $h'(t) > 0$ and $\lim\limits_{t\to\infty} h(t) = \infty$.

To obtain such results, it is sufficient to reduce the study of the oscillatory properties of equation (41) to that of equation (42).

To this end, we let $y(t)$ be a nonoscillatory solution of equation (41) and assume that $y(t) > 0$ for $t \geqslant t_0 > 0$. By Lemma 2 in [2], there exists a $t_1 \geqslant t_0$ such that $y'(t) > 0$ for $t \geqslant t_1$. Now,

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^t \big(r(s)y'(s)\big)\frac{1}{r(s)}\,\mathrm{d}s \\ &= y(t_1) + \left(\int_{t_1}^t \frac{1}{r(s)}\,\mathrm{d}s\right)\big(r(t)y'(t)\big) - \int_{t_1}^t \left(\int_{t_1}^u \frac{1}{r(s)}\,\mathrm{d}s\right)\big(r(u)y'(u)\big)\,\mathrm{d}u \\ &\geqslant r(t)y'(t)\left(\int_{t_1}^t \frac{1}{r(s)}\,\mathrm{d}s\right). \end{aligned}$$

There exists a $t_2 \geqslant t_1$ such that

$$(43) \qquad y(t) \geqslant y\big(h(t)\big) \geqslant r\big(h(t)\big)y'\big(h(t)\big)\left(\int_{t_1}^{h(t)} \frac{1}{r(s)}\,\mathrm{d}s\right) \quad \text{for } t \geqslant t_2.$$

Using (43) in equation (41), we have

$$(44) \qquad w'(t) + q_1(t)\left(\int_{t_1}^{h(t)} \frac{1}{r(u)}\,\mathrm{d}u\right)w\big(h(t)\big) \leqslant 0 \quad \text{for } t \geqslant t_2,$$

where $w(t) = r(t)y'(t)$. Integrating (44) from t to z and letting $z \to \infty$, we obtain

$$(45) \qquad w(t) \geqslant \int_t^\infty q_1(s)\left(\int_{t_1}^{h(s)} \frac{1}{r(u)}\,\mathrm{d}u\right)w\big(h(s)\big)\,\mathrm{d}s.$$

The function w is obviously strictly decreasing for $t \geqslant t_2$. Hence, by Theorem 1 in [16], we conclude that there exists a positive solution v of equation (42) with $\lim_{t\to\infty} v(t) = 0$. This contradicts the assumption that equation (42) is oscillatory.

From the above discussion, one can reformulate the results of this section by replacing the equation of type (41) with equations of type (42). Here we omit the details.

4. Oscillation of equation (2)

The oscillatory behavior of even order forced equations of type (2) with $p = 0$ and/or related equations has received intensive study in recent years. For general discussion on this subject, we refer to [9, 18] and the references cited therein. We observe that most of these criteria depend heavily on the assumption that the function $f(x)$ is nondecreasing for $x \neq 0$, and as pointed out by Wong [19], it is useful to study the oscillatory properties of equation (2) and/or related equations when $n > 2$ and without the assumption that $f(x)$ is a monotonic function. Therefore, the purpose of this section is to show that under the effect of certain forcing term, the study of the oscillatory behavior of equation (2) with f is locally of bounded variation is reduced to the oscillation of some homogeneous linear second order ordinary differential equations of type presented in Section 3.

We assume the following hypothesis on the forcing term:

(46) There exists an n-times differentiable function $k\colon [t_0, \infty) \to \mathbb{R}$ such that $k^{(n)}(t) = e(t)$ and $k(t)$ is oscillatory;

and

(47) there exist sequences $\{u_j\}$ and $\{u_j^*\}$ such that $\lim_{j\to\infty} u_j = \infty = \lim_{j\to\infty} u_j^*$ and $k(u_j) = \inf\{k(t)\colon t \geqslant u_j\}$ and $k(u_j^*) = \sup\{k(t)\colon t \geqslant u_j^*\}$.

Theorem 5. *Let $p(t) = 0$, $f \in C(\mathbb{R}_{t_0})$, $t_0 > 0$, and let the functions G and H be defined as in Theorem 1. Moreover, assume that conditions (3), (4), (46) and (47) hold and for every constant $C \geqslant 1$, the equation (5) with Q defined by (6), is oscillatory, then equation (2) is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (2), say $x(t) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_0 > 0$. Set $x(t) = v(t) + k(t)$, then from equation (2)

(48) $$v^{(n)}(t) = -q(t)G(x(g(t)))H(v(g(t)) + k(g(t))) \leqslant 0 \quad \text{for } t \geqslant t_0,$$

and hence, $v^{(i)}(t)$, $i = 0, 1, \dots, n-1$ are of constant sign for $t \geqslant t_0$. Now, we show that $v(t)$ is eventually positive. Otherwise, there exists a $t_1^* \geqslant t_0$ such that $v(t) < 0$ for $t \geqslant t_1^*$. Since $x(t) > 0$, it follows that $0 < v(t) + k(t)$; or $0 < -v(t) < k(t)$ for $t \geqslant t_1^*$, a contradiction. Hence, we must have $v(t) > 0$ for $t \geqslant t_0$. By Lemma 1, there exist a $t_1 \geqslant t_0$ and a constant $d_1 > 0$ such that

$$0 < v^{(n-1)}(t) \leqslant d_1 \quad \text{and} \quad v'(t) > 0 \quad \text{for } t \geqslant t_1. \tag{49}$$

Integrating the first inequality in (49) $(n-1)$-times from t_1 to t, we conclude that there exist a $t_2^* \geqslant t_1$ and a $d_2 > 0$ such that $v(t) \leqslant d_2 t^{n-1}$ for $t \geqslant t_2^*$, and hence, $x(t) \leqslant d_2 t^{n-1} + |k(t)|$ for $t \geqslant t_2^*$. By (47), there exist a constant $d \geqslant 1$ and a $t_2 \geqslant t_2^*$ such that

$$x\big(g(t)\big) \leqslant d g^{n-1}(t) \quad \text{for } t \geqslant t_2. \tag{50}$$

Next, by (47), there exists N such that for $t \geqslant u_N \geqslant t_2$

$$x(t) = v(t) + k(t) \geqslant v(t) + k(u_N) = w(t). \tag{51}$$

Clearly, $v'(t) = w'(t)$ and $v^{(n)}(t) = w^{(n)}(t)$. Moreover,

$$w(t) = v(t) + k(u_N) \geqslant v(u_N) + k(u_N) = x(u_N) > 0.$$

Thus, by (4), (50) and (51), equation (48) is reduced to

$$w^{(n)}(t) + q(t) G\big(d g^{n-1}(t)\big) w^c\big(g(t)\big) \leqslant 0 \quad \text{for } t \geqslant t_2.$$

The remainder of the proof proceed as in the proof of Theorem 1.

In the following oscillation results for equation (2), we assume that

(52) $0 < p(t) = p = \text{constant} < 1$ and $g_*(t) = t - m$, m is a positive constant;

or

(53) $p(t) = p = \text{ constant } > 1$ and $g_*(t) = t + m$, m is a postive constant,

and condition (47) is replaced by

(54) the function $k(t)$ is periodic of period m, i.e., $k(t \pm m) = k(t)$, where the constant m is defined as in (52) or (53).

□

Theorem 6. *Let $f \in C(\mathbb{R}_{t_0})$, $t_0 > 0$, and let the functions G and H be defined as in Theorem 1. Suppose that conditions (3), (4), (46), (52) and (54) hold. If for every $C \geqslant 1$, the equation*

$$\big(r(t)y'(t)\big)' + \frac{1-p}{2((n-2)!)} G\big(Cg^{n-1}(t)\big)q(t)Q(t)y(t) = 0,$$

is oscillatory, where Q is defined by (6), then equation (2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (2) and assume that $x(t) > 0$, $x(t-m) > 0$ and $x\big(g(t)\big) > 0$ for $t \geqslant t_0 > 0$. Set

$$v(t) + k(t) = x(t) + px(t-m). \tag{55}$$

Thus, as in the proof of Theorem 5, we see that $v(t) > 0$ and (49) holds for $t \geqslant t_1$ and there exist a constant $d \geqslant 1$ and a $t \geqslant t_2$ such that (50) holds for $t \geqslant t_2$.

Next, by (49) and (54) in (55), we have

$$\begin{aligned} x(t) &= v(t) + k(t) - px(t-m) \geqslant v(t) + k(t) - p[v(t-m) + k(t-m) - px(t-2m)] \\ &\geqslant (1-p)\big(v(t) + k(t)\big) \quad \text{for } t \geqslant t_2. \end{aligned}$$

By (54), there exists a $t_3 \geqslant t_2$ such that $k(t_3) = \inf\limits_{t_2 \leqslant s \leqslant t_2+m} k(s)$ and for $t \geqslant t_3$

$$x(t) \geqslant (1-p)\big(v(t) + k(t_3)\big) = w(t). \tag{56}$$

Clearly, $w'(t) = (1-p)v'(t)$ and $w^{(n)}(t) = (1-p)v^{(n)}(t)$ and $w(t) > 0$ for $t \geqslant t_3$. Thus, by (4), (50) and (56), we have

$$w^{(n)}(t) + (1-p)G\big(dg^{n-1}(t)\big)q(t)w^c\big(g(t)\big) \leqslant 0 \quad \text{for } t \geqslant t_3.$$

The rest of the proof proceeds as in the proof of Theorem 1. □

The following theorem, condition (26) of Theorem 3 takes the form:

$$h_1(t) = \min\{t, g(t) + m\} \quad \text{and} \quad h_1'(t) > 0 \quad \text{for } t \geqslant t_0. \tag{57}$$

Theorem 7. *Let $f \in C(\mathbb{R}_{t_0})$, $t_0 > 0$, and assume that the functions G and H be defined as in Theorem 1. Moreover, suppose that conditions (4), (47), (52), (54) and (57) hold. If for every constant $C \geqslant 1$, the equation*

$$\big((h_1^{n-2}(T)h_1'(t))^{-1}y'(t)\big)' + \frac{p-1}{2p^2((n-2)!)} G\big(Cg^{n-1}(t)\big)Q_1(t)q(t)y(t) = 0,$$

is oscillatory, where Q_1 is the same as Q defined by (6) *with h replaced by h_1, then equation* (2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (2), say $x(t) > 0$, $x(t+m) > 0$ and $x(g(t)) > 0$ for $t \geqslant t_0 > 0$. Define the function v by (55) and proceeds as in the proof of Theorems 5 and 6, we see that (50) holds for $t \geqslant t_2$. Using (49) and (54) in (55) we have

$$\begin{aligned} x(t) &= \frac{1}{p}\big(v(t-m) + k(t-m) - x(t-m)\big) \\ &= \frac{1}{p}\big(v(t-m) + k(t-m)\big) - \frac{1}{p^2}\big(v(t-2m) + k(t-2m) - x(t-2m)\big) \\ &\geqslant \frac{p-1}{p^2}\big(v(t-m) + k(t-m)\big), \quad t \geqslant t_2. \end{aligned}$$

By (54), there exists a $t_3 \geqslant t_2$ such that $k(t_3) = \inf\limits_{t_2 \leqslant s \leqslant t_2+m} k(s)$ and for $t \geqslant t_3$,

$$x(t) \geqslant \frac{p-1}{p^2}\big(v(t-m) + k(t-m)\big) = w(t-m).$$

It is easy to check that $w(t) > 0$, $w'(t) = \frac{p-1}{p^2}v'(t)$ and $w^{(n)}(t) = \frac{p-1}{p^2}v^{(n)}(t)$ for $t \geqslant t_3$ and equation (2) takes the form

$$w^{(n)}(t) + \frac{p-1}{p^2}G\big(dg^{n-1}(t)\big)q(t)w^c\big(g(t)-m\big) \leqslant 0.$$

The remainder of the proof proceeds as in the proof of Theorem 1. □

Some general remarks

1. The results of this paper are new, easily verifiable and can be extended to more general cases when the function H satisfies superlinear or sublinear conditions given in [3] and [18].

2. The results of this paper are applicable to equations (1) and (2) when the function $f(x)$ is nondecreasing for $x \neq 0$. In this case, we take $f(x) = H(x)$ and $G(x) = 1$. Also, we do not stipulate that the function g in equations (1) and (2) be either retarded, advanced and mixed type. Hence our results may hold for ordinary, retarded, advanced and mixed type equations.

3. The results of this paper are the complement of the results obtained by Kwong and Wong [12]. Also, we note that the results in Sec. 4 answer some problems raised by Wong [18].

4. As in the remark in Sec. 3, the oscillatory properties of equation (2) that given in Sec. 4 can be reduced to that of first order delay equations of type (42). Here we omit the details.

5. It is interesting to obtain results similar to Theorem 4 for equation (1) when f is locally of bounded variation, also to obtain criteria similar to Theorems 6 and 7 when the functions p and g_* are defined as in equation (1).

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