Ivan Chajda

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SEMILATTICES OF FINITE ARITHMETICAL ALGEBRAS

IVAN CHAJDA, Olomouc

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An algebra A is *arithmetical* if the congruence lattice $\operatorname{Con} A$ is distributive and every two congruences $\Theta, \Phi \in \operatorname{Con} A$ permute, i.e. $\Theta \circ \Phi = \Phi \circ \Theta$. It is easy to see that A is *arithmetical* if and only if the identity

$$(*) \qquad \Theta \cap (\Phi \circ \Psi) \subseteq (\Theta \cap \Psi) \circ (\Theta \cap \Phi)$$

holds for every $\Theta, \Phi, \Psi \in \operatorname{Con} A$, see e.g. [3], [4] (the symbol $\circ$ denotes relational product). As was shown in [3], an algebra A is arithmetical if and only if it satisfies the *Chinese remainder theorem.* The famous characterization was given by A. F. Pixley [3] for finite algebras:

Proposition 1. *A finite algebra A is arithmetical if and only if there exists a ternary function t: $A^3 \to A$ compatible with* $\operatorname{Con} A$ *satisfying*

$$(**) \qquad t(x,x,z) = z, \quad t(x,y,x) = x, \quad t(x,z,z) = x.$$

This function t is called a *Pixley function.* Recall that an n-ary function f: $A^n \to A$ is compatible with $\operatorname{Con} A$ if for each $\Theta \in \operatorname{Con} A$, $\langle x_i, y_i\rangle \in \Theta$ for $i = 1, \ldots, n$ imply

$$\langle f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)\rangle \in \Theta.$$

A similar characterization was given by A. F. Pixley also for varieties of arithmetical algebras. Using it, we gave in [1] a characterization of algebras in such a variety as algebras whose every two-element subset forms a (semi)lattice with respect to polynomial operations. This cannot be immediately used for a single algebra since the proof depends on the construction of the free algebra $F_{\mathcal{V}}(x,y)$. However, we can modify it to

Theorem 1. *Let A be a finite algebra. The following conditions are equivalent:*

(1) *A is arithmetical;*

(2) *for every element b of A there exists a binary function $\wedge$ compatible with* $\operatorname{Con} A$ *such that for each $a \in A$ the algebra $(\{a,b\},\wedge)$ is a semilattice with the greatest element b.*

Proof. (1)⇒(2): Put $x \wedge y = t(x,b,y)$ for a Pixley function $t(x,y,z)$. It is routine to show that for any $a \in A$, $(\{a,b\},\wedge)$ is a semilattice with the greatest element b.

(2)⇒(1): Suppose $\Theta, \Phi, \Psi \in \operatorname{Con} A$ and

$$\langle a,c\rangle \in \Theta \cap (\Phi \circ \Psi)$$

for some a, c of A, i.e. there exists an element $b \in A$ such that

$$\langle a,c\rangle \in \Theta, \quad \langle a,b\rangle \in \Phi, \quad \langle b,c\rangle \in \Psi.$$

Let $\wedge\colon A^2 \to A$ be a binary function compatible with $\operatorname{Con} A$ such that $(\{a,b\},\wedge)$, $(\{c,b\},\wedge)$ are $\wedge$-semilattices with the (common) greatest element b. Then

$$\begin{aligned} \langle a, a\wedge c\rangle &= \langle a\wedge a, a\wedge c\rangle \in \Theta(a,c) \subseteq \Theta,\\ \langle a, a\wedge c\rangle &= \langle a\wedge b, a\wedge c\rangle \in \Theta(b,c) \subseteq \Psi,\\ \text{i.e. } \langle a, a\wedge c\rangle &\in \Theta \cap \Psi, \end{aligned}$$

and further

$$\begin{aligned} \langle a\wedge c, c\rangle &= \langle a\wedge c, c\wedge c\rangle \in \Theta(a,c) \subseteq \Theta,\\ \langle a\wedge c, c\rangle &= \langle a\wedge c, b\wedge c\rangle \in \Theta(a,b) \subseteq \Phi,\\ \text{i.e. } \langle a\wedge c, c\rangle &\in \Theta \cap \Phi \end{aligned}$$

whence $\langle a,c\rangle \in (\Theta \cap \Psi) \circ (\Theta \cap \Phi)$. By the above quoted result of [3], [4] concerning (∗) we obtain (1). □

Remark 1. Theorem 1 can be paraphrased: A finite algebra A is arithmetical iff A is the union of two-element $\wedge$-semilattices with a common greatest element, where the semilattice operation is compatible with $\operatorname{Con} A$. This can be considered a description of the structure of a finite arithmetical algebra.

Remark 2. If A is a finite arithmetical algebra and $a,b,c \in A$ and if $\wedge$ is defined as above, then $(\{a,b\},\wedge)$, $(\{c,b\},\wedge)$ are semilattices with the greatest element b but the four-element subset $\{a\wedge c, a, c, b\}$ is not a lattice in general since the operation $\wedge$ need be neither commutative nor associative for the elements a, c.

Another nice characterization was established by K. Kaarli [2]:

Proposition 2. *An algebra A is arithmetical if and only if the following compatible function property holds: For any positive integer k and finite subsets $X, Y \subseteq A^k$ with $X \subseteq Y$, any function $f\colon X \to A$ compatible with* $\operatorname{Con} A$ *has an extension from Y to A which is also compatible with* $\operatorname{Con} A$.

A ternary function $d(x, y, z)$ satisfying $d(x, x, z) = z$ and $d(x, y, z) = x$ for $x \neq y$ is called a *discriminator*.

If A is an algebra and a, b are different elements of A, then we can uniquely define a discriminator d on $\{a, b\}$, and it is evidently a partial function compatible with $\operatorname{Con} A$. We are going to show that in a finite case the extension property of discriminator on two non-disjoint two-element subsets characterizes the arithmeticity as well:

Theorem 2. *For a finite algebra A, the following conditions are equivalent:*

(1) *A is arithmetical;*

(2) *for every three elements x, y, z of A and the discriminator d_1 on $\{x, y\}$ and d_2 on $\{y, z\}$ there exists a common extension $t\colon A^3 \to A$ of d_1, d_2 which is compatible with* $\operatorname{Con} A$.

Proof. (1)⇒(2): It is easy to show that a Pixley function compatible with $\operatorname{Con} A$ is an extension of both d_1, d_2.

(2)⇒(1): Similarly to the proof of Theorem 1, if $\langle x, z\rangle \in \Theta \cap (\Phi \circ \Psi)$ for some $\Theta, \Phi, \Psi \in \operatorname{Con} A$, i.e. $\langle x, z\rangle \in \Theta$, $\langle x, y\rangle \in \Phi$, $\langle y, z\rangle \in \Psi$ for some $y \in A$, we conclude

$$\begin{aligned}
\langle x, t(x, y, z)\rangle &= \langle d_1(x, y, x), t(x, y, z)\rangle = \langle t(x, y, x), t(x, y, z)\rangle \in \Theta,\\
\langle x, t(x, y, z)\rangle &= \langle d_1(x, y, y), t(x, y, z)\rangle = \langle t(x, y, y), t(x, y, z)\rangle \in \Psi,\\
\langle t(x, y, z), z\rangle &= \langle t(x, y, z), d_2(z, y, z)\rangle = \langle t(x, y, z), t(z, y, z)\rangle \in \Theta,\\
\langle t(x, y, z), z\rangle &= \langle t(x, y, z), d_2(y, y, z)\rangle = \langle t(x, y, z), t(y, y, z)\rangle \in \Phi,
\end{aligned}$$

thus $\langle x, z\rangle \in (\Theta \cap \Psi) \circ (\Theta \cap \Phi)$, where $t(x, y, z)$ is the extension of d_1, d_2 which is compatible with $\operatorname{Con} A$. By the above quoted result of [3], [4] concerning (*) we obtain (1). □

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Author's address: Dept. of Algebra and Geometry, Palacký University Olomouc, Faculty of Sci., Tomkova 38, 779 00 Olomouc, Czech Republic.