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COVERABLE STANDARD MEASURES WITH THE CHAIN CONDITION AND THE LEBESGUE DECOMPOSITION

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1. INTRODUCTION

Let $(X, \mathcal{S}, \mu)$ be a measure space with a σ-algebra and a countably additive measure on $(X, \mathcal{S})$. A measure μ is said to be semifinite if $\mu(E) = \sup\{\mu(F)\colon E \supset F, \mu(F) < \infty\}$ for any set E in $\mathcal{S}$. We shall consider only semifinite measures. If a measure ν on $(X, \mathcal{S})$ is absolutely continuous (singular) with respect to μ, then we write $\nu \ll \mu (\nu \perp \mu)$. ν is said to have the Lebesgue decomposition with respect to μ if there exist measures ν_1 and ν_2 such that $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$. Ficker [4] showed that any (not necessarily semifinite) measure ν has the Lebesgue decomposition with respect to any (not necessarily semifinite) measure if ν satisfies the countable chain condition (ccc), where we say that a measure ν satisfies the ccc if C is countable for any family $\{X_\gamma \in \mathcal{S}\colon \mu(X_\gamma) > 0\ (\gamma \in \mathsf{C}),\ X_\gamma \cap X_{\gamma'} = \emptyset\ (\gamma \neq \gamma')\}$ of sets (see also Capek [2]). The Lebesgue decomposition is a beautiful theorem in measure theory. The Lebesgue decomposition in more general measure spaces, that is, in not necessarily σ-finite measure spaces is important, for example, in statistics. We shall study such generalizations. Let κ be any infinite cardinal. A measure μ is said to satisfy the κ-chain condition (κ-cc) if the cardinality of C is less than κ ($|\mathsf{C}| < \kappa$) for any family $\{X_\gamma \in \mathcal{S}\colon 0 < (X_\gamma) < \infty\ (\gamma \in \mathsf{C}),\ \mu(X_\gamma \cap X_{\gamma'}) = 0\ (\gamma \neq \gamma')\}$ of sets. If the family is maximal and μ is infinite, then the cardinality of C is independent of the choice of families and it is called the magnitude of μ. The magnitude of a finite measure is defined to be ω. Here we denote the least infinite and the least uncountable cardinal by ω and ω_1 respectively. A measure μ is said to be μ^*-semifinite if $\mu^*(E) = \sup\{\mu^*(F)\colon E \supset F,\ \mu^*(F) < \infty\}$, where μ^* is the outer measure induced by μ. A measure μ is said to be measurable if there exists a non-trivial finite measure ν on $(X, \mathcal{S})$ such that $\nu(E) = 0$ for any set E with $\mu(E) < \infty$ (Kubokawa [14]). A measure μ is said to be non-measurable if it is not measurable.

A measure μ is said to be κ-additive if $\bigcup\{E_\alpha\colon \alpha < \lambda\} \in \mathcal{S}$ and $\mu(\bigcup_\alpha E_\alpha) = \sum_\alpha \mu(E_\alpha)$ for any cardinal $\lambda < \kappa$ and any disjoint family $\{E_\alpha \in \mathcal{S}\colon \alpha < \lambda\}$ of sets. Let $Y = \kappa$ and $\mathcal{T}$ the power set of $Y(P(Y))$. We assume that there exists a nontrivial diffused finite measure m on $(Y, \mathcal{T})$. Then κ is said to be Ulam measurable (Ulam [20]). κ is said to be measurable or real-valued measurable if in addition m is κ-additive and, two-valued or nonatomic respectively. The least Ulam measurable cardinal κ_* is a real-valued cardinal and satisfies $\omega_\omega < \kappa_* \leqslant 2^\omega$ if it exists (Jech [11, Theorem 66, p. 297], Ulam [20]). A cardinal κ is said to be Ulam non-measurable if it is not Ulam measurable. This means $\kappa < \kappa_*$. A measure μ is said to be coverable if there exists a measurable cover A_0 for any subset A of X, where A_0 is said to be a measurable cover of A if $A_0 \supset A$, $A_0 \in \mathcal{S}$ and $\mu(E) = 0$ for any set E in $\mathcal{S}$ with $A_0 - A \supset E$. A measure μ is said to be divisible if there exists a family $\{X_\gamma \in \mathcal{S}\colon \mu(X_\gamma) < \infty(\gamma \in \mathsf{C}),\ X_\gamma \cap X_{\gamma'} = \emptyset\ (\gamma \neq \gamma')\}$ of sets such that $\mu(E) = \sum_\gamma \mu(E \cap X_\gamma)$ for any set E in $\mathcal{S}$. A family $\{X_\gamma\}$ of sets satisfying all these conditions is called a division of μ. A measure μ is said to be strictly localizable or decomposable if there exists a division $\{X_\gamma\colon \gamma \in \mathsf{C}\}$ such that $X = \bigcup_\gamma X_\gamma$ (disjoint) and $E \in \mathcal{S}$ if $E \cap X_\gamma \in \mathcal{S}$ for all γ in C. A measure μ is said to be standard if there exists a division $\{X_\gamma\colon \gamma \in \mathsf{C}\}$ of μ such that for any subset B of C, $\bigcup_{\gamma \in \mathsf{B}} X_\gamma \in \mathcal{S}$. The notions of coverable and standard measures were first introduced in the paper. Strictly localizable measures are coverable and standard. We write $\tilde{E} \leqslant \tilde{F}$ if $\mu(E - F) = 0$. $\mathcal{S}(\mu) = \{\tilde{E}\colon E \in \mathcal{S}\}$ become a partially ordered set by this order ($\tilde{E} = \tilde{F}$ if and only if $\mu(E \Delta F) = 0$). A measure μ is said to be localizable or Maharam if there exists a supremum (and an infimum) for any non-empty subset of $\mathcal{S}(\mu)$ (Segal [19]). A strictly localizable measure is localizable. Let E be a non-empty measurable set. We define a measure μ_E on $(E, \mathcal{S}_E)$ for each F in $\mathcal{S}_E$ by $\mu_E(F) = \mu(F)$, where $\mathcal{S}_E = \{F \in \mathcal{S}\colon F \subset E\}$. We consider the following problem:

(*) *Let $(X, \mathcal{S}, \nu)$ be any measure space. Under which conditions does a measure ν have the Lebesgue decomposition with respect to any measure on $(X, \mathcal{S})$?*

In section 2 we shall study necessary conditions for (*). We first show that localizability and non-measurability is necessary and therefore divisibility is also so if we assume the existence of a real-valued measurable cardinal (Theorem 2.0). The former two conditions are necessary and sufficient for the validity of the Radon-Nikodym theorem (Kubokawa [14]). Under mild conditions concerning measures coverability is necessary (Theorem 2.1). We shall give examples of divisible, standard, strictly localizable and coverable measures at the opening of section 3. A coverable and standard measure is one of the most important notions in the paper. The problem has been solved almost completely for a complete measure, i.e., a complete measure

ν which is not pathological is non-measurable and strictly localizable if and only if ν has the Lebesgue decomposition with respect to any measure on $(X, \mathcal{S})$ (Theorem 3.1, Corollary 4.1). We shall give answers for the problem in section 4. The problem was almost solved for standard measures (Theorem 4.0). The author can not decide whether non-measurable coverable (or localizable) divisible measures are standard. Radon measures are divisible (Schwartz [18]) and those on generalized paracompact spaces are localizable and standard (Gardner and Pfeffer [8]). Localizable measures are important in the Radon-Nikodym theorem and they play a role in statistics. We shall study the Lebesgue decomposition of localizable measures. In section 2 we shall treat the Lebesgue decomposition of ν with respect to μ assuming that ν and μ are both localizable and non-measurable. In Theorem 4.4 we shall show that ν^*-semifinite Radon measure ν on a Hausdorff space X has the Lebesgue decomposition with respect to any Radon measure on X if and only if ν is coverable. Lastly we shall give examples which show the significance of conditions in the above theorems.

2. Necessary conditions

Let $(X, \mathcal{S}, \nu)$ be a measure space. We assume that ν has the Lebesgue decomposition $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$, $\nu_2 \perp \mu$ for any measure μ on $(X, \mathcal{S})$. What conditions does ν satisfy?

Theorem 2.0. *If ν has the Lebesgue decomposition with respect to any measure μ on $(X, \mathcal{S})$, then ν is non-measurable and localizable. If moreover there exists a real-valued measurable cardinal, then ν is divisible.*

Proof. We first show that ν is non-measurable. Let μ be any finite measure on $(X, \mathcal{S})$ such that $\mu(E) = 0$ for any set with $\nu(E) < \infty$. If $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$, $\perp \mu$ be the Lebesgue decomposition of ν with respect to μ, then there exist sets B such that $\nu_2(A) = \mu(\mathsf{B}) = 0$, $X = A \cup \mathsf{B}$ (disjoint). Then for any set E in $\mathcal{S}$ we have $\nu_1(E) = \nu(A \cap E)$, $\nu_2(E) = \nu(\mathsf{B} \cap E)$. Indeed $\nu_1(E) = \nu_1(A \cap E) = \nu_1(A \cap E) + \nu_2(A \cap E) = \nu(A \cap E)$. If $E \subset A$ and $\nu(E) < \infty$, then $\mu(E) = 0$ which implies $\nu(E) = \nu(A \cap E) = \nu_1(E) = 0$. Since ν is semifinite, $\nu(A) = 0$. $\nu \gg \mu$ implies $\mu(A) = 0$. Therefore $\mu(X) = \mu(A) + \mu(\mathsf{B}) = 0$.

Next we show that ν is localizable. We choose a maximal family $\{X_\gamma \in \mathcal{S}\colon 0 < \nu(X_\gamma) < \infty (\gamma \in \mathsf{C}),\ \nu(X_\gamma \cap X_{\gamma'}) = 0 (\gamma \neq \gamma')\}$ of sets. Then $\tilde{X} = \sup\limits_\gamma \tilde{X}_\gamma$. We first show that a set $\{\tilde{A}_\gamma \in \mathcal{S}(\nu)\colon A_\gamma \subset X_\gamma (\gamma \in \mathsf{C})\}$ has a supremum. We define a measure μ on $(X, \mathcal{S})$ for each E in $\mathcal{S}$ by $\mu(E) = \sum\limits_\gamma \nu(A_\gamma \cap E)$. By assumption

ν has the Lebesgue decomposition $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$, $\nu_2 \perp \mu$. Therefore there exist sets A, B such that $\nu_2(A) = \mu(B) = 0$, $X = A \cup B$ (disjoint). Hence $\nu_1(E) = \nu(A \cap E)$, $\nu_2(E) = \nu(B \cap E)$ for all E in $\mathcal{S}$. We fix γ in C. If $E(E \subset X_\gamma)$ is measurable, then $\nu(E) = \nu(A \cap X_\gamma \cap E) + \nu(B \cap X_\gamma \cap E)$. On the other hand $\nu(E) = \nu(A_\gamma \cap E) + \nu(B_\gamma \cap E)$, where $B_\gamma = X_\gamma - A_\gamma$, is the Lebesgue decomposition of ν_γ with respect to μ_γ, where we denote $\nu \mid X_\gamma$ and $\mu \mid X_\gamma$ by ν_γ and μ_γ respectively. Uniqueness of Lebesgue decompositions implies $\nu(A \cap X_\gamma \cap E) = \nu(A_\gamma \cap E)$ for all $E \in \mathcal{S} \mid X_\gamma$. Putting $E = (A \cap X_\gamma) - A_\gamma$ or $A_\gamma - (A \cap X_\gamma)$ we get $\nu[(A \cap X_\gamma) \Delta A_\gamma] = 0$. Hence $\tilde{A} = \tilde{A} \cap \big(\sup_\gamma \tilde{X}_\gamma\big) = \sup_\gamma \big(\tilde{A} \cap \tilde{X}_\gamma\big) = \sup_\gamma \tilde{A}_\gamma$. We show that any non-empty set $\{\tilde{E}_\delta \in \mathcal{S}(\nu) \colon \delta \in \Delta\}$ has a supremum. Since $\nu(X_\gamma) < \infty$, there exists a supremum $\tilde{A}_\gamma = \sup\{\tilde{E}_\delta \cap \tilde{X}_\gamma \colon \delta \in \Delta\}$ for each γ in C. We may choose A_γ with $A_\gamma \subset X_\gamma$. From the first half there exists a supremum $\tilde{A} = \sup_\gamma \tilde{A}_\gamma$, which is easily seen to be a supremum of the desired set.

If moreover there exists a real-valued measurable cardinal κ, then it satisfies $\omega_\omega < \kappa \leqslant 2^\omega$ (Jech [11, Theorem 66, p. 297]). A measurable cardinal is inaccessible and it is greater than 2^ω if it exists. Therefore the least Ulam measurable cardinal κ_* is real-valued measurable and $\kappa_* \leqslant 2^\omega$. A localizable measure ν is non-measurable if and only if the magnitude κ' of ν is Ulam non-measurable (Kubokawa [14, 2.13 Theorem]), which implies $\kappa' < \kappa_*$. Therefore ν satisfies the κ_*-cc and the $(2^\omega)^+$-cc, where κ^+ denote the next cardinal of κ. A localizable measure with the $(2^\omega)^+$-cc is divisible (Fell [3], See also Fremlin [6]), which implies that ν is divisible. □

Theorem 2.1. *We assume that a measure ν is ν^*-semifinite or $\{x\}$ is measurable for each x in X. If ν has the Lebesgue decomposition with respect to any measure μ on $(X, \mathcal{S})$, then ν is coverable.*

Remark. We have the following proposition: Any localizable ν^*-semifinite measure ν is coverable.

Proof. We first assume that ν is ν^*-semifinite. Let A be any subset of X. We define a measure μ on $(X, \mathcal{S})$ for each E is $\mathcal{S}$ by $\mu(E) = \nu^*(A \cap E)$. μ is semifinite because ν is ν^*-semifinite. By assumption ν has the Lebesgue decomposition with respect to μ. $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$, $\nu_2 \perp \mu$. Therefore there exist sets A_0, B_0 such that $\nu_2(A_0) = \mu(B_0) = 0$ and $X = A_0 \cup B_0$ (disjoint). Since $\nu^*(A \cap B_0) = \mu(B_0) = 0$, there exists a set N with $\nu(N) = 0$ and $A \cap B_0 \subset N$. $A_0 \cup N$ is a measurable cover of A. Indeed if $A_0 \cup N - A \supset E$ for a measurable set E, then $\nu(E) = \nu_1(E) + \nu_2(E) = 0$ because $\mu(E) = 0$ and $\nu_2(A_0 \cup N) = 0$.

Next we assume that $\{x\}$ is measurable for each x in X. We define a measure μ for each E in $\mathcal{S}$ by $\mu(E) = |A \cap E|$. By the similar method mentioned above we get the conclusion. □

We first give examples of measures. Localizable measures with the $(2^\omega)^+$-cc and any measures with the ω_2-cc is divisible (Fell [3], See Fremlin [6] for another proof). Radon measures μ are divisible (Schwartz [18, Theorem 13, p. 46]). A division of μ consisting of compact subsets is called a concassage of μ. A Radon measures of type $(\mathcal{H})$ is divisible. A Radon measure space and a quasi-Radon measure space are strictly localizable (Fremlin [5, 72B]). A strictly localizable measure is important in the theory of lifting (Ionescu Tulcea [10]). A coverable standard measure μ is localizable and μ^*-semifinite. Localizable measures are connected with the Radon-Nikodym theorem (Segal [19], Kubokawa [14]) and important in the classical theory of Banach spaces, i.e., the conjugate space of $L_1(X)$ is $L_\infty(X)$ if and only if the measure is localizable. They play a role in statistics (Lushgy and Mussmann [16], Ramamoorthi and Yamada [17]). Localizable measures whose magnitudes are non-measurable are used in the theory of domination in statistics (ibid.). The Lebesgue decomposition of localizable measures plays a role in the same theory (Ramamoorthi and Yamanda [17, Theorem, p. 259]).

Theorem 3.1. *If a complete measure μ is divisible and coverable, then it is strictly localizable. In particular it is standard.*

Proof. Let $\{X_\alpha \in \mathcal{S}\colon \alpha \in A\}$ be a division of μ. Let μ^* be the outer measure induced by μ. For any subset D of X we have (It is not necessary that $\{X_\alpha\}$ is a division in the next equality.)

$$\mu^*(D) = \sum_\alpha \mu^*(D \cap X_\alpha). \tag{3.1.1}$$

(This implies that coverable measures are μ^*-semifinite.) If D^0 is a measurable cover of D, then $D^0 \cap X_\alpha$ is a measurable cover of $D \cap X_\alpha$ for each α in A, which implies

$$\mu^*(D) = \mu(D^0) = \sum_\alpha \mu(D^0 \cap X_\alpha) = \sum_\alpha \mu^*(D \cap X_\alpha).$$

We assume that $E \cap X_\alpha$ is measurable for each α in A. If E^0 is measurable cover of E, then

$$\mu^*(E^0 - E) = \sum_\alpha \mu^*(E^0 \cap X_\alpha - E \cap X_\alpha) = 0,$$

since $E^0 - E \supset E^0 \cap X_\alpha - E \cap X_\alpha \in \mathcal{S}$. This implies that $E^0 - E$ is measurable, which completes the proof. $\square$

Theorem 3.2. *A standard measure μ is non-measurable if and only if the magnitude of μ is Ulam non-measurable.*

Proof. We assume that the magnitude of μ is Ulam non-measurable. We may assume that a division of μ $\{X_\gamma \in \mathcal{S} : \gamma \in \mathsf{C}\}$ satisfies $\mu(X_\gamma) > 0$ for all γ in C. Let ν be a finite measure on $(X, \mathcal{S})$ such that $\nu(E) = 0$ for any set E with $\mu(E) < \infty$. We show $\nu = 0$. Let $\mathcal{T}$ be the power set of C and we define a finite measure m on $(\mathsf{C}, \mathcal{T})$ for each D in $\mathcal{T}$ by $m(\mathsf{D}) = \nu(\bigcup_{\gamma \in \mathsf{D}} X_\gamma)$. Then we have $m(\{\gamma\}) = 0$ for each γ in C. Since $|\mathsf{C}|$ is Ulam non-measurable, $\nu(X) = m(\mathsf{C}) = 0$.

Conversely if the magnitude of μ is Ulam measurable, then μ is measurable (Kubokawa [14, 2.11 Proposition]). □

Remark. Ulam [20] showed that $\omega_1, \omega_2, \ldots, \omega_\omega, \ldots$ are all Ulam non-measurable.

4. The Lebesgue decomposition

We give necessary and sufficient conditions for the Lebesgue decomposition: first for a standard measure (Theorem 4.0) and next for measures that are both localizable and non-measurable (Theorem 4.2). Lastly we treat the Lebesgue decomposition of Radon measures. In Corollary 4.1 we almost completely determine a complete measure which has the Lebesgue decomposition with respect to any measure. 4.5 and 4.6 are examples.

Theorem 4.0. *Let ν be a standard measure which satisfies either* (a) *or* (b). *ν has the Lebesgue decomposition with respect to any measure if and only if ν is coverable and non-measurable or coverable and the magniture of ν is Ulam non-measurable.*

(a) *ν is ν^*-semifinite;*

(b) *$\{x\}$ is measurable for all x in X.*

Proof. If ν has the Lebesgue decomposition with respect to any measure μ on $(X, \mathcal{S})$, then by Theorem 2.1 ν is coverable and by Theorem 2.0 it is non-measurable. Therefore by Theorem 3.2 the magnitude of μ is Ulam non-measurable. By the same Theorem it suffices for the proof to show that a coverable, non-measurable, standard measure ν has the Lebesgue decomposition with respect to any measure. We can choose a division of ν such that $X = \bigcup_\gamma X_\gamma$ (disjoint), $\nu(X_\gamma) > 0$ $(\gamma \in \mathsf{C})$ and $\bigcup_{\gamma \in \mathsf{C}_0} X_\gamma \in \mathcal{S}$ for each $\mathsf{C}_0 \subset \mathsf{C}$. We put for each γ in C, $\mathcal{S}_\gamma = \mathcal{S} \mid X_\gamma$, $\nu_\gamma = \nu \mid X_\gamma$, $\mu_\gamma = \mu \mid X_\gamma$. ν_γ has the Lebesgue decomposition with respect to μ_γ. $\nu_\gamma = (\nu_\gamma)_1 + (\nu_\gamma)_2$, $(\nu_\gamma)_1 \ll \mu_\gamma$, $(\nu_\gamma)_2 \perp \mu_\gamma$. Hence there exist sets A_γ, B_γ such that

$(\nu_\gamma)_2(A_\gamma) = \mu_\gamma(\mathsf{B}_\gamma) = 0$ and $X_\gamma = A_\gamma \cup B_\gamma$ (disjoint). Putting $A = \bigcup\limits_\gamma A_\gamma$ and let A^0 be a measurable cover of A and let $B^0 = X - A^0$. We define measures ν_1, ν_2 on $(X, \mathcal{S})$ for each E in $\mathcal{S}$ by $\nu_1(E) = \nu(A^0 \cap E)$, $\nu_2(E) = \nu(B^0 \cap E)$. Then we have $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$, $\nu_2 \perp \mu$. By the definition of a measurable cover $\nu(A^0 \cap X_\gamma - A_\gamma) = 0$. If $\mu(E) = 0$, then $\nu(A^0 \cap X_\gamma \cap E) = \nu_\gamma(A_\gamma \cap E \cap X_\gamma) = (\nu_\gamma)_1(E \cap X_\gamma) = 0$ which implies $\nu_1(E) = \nu(A^0 \cap E) = \sum\limits_\gamma \nu(A^0 \cap E \cap X_\gamma) = 0$. Hence $\nu_1 \ll \mu$. We have for each E in $\mathcal{S}$

$$\mu(E) = \sum_\gamma \mu(E \cap X_\gamma). \tag{4.0.0}$$

We may assume that $\mathsf{C}_0 = \{\gamma \in \mathsf{C} \colon \mu(E \cap X_\gamma) > 0\}$ is countable. If $E_0 = \bigcup\limits_{\gamma \in \mathsf{C}_0} (E \cap X_\gamma)$, then $\mu[(E - E_0) \cap X_\gamma] = 0$ for all γ in C. We choose any finite measure ϱ with $\mu \geqslant \varrho$. Let $\mathcal{T}$ be the power set C and we define a finite measure m on $(\mathsf{C}, \mathcal{T})$ for each D in $\mathcal{T}$ by

$$m(\mathsf{D}) = \varrho\Big[(E - E_0) \cap \Big(\bigcup_{\gamma \in \mathsf{D}} X_\gamma\Big)\Big].$$

Then $m(\{\gamma\}) = 0$ for each γ in C. Since ν is non-measurable, by Theorem 3.2 $|\mathsf{C}|$ is Ulam non-measurable and therefore $\varrho(E - E_0) = m(\mathsf{C}) = 0$. Hence $\mu(E - E_0) = 0$ since ϱ is arbitrary, which implies the equality (4.0.0). We have $\mu(B^0) = \sum\limits_\gamma \mu(B^0 \cap X_\gamma) \leqslant \sum\limits_\gamma \mu(B_\gamma) = 0$. $\square$

Corollary 4.1. *A strictly localizable measure ν on $(X, \mathcal{S})$ has the Lebesgue decomposition with respect to any measure μ on $(X, \mathcal{S})$ if and only if ν is non-measurable or the magnitude of ν is Ulam non-measurable.*

Proof. A strictly localizable measure ν is coverable: Let $\{X_\gamma \in \mathcal{S} \colon \gamma \in \mathsf{C}\}$ be a division of ν and let A any subset of X. If $A_\gamma (A_\gamma \subset X_\gamma)$ is a measurable cover of $A \cap X_\gamma$ for each γ in C, then $A^0 = \bigcup\limits_\gamma A_\gamma$ is a measurable cover of A. By (3.1.1) ν is ν^*-semifinite. A measure ν satisfies (a) of Theorem 4.0 and is standard. Therefore by the same Theorem we have Corollary. $\square$

We have considered measures which have the Lebesgue decomposition with respect to any measure. By Theorem 2.0 the measures are necessary to be localizable and non-measurable. These measures are important in applications. Gardner and Pfeffer [8, 3.4. Theorem] showed that Radon measures on a fairly general topological space (See Burke [1]) are localizable. Volčič proved a weak form of the Lebesgue decomposition assuming $\nu + \mu$ is localizable [21, Theorem 2.4 and 3.1].

Theorem 4.2. *Let ν and μ be non-measurable and localizable measures on a measurable space $(X, \mathcal{S})$. A measure ν has the Lebesgue decomposition with respect to μ if and only if $\nu + \mu$ is localizable.*

Proof. Sufficiency: We assume that $\lambda = \nu + \mu$ is localizable. By assumption the magnitude of ν and μ are both Ulam non-measurable (Kubokava [14, 2.13. Theorem]). It is easy to see that the magnitude of λ is Ulam non-measurable. We use a result of Kubokawa [13, Lemma 4]. □

Lemma 4.3. *Let ϱ be a localizable measure whose magnitude is Ulam non-measurable and let μ a measure with $\varrho \gg \mu$. If*

$$\tilde{B} = \sup\{\tilde{E} \in \mathcal{S}(\varrho) \colon \mu(E) = 0\},$$

then $\mu(B) = 0$.

Let $\varrho = \lambda$. Then by Lemma $\mu(B) = 0$. We define measures ν_1, ν_2 on $(X, \mathcal{S})$ for each E in $\mathcal{S}$ by $\nu_1(E) = \nu(A \cap E)$, $\nu_2(E) = \nu(B \cap E)$, where $A = X - B$. We have $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$, $\nu_2 \perp \mu$: If $\mu(E) = 0$, then $\tilde{E} \leqslant \tilde{B} \in \mathcal{S}(\lambda)$, which implies $(A \cap E)^{\sim} \leqslant (A \cap B)^{\sim} = 0$ and we get $\nu_1(E) = 0$. Therefore $\nu_1 \ll \mu$.

Necessity: We assume that $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$, $\nu_2 \perp \mu$. Then $\nu + \mu = (\nu_1 + \mu) + \nu_2$, $\nu_1 + \mu \gg \mu \gg \nu_1 + \mu$, $(\nu_1 + \mu) \perp \nu_2$. Measures ν_2 and $\nu_1 + \mu$ are localizable which implies the conclusion.

Remark. We have the following: Let ν and μ be localizable Radon measures on a Hausdorff topological space X. A measure ν has the Lebesgue decomposition with respect to μ if and only if $\nu + \mu$ is localizable.

In the above theorem the assumption that measures are non-measurable is necessary in general.

Kubokawa [15] proved that Radon measures on a fairly general topological space are coverable. By the following theorem we have the Lebesgue decompositions for such measures.

Theorem 4.4. *Let ν be a ν^*-semifinite Radon measure on a Hausdorff topological space X. A measure ν has the Lebesgue decomposition with respect to any Radon measure μ on X if and only if ν is coverable.*

Proof. Necessity: Let A be any subset of X. The measure μ defined by $\mu(E) = \nu^*(A \cap E)$ for any set E in $\mathcal{S}$ in Theorem 2.1 is Radon. It follows from this similarly as in the proof of Theorem 2.1 that A has a measurable cover.

Sufficiency: A Radon measure ν has a division $\{X_\gamma \colon 0 < \nu(X_\gamma) < \infty (\gamma \in \mathrm{C})\}$ (called a concassage of ν) and this is a division of μ. We can apply the proof of Theorem 4.0. □

Lastly we give two examples of measures which do not have the Lebesgue decomposition.

Example 4.5. A strictly localizable measure which does not have the Lebesgue decomposition with respect to a finite measure. We assume the existence of an Ulam measurable cardinal κ. Let $X = \kappa$ and ν be counting measure on X. Then there exists a nontrivial diffused finite measure μ on X. It is easy to see that a strictly localizable measure ν does not have the Lebesgue decomposition with respect to μ.

Example 4.6. Localizable, non-measurable standard measures which do not have the Lebesgue decomposition. Let X and Y be uncountable sets. Let the cardinality of Y be Ulam non-measurable, for example, $|Y| = \omega_1$. Let $\mathcal{T}$ be the smallest σ-algebra which includes $\{y\}$ for each y in Y. Let $Z = X \times Y$ and we define a σ-algebra $\mathcal{S}$ of subsets of Z by $\mathcal{S} = \{E \subset Z\colon E_x \in T \text{ for all } x \text{ in } X \text{ and } \{x \in X\colon E_x \neq \emptyset \text{ or } Y\} \text{ is countable}\}$, where $E_x = \{y\colon (x,y) \in E\}$. Let ϱ and λ be measures on $(Y, \mathcal{T})$ defined by $\varrho(\{y\}) = 0$ for each y in Y and $\varrho(Y) = 1$, and $\lambda(F) = 1$ or 0 according as $y^* \in F$ or $y^* \notin F$ respectively, where y^* is a fixed element of Y. Lastly we define measures ν and μ on $(Z, \mathcal{S})$ for each E in $\mathcal{S}$ by

$$\nu(E) = \sum_x \varrho(E_x),$$

$$\mu(E) = \sum_x \lambda(E_x).$$

ν and μ are non-measurable, localizable standard measures. A measure $\nu + \mu$ is not localizable and ν and μ are both not coverable. A measure $\nu(\mu)$ does not have the Lebesgue decomposition with respect to $\mu(\nu)$. It may be worth mentioning that ν is complete and neither ν nor μ is strictly localizable.

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