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THE FUNDAMENTAL THEOREM FOR THE ν_1-INTEGRAL ON MORE GENERAL SETS AND A CORRESPONDING DIVERGENCE THEOREM WITH SINGULARITIES

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INTRODUCTION

The authors have recently introduced an axiomatic theory of non-absolutely convergent integrals in $\mathbb{R}^n$ which was specialized to ν_1-integrals on intervals, cf. [Ju-No 1], [Ju-No 2]. The ν_1-theory is relatively elementary and yielded a strong form of the divergence theorem with respect to the analytic assumptions on the vector function $\vec{v}$ involved.

On one hand we allowed certain *exceptional* points where $\vec{v}$ is not differentiable, but still bounded, and on the other hand we were able to treat certain *singularities*, where $\vec{v}$ is not bounded, the latter being the essential progress. At these singularities $\vec{v}$ was assumed to be of Lipschitz-type with a negative exponent $\beta > 1 - n$.

Countably many types β were allowed and singularities of type β were restricted to lie on sets of finite outer α- dimensional Hausdorff measure with $\alpha = \beta + n - 1$. Similar singularities were discussed before by [Pf 1], but here they were restricted to lie on hyperplanes. Also [Jar-Ku 3] discussed singularities, but only at isolated points.

Of course, one would like to have results of this generality also for vector functions on more general sets, not just intervals. Such results exist including exceptional points but no singularities, cf. [Jar-Ku 1-2], [Pf 2], [Ju], [No]. The goal of this paper is to treat singularities of the type mentioned on relatively general sets. It is remarkable that this can be done using our ν_1-integral which was originally restricted to intervals.

There is another aspect of more theoretical interest: Our integral is related to interval functions and corresponding interval derivatives, and the divergence theorem can be seen to be a special case of a fundamental theorem which gives sufficient conditions for the interval derivative to be integrable to the expected value. Such fundamental theorems were given in [Ju], [No] and could have been formulated also for the ν_1-integral on intervals. Here we give a general result of this sort and obtain our general form of the divergence theorem as a simple consequence. Moreover, this arrangement makes the proof particularly lucid.

0. Preliminaries

The set of all real numbers is denoted by $\mathbb{R}$, and throughout this paper n is a fixed positive integer. We work in $\mathbb{R}^n$ with the usual inner product $x \cdot y = \sum_{i=1}^{n} x_i y_i$ ($x = (x_i)_{1 \leqslant i \leqslant n}$, $y = (y_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n$) and the associated norm $\| \cdot \|$.

If $x \in \mathbb{R}^n$ and $r > 0$ we set $B(x,r) = \{y \in \mathbb{R}^n : \|y - x\| \leqslant r\}$, and for $E \subseteq \mathbb{R}^n$ we denote by $E^\circ, \partial E$ and $d(E)$ the interior, boundary and diameter of E.

For $A, B \subseteq \mathbb{R}^n$ we denote the set difference by $A - B$ and the complement of A in $\mathbb{R}^n$ by A^c.

By $| \cdot |_s$ ($0 \leqslant s \leqslant n$) we denote the s-dimensional normalized outer Hausdorff measure in $\mathbb{R}^n$ which coincides for integral s on $\mathbb{R}^s$ ($\subseteq \mathbb{R}^n$) with the s-dimensional outer Lebesgue measure ($| \cdot |_0$ being the counting measure). Instead of $| \cdot |_{n-1}$ we also write $\mathscr{H}(\cdot)$, and the term almost everywhere (a.e.) will always refer to the Lebesgue measure $| \cdot |_n$. A set $E \subseteq \mathbb{R}^n$ is called σ_s-finite if it can be represented as a countable union of sets with finite s-dimensional outer Hausdorff measure, and E is called an s-null set if $|E|_s = 0$.

An interval I in $\mathbb{R}^n$ is always assumed to be compact and non-degenerate, and finitely many intervals are said to be non-overlapping if they have pairwise disjoint interiors.

1. The ν_1-integral on general sets and the formulation of the Fundamental Theorem

We begin this section by extending the definition of the ν_1-integral (see [Ju-No 2]) to relatively general sets A, and we then discuss local Lipschitz conditions for an interval function relative to such a set A. Next will be the formulation of the Fundamental Theorem which essentially says that these Lipschitz conditions are sufficient for the ν_1-integrability on A of the derivative of an a.e. differentiable additive interval function to the excepted value.

By $\mathscr{A}$ we denote the system of all compact sets $A \subseteq \mathbb{R}^n$ such that $|\partial A|_{n-1}$ is finite.

If $A \in \mathscr{A}$ and if the real-valued function f is defined at least (almost everywhere) on A, we define the function $f_A \colon \mathbb{R}^n \to \mathbb{R}$ by $f_A(x) = f(x)$ if $x \in A$ and by $f_A(x) = 0$ else.

Given a set $A \in \mathscr{A}$ and a function $f \colon A \to \mathbb{R}$ we call f *ν_1-integrable on A* iff there exists an interval I containing A such that f_A is ν_1-integrable on I, and in that case we write

$$\,^{\nu}\!\!\int_A f = \,^{\nu}\!\!\int_I f_A.$$

Remark 1.1. Note that $^{\nu}\!\!\int_A f$ is independent of the interval I. For, if J denotes an interval containing I we can express J as a finite non-overlapping union of intervals I_k and the interval I. Obviously $f_A = 0$ a.e. on I_k and therefore $^{\nu}\!\!\int_J f_A = {}^{\nu}\!\!\int_I f_A$ since the ν_1-integral is an extension of the Lebesgue integral and because of its additivity property (see [Ju-No 2, Prop. 1.1]).

By an *interval function* (on $\mathbb{R}^n$) we mean a function F which associates with each interval $I \subseteq \mathbb{R}^n$ a real number $F(I)$.

Such an interval function is called *additive* if for any interval I and any decomposition $\{I_k\}$ of I (i.e. a finite sequence of non-overlapping intervals I_k whose union is I) the equality $F(I) = \sum F(I_k)$ holds.

An interval function F is said to be *differentiable at a point* $x \in \mathbb{R}^n$ if F is derivable in the ordinary sense at x according to [Saks], and in that case $\dot{F}(x)$ denotes the ordinary derivative of F at x.

Let F be an additive interval function and assume that there is an interval I such that $F(J) = 0$ for each interval $J \subseteq \mathbb{R}^n - I^\circ$. Then, using the argument of Remark 1.1, the real number $F(I)$ does not depend on the choice of I, and this unique number will be denoted by $F(\mathbb{R}^n)$.

Let $A \in \mathscr{A}$, $x \in A$ and let F be an interval function. We now introduce Lipschitz conditions (relative to A) for F at x, and in what follows the limit process $I \to x$ means that I is any interval containing x with $d(I) \to 0$.

By definition F satisfies at x the condition

$$(\boldsymbol{A}_n) \quad \text{if} \quad F(I) = O(1)d(A \cap I)|\partial(A \cap I)|_{n-1} \quad (I \to x \quad \text{and} \quad d(I)^n = O(|I|_n)),$$

more precisely this requires

$$\exists\, K' > 0\ \forall K > 0\ \exists\, \delta > 0 \text{ such that } |F(I)| \leqslant K' d(A \cap I)|\partial(A \cap I)|_{n-1}$$

holds for each interval I with $x \in I$, $d(I) < \delta$ and $d(I)^n \leqslant K|I|_n$. (Note that K' depends only on F, A and x.)

$(\boldsymbol{\Lambda}_\beta)$ $(n-1 < \beta < n)$ if in the same sense

$$F(I) = O(1) d(A \cap I)^{\beta+1-n} |\partial(A \cap I)|_{n-1} \quad (I \to x \text{ and } d(I)^n = O(|I|_n)),$$

$(\boldsymbol{\lambda}_\beta)$ $(n-1 < \beta < n)$ if similarly

$$F(I) = o(1) d(A \cap I)^{\beta+1-n} |\partial(A \cap I)|_{n-1} \quad (I \to x \text{ and } d(I)^n = O(|I|_n)),$$

$(\boldsymbol{\Lambda}_{n-1})$		if	$F(I) = O(1)\lvert\partial(A \cap I)\rvert_{n-1}$	$(I \to x)$,
$(\boldsymbol{\lambda}_{n-1})$		if	$F(I) = o(1)\lvert\partial(A \cap I)\rvert_{n-1}$	$(I \to x)$,
$(\boldsymbol{\Lambda}_\beta)$	$(0 \leqslant \beta < n-1)$	if	$F(I) = O(1) d(A \cap I)^\beta$	$(I \to x)$,
$(\boldsymbol{\lambda}_\beta)$	$(0 \leqslant \beta < n-1)$	if	$F(I) = o(1) d(A \cap I)^\beta$	$(I \to x)$.

For reasons of simplicity we set $\boldsymbol{\lambda}_n = \boldsymbol{\Lambda}_n$.

Remark 1.2. The reader should pay attention to the fact that the Lipschitz conditions defined above heavily depend on the set A. But since there will be no danger of misunderstanding we simply write Λ_β and λ_β.

Theorem 1.1 (Fundamental Theorem). *Let $A \in \mathscr{A}$, $D \subseteq A$ and let F be an additive interval function on $\mathbb{R}^n$ such that $F(I) = 0$ for each interval $I \subseteq A^c$. Furthermore, assume $A - D$ to be an n-null set represented as the disjoint countable union of σ_{α_i}-finite sets M_i and α_i-null sets N_i with $0 \leqslant \alpha_i \leqslant n$ $(i \in \mathbb{N})$.*

If in addition F is differentiable on D, and if F satisfies the Lipschitz condition λ_{α_i} resp. Λ_{α_i} (relative to A) at each point of M_i resp. N_i as well as the Lipschitz condition λ_{n-1} (relative to A) at each point of $A - \bigcup_{\alpha_i \leqslant n-1} (M_i \cup N_i)$ then $\dot{F}$ is ν_1-integrable on A and

$$F(\mathbb{R}^n) = {}^{\nu_1}\!\!\int_A \dot{F}.$$

2. Proof of the Fundamental Theorem

Note that if E is an $(n-1)$-null set, and if $\varepsilon > 0$ is arbitrary there exists an open set G containing E such that $|G \cap \partial A|_{n-1} < \varepsilon$. For, as is well-known, we can determine an $(n-1)$-null set G' containing E and being the countable intersection of monotone decreasing open sets G_j. Consequently a standard argument yields $0 = |G' \cap \partial A|_{n-1} = \lim_{j \to \infty} |G_j \cap \partial A|_{n-1}$.

Set $\dot{F} = 0$ on $A - D$, and to prove ν_1-integrability of $\dot{F}$ on A we choose an interval I containing A in its interior. We will show that $\dot{F}_A$ is ν_1-integrable on I, and by definition we only have to find a suitable division $\dot{E}, (E_i, C_i)_{i\in\mathbb{N}}$ of I (i.e. I is the disjoint union of all the sets $\dot{E}$ and E_i, $\dot{E} \subseteq I^\circ$, $|I - \dot{E}|_n = 0$; the C_i are admissible control conditions, and each set E_i is related to C_i) and to check the validity of corresponding null conditions for the interval function F restricted to the subintervals of I, see [Ju-No 2, Sec. 1]. Further explanations occur later in the proof.

Then, again by definition, we have $F(\mathbb{R}^n) = F(I) = {}^{\nu}\!\!\int_I \dot{F}_A = {}^{\nu}\!\!\int_A \dot{F}$, as desired.

Since $|M_i|_{\alpha_i} = 0$ if $\alpha_i = n$ and since $\Lambda_n = \lambda_n$ we may assume $M_i = \emptyset$ for those i. Without loss of generality we also assume $|M_i|_{\alpha_i}$ to be finite ($i \in \mathbb{N}$), and in addition we assume the O-constant $K(x)$ occuring in the definition of Λ_{α_i} to be bounded on N_i by $K_i > 0$ ($i \in \mathbb{N}$).

Now our division of I is given by

$$\dot{E} = D \cup (I^\circ - A), \quad (\partial I, C^*), \quad (M_i, C_1^{\alpha_i})_{\alpha_i < n}, \quad (N_i, C_2^{\alpha_i})_{i\in\mathbb{N}}$$

with the understanding that $C_2^{\alpha_i} = C^n$ if $\alpha_i = n$.

We proceed by proving that F satisfies the null condition corresponding to $C_2^{\alpha_i}$ on N_i, in short that F satisfies $\mathscr{N}(C_2^{\alpha_i}, N_i)$, if $\alpha_i = n$:

Let $\varepsilon > 0$, $K > 0$ be given. Since F satisfies Λ_n on N_i we can determine for $x \in N_i$ a $\delta(x) > 0$ such that

$$|F(I)| \leqslant K_i d(A \cap I) |\partial(A \cap I)|_{n-1}$$

holds for each interval I containing x with $d(I) < \delta(x)$ and $d(I)^n \leqslant K|I|_n$.

Because of $|N_i|_n = 0$ there is an open set $G \supseteq N_i$ with $|G|_n < \varepsilon/4nKK_i$, and we may assume $B(x, \delta(x)) \subseteq G$ for $x \in N_i$ as well as $\delta(\cdot) \leqslant \varepsilon_i = \varepsilon/2(1 + K_i|\partial A|_{n-1})$ on N_i.

Now let $\{(x_k, I_k)\}$ be a (N_i, δ)-fine sequence (i.e. a finite sequence of pairs (x_k, I_k) with $x_k \in I_k \cap N_i$, $d(I_k) < \delta(x_k)$ and the I_k being non-overlapping subintervals of I) with $\{I_k\} \in C_2^n(K)$ (i.e. $d(I_k)^n \leqslant K|I_k|_n$ for all k). Then, reminding that $\partial(A \cap I_k) \subseteq (I_k^\circ \cap \partial A) \cup \partial I_k$, we get:

$$\begin{aligned}\sum |F(I_k)| &\leqslant K_i \sum d(A \cap I_k)\Big(|I_k^\circ \cap \partial A|_{n-1} + |\partial I_k|_{n-1}\Big) \\ &\leqslant K_i \varepsilon_i \sum |I_k^\circ \cap \partial A|_{n-1} + K_i \sum 2nd(I_k)^n \\ &\leqslant \frac{\varepsilon}{2} + 2nK_iK|G|_n \leqslant \varepsilon.\end{aligned}$$

Similarly (even simpler) one shows that F satisfies $\mathscr{N}(C_2^{\alpha_i}, N_i)$ if $n - 1 < \alpha_i < n$, and a glance shows that F also satisfies $\mathscr{N}(C_2^{\alpha_i}, N_i)$ if $0 \leqslant \alpha_i < n - 1$. We prove that F also satisfies $\mathscr{N}(C_2^{\alpha_i}, N_i)$ if $\alpha_i = n - 1$:

Let again $\varepsilon > 0$ and $K > 0$ be given. For $x \in N_i$ we can find a $\delta(x) > 0$ such that

$$|F(I)| \leqslant K_i |\partial(A \cap I)|_{n-1}$$

for each interval I containing x and having diameter less than $\delta(x)$. We may assume $B(x,\delta(x)) \subseteq A^\circ$ if $x \in N_i \cap A^\circ$. Since $|N_i \cap \partial A|_{n-1} = 0$ we can determine an open set $G_i \supseteq N_i \cap \partial A$ with $|G_i \cap \partial A|_{n-1} < \varepsilon/2K_i$ by the note at the beginning of the proof. We assume that $B(x,\delta(x)) \subseteq G_i$ if $x \in N_i \cap \partial A$, and we set $\Delta = \varepsilon/4nK_i$. Given a (N_i,δ)-fine sequence $\{(x_k,I_k)\}$ with $\sum d(I_k)^{n-1} \leqslant \Delta$ we conclude

$$\begin{aligned}\sum |F(I_k)| &\leqslant K_i \sum (|I_k^\circ \cap \partial A|_{n-1} + 2n\, d(I_k)^{n-1}) \\ &\leqslant K_i |G_i \cap \partial A|_{n-1} + 2nK_i\Delta \leqslant \varepsilon.\end{aligned}$$

Next we prove that F satisfies $\mathscr{N}(C_1^{\alpha_i}, M_i)$ if $n-1 < \alpha_i < n$: Given $\varepsilon > 0$, $K > 0$ and $x \in M_i$ we choose a $\delta(x) > 0$ such that

$$|F(I)| \leqslant \varepsilon_i d(A \cap I)^{\alpha_i+1-n} |\partial(A \cap I)|_{n-1}, \quad \varepsilon_i = \varepsilon/(2nK + |\partial A|_{n-1})$$

holds for any interval I containing x with $d(I) < \delta(x)$ and $d(I)^n \leqslant K|I|_n$. We may assume $\delta(x) \leqslant 1$, and if $\{(x_k,I_k)\}$ denotes a (M_i,δ)-fine sequence with $\{I_k\} \in C_1^{\alpha_i}(K)$ (i.e. $\sum d(I_k)^{\alpha_i} \leqslant K$ and $d(I_k)^n \leqslant K|I_k|_n$ for all k) we conclude

$$\begin{aligned}\sum |F(I_k)| &\leqslant \varepsilon_i \sum d(A \cap I_k)^{\alpha_i+1-n} \Big(|I_k^\circ \cap \partial A|_{n-1} + 2nd(I_k)^{n-1} \Big) \\ &\leqslant \varepsilon_i \sum |I_k^\circ \cap \partial A|_{n-1} + 2n\varepsilon_i \sum d(I_k)^{\alpha_i} \leqslant \varepsilon_i (|\partial A|_{n-1} + 2nK) = \varepsilon.\end{aligned}$$

Similarly one shows that F satisfies $\mathscr{N}(C_1^{\alpha_i}, M_i)$ if $0 \leqslant \alpha_i \leqslant n-1$, and that F satisfies $\mathscr{N}(C^*, A - \bigcup_{\alpha_i \leqslant n-1} (M_i \cup N_i))$ what completes the proof. □

3. The Divergence Theorem

In [Ju-No 2] a strong form of the divergence theorem for vector functions $\vec{v}$ defined on an n-dimensional interval was proved by using the ν_1-integral. Here we will generalize the geometric aspect of this theorem by allowing any set $A \in \mathscr{A}$ for the domain of $\vec{v}$.

Assume $A \in \mathscr{A}$, $x \in A$, $1-n \leqslant \beta \leqslant 1$ and let a vector function $\vec{v}\colon A \to \mathbb{R}^n$ be given. We say that $\vec{v}$ satisfies at x the condition

(ℓ_1) if there exists a real n by n matrix M such that

$$\vec{v}(y) - \vec{v}(x) - M \cdot (y-x) = o(1)\|y-x\| \quad (y \to x, y \in A),$$

(ℓ_β) $(\beta \neq 1)$ if $\vec{v}(y) - \vec{v}(x) = o(1)\|y-x\|^\beta \quad (y \to x,\ y \neq x,\ y \in A)$,

(L_β) if $\vec{v}(y) - \vec{v}(x) = O(1)\|y-x\|^\beta \quad (y \to x,\ y \neq x,\ y \in A)$.

If $x \in A^\circ$ and $\vec{v} = (v_i)_{1 \leqslant i \leqslant n}$ is totally differentiable at x we set $\operatorname{div} \vec{v}(x) = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}(x)$, and at all other points $x \in A$ we set $\operatorname{div} \vec{v}(x) = 0$.

By [Fed] there exists for each $A \in \mathscr{A}$ a $\mathscr{H}$-measurable vector function $\vec{n}_A \colon \partial A \to \mathbb{R}^n$, the so called exterior normal, with $\|\vec{n}_A\| \leqslant 1$. Furthermore, for any vector function $\vec{v}$ being totally differentiable in a neighborhood of A we have $\int_{\partial A} \vec{v} \cdot \vec{n}_A \, d\mathscr{H} = \int_A \operatorname{div} \vec{v}$, where the integral on the right is a simple n-dimensional Lebesgue integral.

Theorem 3.1 (Divergence Theorem). *Let A be a compact subset of $\mathbb{R}^n$ with $|\partial A|_{n-1} < \infty$ and $\vec{v} \colon A \to \mathbb{R}^n$ be a vector function. By D we denote the set of all points from the interior of A where $\vec{v}$ is totally differentiable, and we write $A - D$ as a disjoint countable union of σ_{α_i}-finite sets M_i and α_i-null sets N_i with $0 < \alpha_i \leqslant n$ ($i \in \mathbb{N}$) such that $\bigcup_{\alpha_i < n-1} (M_i \cup N_i)$ lies in the interior of A. If we assume that $\vec{v}$ satisfies the condition (ℓ_{α_i+1-n}) resp. (L_{α_i+1-n}) at each point of M_i resp. N_i then $\vec{v}$ is $\mathscr{H}$-measurable and bounded on ∂A,* $\operatorname{div} \vec{v}$ *is ν_1-integrable on A and*

$$\int_{\partial A} \vec{v} \cdot \vec{n}_A \, \mathrm{d}\mathscr{H} = \overset{\nu_1}{\int_A} \operatorname{div} \vec{v}.$$

Proof. Note that $M_i \subseteq \partial A$ for $\alpha_i = n$ (since $M_i \cap A^\circ \subseteq D$), hence $|A - D|_n = 0$. Furthermore, $\vec{v}$ is continuous on A except for an $(n-1)$-null set, and thus the $\mathscr{H}$-measurability of $\vec{v}$ on A follows. Since $\vec{v}$ is locally bounded at each point of ∂A we also see that $\vec{v}$ is bounded on ∂A.

Extend $\vec{v}$ to whole of $\mathbb{R}^n$ by setting $\vec{v}(x) = 0$ if $x \in A^c$ and fix an interval I. Within the proof of Thm 2.1 in [Ju-No 2] it is shown that $\int_{\partial I} \|\vec{v}\| \, \mathrm{d}\mathscr{H} < \infty$, and since $\partial(A \cap I) \subseteq \partial A \cup \partial I$ we have a finite integral $\int_{\partial(A \cap I)} \|\vec{v}\| \, \mathrm{d}\mathscr{H}$. Consequently, we define an additive interval function F on $\mathbb{R}^n$ by setting $F(I) = \int_{\partial(A \cap I)} \vec{v} \cdot \vec{n}_{A \cap I} \, \mathrm{d}\mathscr{H}$ for each interval I (using standard additive properties of $\vec{n}_B$, $B \in \mathscr{A}$).

Using the divergence theorem for linear vector functions one easily sees that F is differentiable on D with $\dot{F} = \operatorname{div} \vec{v}$, and that F satisfies the condition λ_{α_i} resp. Λ_{α_i} (relative to A) on M_i resp. N_i if $n - 1 \leqslant \alpha_i \leqslant n$ as well as the condition λ_{n-1} (relative to A) on $A - \bigcup_{\alpha_i \leqslant n-1} (M_i \cup N_i)$. E.g., take $x \in M_i \cup N_i$ where $\alpha_i = n$ and determine $K > 0$, $\delta > 0$ such that $\|\vec{v}(y) - \vec{v}(x)\| \leqslant K\|y - x\|$ for all $y \in B(x, \delta) \cap A$. Then for any interval I containing x and having diameter less than δ we conclude:

$$|F(I)| = \left| \int_{\partial(A \cap I)} (\vec{v} - \vec{v}(x)) \cdot \vec{n}_{A \cap I} \, \mathrm{d}\mathscr{H} \right| \leqslant K d(A \cap I) |\partial(A \cap I)|_{n-1}.$$

Now take $x \in M_i$ with $0 < \alpha_i < n-1$ and let us show that F satisfies λ_{α_i} at x relative to A. In the proof of Thm 2.1 in [Ju-No 2] it was shown that the inequality $\int_{\partial I} \|y-x\|^{\alpha_i+1-n}\,\mathrm{d}\mathscr{H}(y) \leqslant c(n,\alpha_i)\,d(I)^{\alpha_i}$ holds for each interval I containing x, where $c = c(n,\alpha_i)$ is a positive constant depending only on n and α_i. Given $\varepsilon > 0$ we determine a $\delta > 0$ such that $\|\vec{v}(y) - \vec{v}(x)\| \leqslant \frac{\varepsilon}{c}\|y-x\|^{\alpha_i+1-n}$ holds for all $y \in B(x,\delta) \subseteq A^\circ$, $y \neq x$. If now I denotes an interval with $x \in I$ and $d(I) < \delta$ we get

$$|F(I)| = \left|\int_{\partial I} (\vec{v} - \vec{v}(x)) \cdot \vec{n}_I\,\mathrm{d}\mathscr{H}\right| \leqslant \frac{\varepsilon}{c}\,c\,d(I)^{\alpha_i} = \varepsilon d(A\cap I)^{\alpha_i}.$$

Analogously one shows that F satisfies Λ_{α_i} on N_i (relative to A) for $0 < \alpha_i < n-1$. Thus the ν_1-integrability of $\dot{F}$ on A follows by the Fundamental Theorem, and since $\dot{F} = \operatorname{div}\vec{v}$ a.e. on A we have

$$\overset{\nu_1}{\int_A} \operatorname{div}\vec{v} = \overset{\nu_1}{\int_A} \dot{F} = F(\mathbb{R}^n) = \int_{\partial A} \vec{v}\cdot\vec{n}_A\,\mathrm{d}\mathscr{H}.$$

□

Remark 3.1. (i) For $n \geqslant 2$ the case $\alpha_i = 0$ has to be excluded since otherwise the integral $\int_{\partial(A\cap I)} \vec{v}\cdot\vec{n}_{A\cap I}\,\mathrm{d}\mathscr{H}$ can fail to exist. But if $n = 1$ the case $\alpha_i = 0$ can obviously be included since $\vec{v}$ remains continuous on A.

(ii) The analytic assumptions in our Divergence Theorem do not completely cover the situation of the corresponding theorem for intervals in [Ju-No 2] since we here require the singularities $\bigcup_{\alpha_i<n-1} (M_i \cup N_i)$ not to lie on the boundary of A. Anyhow, imposing certain regularity conditions on ∂A which, in particular, are fulfilled by intervals, it is possible to include this situation.

References

[Fed] *H. Federer*: Geometric Measure Theory. Springer, New York, 1969.

[Jar-Ku 1] *J. Jarnik and J. Kurzweil*: A non-absolutely convergent integral which admits C^1-Transformations. Časopis pro Pěstovaní Mat. *109* (1984), 157–167.

[Jar-Ku 2] *J. Jarnik and J. Kurzweil*: A non-absolutely convergent integral which admits transformation and can be used for integration on manifolds. Czech. Math. J. *35 (110)* (1985), 116–139.

[Jar-Ku 3] *J. Jarnik and J. Kurzweil*: A new and more powerful concept of the *PU*-integral. Czech. Math. J. *38 (113)* (1988), 8–48.

[Ju] *W.B. Jurkat*: The Divergence Theorem and Perron integration with exceptional sets. Czech. Math. J. *43* (1993), 27–45.

[Ju-No 1] *W.B. Jurkat and D.J.F. Nonnenmacher*: An axiomatic theory of non-absolutely convergent integrals in $\mathbf{R}^n$. Fund. Math. *145* (1994), 221–242.

[Ju-No 2] *W.B. Jurkat and D.J.F. Nonnenmacher*: A generalized n-dimensional Riemann integral and the Divergence Theorem with singularities. Acta Sci. Math. (Szeged) *59* (1994), 241–256.

[No] *D.J.F. Nonnenmacher*: Theorie mehrdimensionaler Perron-Integrale mit Ausnahmemengen. PhD thesis, Univ. of Ulm, 1990.

[Pf 1] *W.F. Pfeffer*: The Divergence Theorem. Trans. Amer. Math. Soc. *295* (1986), 665–685.

[Pf 2] *W.F. Pfeffer*: The Gauß-Green Theorem. Advances in Mathematics *87* (1991), no. 1, 93–147.

[Saks] *S. Saks*: Theory of the integral. Dover, New York, 1964.

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