OSCILLATION CRITERIA FOR FORCED NEUTRAL DIFFERENTIAL EQUATIONS

S. R. GRACE,* Cairo, and B. S. LALLI, Saskatoon

(Received December 23, 1992)

1. INTRODUCTION

In this paper we are concerned with the oscillatory behavior of forced neutral differential equations of the form

$$(1.1;\delta)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)+px[t+\delta\sigma]\big)-q(t)f\big(x[g(t)]\big)=e(t),$$

$$(1.2;\delta)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)+px[t+\delta\sigma]\big)+q(t)f\big(x[g(t)]\big)=e(t),$$

and

$$(1.3;\delta)\qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(x(t)+px[t+\delta\sigma]\big)+q(t)f\big(x[g(t)]\big)=e(t),$$

where $\delta=\pm1$, p and σ are nonnegative real constants. The functions e, g, $q\colon [t_0,\infty)\to\mathbb{R}$, $t_0\geqslant 0$ and $f\colon\mathbb{R}\to\mathbb{R}$ are continuous; $q(t)\geqslant 0$ and is not identically zero on any ray of the form $[t^*,\infty)$, $t^*\geqslant t_0$. The function g is such that $\lim\limits_{t\to\infty} g(t)=\infty$ and f satisfies the condition $xf(x)>0$ for $x\neq 0$.

By a solution of the equation $(1.i;\delta)$, $i=1,2,3$, we mean a function $x\colon [T_x,\infty)\to\mathbb{R}$ such that $x(t)+px[t+\delta\sigma]$ is continuously differentiable and satisfies $(1.i;\delta)$ for all $t\geqslant T_x$. A solution $x(t)$ of $(1.i;\delta)$ is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory. Equation $(1.i;\delta)$ is said to be oscillatory if all of its solutions are oscillatory.

Now we list two assumptions which are needed below:

There exists a function $\eta\in C^i[t_0,\infty)$, $i=1,2$ such that

$$(1.4;\,i)\qquad \frac{\mathrm{d}^i}{\mathrm{d}t^i}\big(\eta(t)\big)=e(t),\quad \eta \text{ is oscillatory,}$$

* The research was started during the summer of 1992 while this author was visiting the University of Saskatchewan as a visiting Professor of Mathematics.

(1.5) η is periodic of period σ i.e, $\eta(t \pm \sigma) = \eta(t)$ for all t and σ.

The oscillatory behavior of neutral equations of the type $(1.i;\delta)$ with $e(t) \equiv 0$ has been extensively studied by many authors, see, for example [1], [2], [5], [6], [11] and [12], and the reference cited therein. When $p = 0$ Kartsatos ([7], [8]) obteined some criteria for $(1.3;\delta)$, however, for the case when $p \neq 0$, very little is known. Therefore the purpose of this paper is to establish some oscillation criteria for $(1.i;\delta)$, $i = 1$, 2, 3.

2. Oscillation of equations $(1.i;\delta)$, $i = 1$, 2.

In this section we establish some sufficient conditions under which equations $(1.i;\delta)$, $i = 1$, 2 are oscillatory.

Theorem 2.1. *Let condition* $(1.4;1)$ *hold. If*

$$\limsup_{t\to\infty} \eta(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} \eta(t) = -\infty, \tag{2.1}$$

then all bounded solutions of Eq. $(1.1;\delta)$ *are oscillatory.*

Proof. Let $x(t)$ be a bounded and nonoscillatory solution of Eq. $(1.1;\delta)$ and assume that there exists a $t_0 \geqslant 0$ such that

$$x(t) > 0, \quad x[t+\delta\sigma] > 0 \quad \text{and} \quad x[g(t)] > 0 \quad \text{for } t \geqslant t_0.$$

Define

$$y(t) = x(t) + px[t+\delta\sigma] \quad \text{and} \quad z(t) = y(t) - \eta(t).$$

Then Eq. $(1.1;\delta)$ takes the form

$$z'(t) = q(t)f\big(x[g(t)]\big) > 0 \quad \text{for } t \geqslant t_0, \quad \Big({}' = \frac{\mathrm{d}}{\mathrm{d}t}\Big).$$

It follows that $z(t)$ is an increasing function on $[t_0, \infty)$. We show that $z(t) > 0$ for $t \geqslant T$ for some $T \geqslant t_0$. If not, then $z(t) < 0$ for $t \geqslant t_1$, for some $t_1 \geqslant t_0$. Hence

$$y(t) - \eta(t) < 0, \quad \text{that is, } y(t) < \eta(t) \quad \text{for } t \geqslant t_1,$$

which is a contradiction, since $\eta(t)$ is oscillatory and $y(t)$ is positive. Thus, we have

$$z(t) > 0 \quad \text{and} \quad z'(t) > 0 \quad \text{for } t \geqslant T. \tag{2.2}$$

Taking limit superior $y(t) > \eta(t)$ we have

$$\limsup_{t\to\infty} y(t) > \limsup_{t\to\infty} \eta(t) = \infty,$$

which contradicts the fact that $y(t)$ is bounded. This completes the proof of the Theorem. □

Our next result is for Eq. $(1.2;\delta)$.

Theorem 2.2. *Assume that conditions* $(1.4;1)$ *and* (2.1) *are satisfied. Then Eq.* $(1.2;\delta)$ *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. $(1.2;\delta)$. We may (and we do) that $x(t)$ is eventually positive. There exists a $t_0 \geqslant 0$ such that $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqslant t_0$. With functions $y(t)$ and $z(t)$ defined as before we have

$$z'(t) = -q(t)f\big(x[g(t)]\big) < 0 \quad \text{for } t \geqslant t_0.$$

This implies that $z(t)$ is eventually of one sign. As in the proof of Theorem 2.1, we have $z(t) > 0$. Thus

$$z(t) > 0 \quad \text{and} \quad z'(t) < 0 \quad \text{for } t \geqslant T. \tag{2.3}$$

Since $z(t) + \eta(t) = y(t) > 0$, we have $z(t) \geqslant -\eta(t)$. From which it follows that

$$\limsup_{t\to\infty} z(t) \geqslant \limsup_{t\to\infty} \big(-\eta(t)\big) = -\liminf_{t\to\infty} \eta(t) = \infty,$$

which contradicts the fact that $z(t)$ is bounded above. Thus the proof of the Theorem is complete. □

For illustration purposes we provide the following examples.

Example 2.1. Consider the forced neutral differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x(t) + px[t+\delta\sigma]\big) - \frac{(1+p\mathrm{e}^{-\delta\sigma})\mathrm{e}^{-t}}{(1-\mathrm{e}^{-g(t)})^{\alpha}}|x[g(t)]|^{\alpha}\operatorname{sgn} x[g(t)] = t\cos t + \sin t, \quad t > 0, \tag{2.4; δ}$$

where $\delta = \pm 1$, p and σ are nonnegative real numbers, α is a positive constant, g; $[t_0,\infty) \to \mathbb{R}$ is continuous and $\lim_{t\to\infty} g(t) = \infty$. If we choose $\eta(t) = t\sin t$, then all the hypotheses of Theorem 2.1 are satisfied and hence every bounded solution of $(2.4;\delta)$ is oscillatory. It is easy to verify that the corresponding unforced equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x(t) + px[t+\delta\sigma]\big) = \frac{(1+p\mathrm{e}^{-\delta\sigma})\mathrm{e}^{-t}}{(1-\mathrm{e}^{-g(t)})^{\alpha}}|x[g(t)]|^{\alpha}\operatorname{sgn} x[g(t)] \tag{2.5; δ}$$

has a bounded nonoscillatory solution $x(t) = 1 - \mathrm{e}^{-t}$.

Example 2.2. Consider the forced neutral differential equation

$$(2.6;\delta)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)+px[t+\delta\sigma]\big)+(1+p\mathrm{e}^{-\delta\sigma})\mathrm{e}^{\alpha g(t)-t}|x[g(t)]|^{\alpha}\operatorname{sgn}x[g(t)] = \mathrm{e}^{t}(\sin t+\cos t),\quad t\geqslant 0,$$

where $\delta=\pm 1$, α, σ are nonnegative constants, $\alpha>0$; $g\colon [t_0,\infty)\to \mathbb{R}$ is continuous and $g(t)\to\infty$ as $t\to\infty$. Here, we choose $\eta(t)=\mathrm{e}^{t}\sin t$ and find that all the conditions of Theorem 2.2 are fulfilled. Thus $(2.6;\delta)$ is oscillatory. We also note that the corresponding unforced equation

$$(2.7;\delta)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)+px[t+\delta\sigma]\big)+(1+p\mathrm{e}^{-\delta\sigma})\mathrm{e}^{\alpha g(t)-t}|x[g(t)]|^{\alpha}\operatorname{sgn}x[g(t)] = 0,\quad t>0$$

has a nonoscillatory solution $x(t)=\mathrm{e}^{-t}$.

Remark 2.1. From these examples it is evident that the presence of a forcing term can generate oscillations in an otherwise nonoscillatory equation.

The following theorem is concerned with the oscillatory behavior of the superlinear equations $(1.1;\delta)$ i.e., the equation when the function f satisfies the condition

$$(2.8)\qquad f'(x)\geqslant 0\quad\text{for } x\neq 0\quad\text{and}\quad \int_{\pm\varepsilon}^{\pm\infty}\frac{\mathrm{d}u}{f(u)}<\infty,\quad \varepsilon>0.$$

For convenience we introduce the following notation:

$$A_{(g,\beta)}=\{t\in[t_0,\infty)\colon g(t)>t+\beta\geqslant t_0\},$$

where β is a nonnegative constant.

Theorem 2.3. *Suppose that conditions* (1.4; 1), (1.5) *and* (2.8) *are satisfied. If, in addition*

$$(2.9)\qquad \int_{A(g,\beta)} q(s)=\infty$$

holds, then,

(i) *equation* $(1.1;-1)$ *is oscillatory provided* $0\leqslant p<1$ *and* $\beta=0$;

(ii) *equation* $(1.1;1)$ *is oscillatory provided* $p>1$ *and* $\beta=\sigma$.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. $(1.1;\delta)$ which is such that

$$x(t)>0,\quad x[t+\delta\sigma]>0\quad\text{and}\quad x[g(t)]>0\quad\text{for } t\geqslant t_0\geqslant 0.$$

With $y(t)$ and $z(t)$ as defined in the proof of Theorem 2.1 we obtain (2.2). We consider two cases.

Case 1: $\delta = -1$ and $0 \leqslant p < 1$.

From the definition of $z(t)$ we have

$$x(t) = z(t) + \eta(t) - p(z[t-\sigma] + \eta[t-\sigma] - px[t-2\sigma]).$$

In view of the fact that η is periodic and z is increasing, it is possible to choose t_1 such that

$$x(t) \geqslant (1-p)\big(z(t) + \eta(t)\big), \quad t \geqslant t_1 \geqslant t_0. \tag{2.10}$$

There exists a $T \geqslant t_1$ such that

$$x(t) \geqslant (1-p)\big(z(t) + \eta(T)\big) = \xi_1(t), \quad t \geqslant T. \tag{2.11}$$

Clearly

$$z'(t) = \frac{1}{1-p}\xi_1'(t), \quad t \geqslant T$$

and

$$\begin{aligned} \xi_1(t) &= (1-p)\big(z(t) + \eta(t)\big) \\ &\geqslant (1-p)\big(z(T) + \eta(T)\big) \\ &> 0 \quad \text{for } t \geqslant T. \end{aligned}$$

Case 2: $\delta = 1$ and $p > 1$.

Once again, from the definition of $z(t)$, we have

$$\begin{aligned} x(t) &= \frac{1}{p}(z[t-\sigma] + \eta[t-\sigma] - x[t-\sigma]) \\ &= \frac{1}{p}\Big(z[t-\sigma] + \eta[t-\sigma] - \frac{1}{p}(z[t-2\sigma] + \eta[t-2\sigma] - x[t-2\sigma])\Big). \end{aligned}$$

Using (1.5) and (2.2), as was done before, we choose a sufficiently large $t_1^* \geqslant t_0$ such that

$$x(t) \geqslant \frac{(p-1)}{p^2}(z[t-\sigma] + \eta[t-\sigma]), \quad t \geqslant t_1^*. \tag{2.12}$$

There exists $T_1 \geqslant t_1$ such that

$$x(t) \geqslant \frac{p-1}{p^2}(z[t-\sigma] + \eta[T_1-\sigma]) = \xi_2(t-\sigma), \quad t \geqslant T_1. \tag{2.13}$$

As in the case 1, we have

$$z'(t) = \frac{p^2}{p-1}\xi_2'(t) \quad \text{and} \quad \xi_2(t) > 0, \quad t \geqslant T_1.$$

In view of (2.11) and (2.13), Eq. $(1.1;\delta)$ reduces to

$$\xi_i'(t) \geqslant \gamma q(t) f\big(\xi_i[g(t)-\beta]\big) \quad t \geqslant T^* \geqslant \max\{T, T_1\}, \quad i = 1, 2, \tag{2.14}$$

where

$$\gamma = \begin{cases} 1-p, \ \beta = 0, & \text{if } i = 1; \\ \dfrac{p-1}{p^2}, \ \beta = \sigma, & \text{if } i = 2. \end{cases} \tag{2.15}$$

Divide (2.14) by $f\big(\xi_i(t)\big)$ and then integrate over $D = A_{(g,\beta)} \cup [T^*, t]$. Since ξ_i is nondecreasing, we have $\xi_i[g(t)-\beta] \geqslant \xi_i(t)$, $i = 1, 2$, on the set D. Hence

$$\int_{T^*}^{t} \frac{\xi_i'(s)}{f\big(\xi_i(s)\big)} \,\mathrm{d}s \geqslant \gamma \int_D q(s)\,\mathrm{d}s\,.$$

Now Letting $t \to \infty$ we get

$$\int_D q(s)\,\mathrm{d}s \geqslant \gamma \int_{\xi_i(T^*)}^{\infty} \frac{\mathrm{d}u}{f(u)} < \infty,$$

which contradicts (2.9). This completes the proof of the Theorem. □

In the following theorem we deal with the case when $(1.1;\delta)$ is almost linear i.e., when f satisfies the condition

$$\frac{f(x)}{x} \geqslant M \quad \text{for } x \neq 0. \tag{2.16}$$

Theorem 2.4. *Suppose that $g(t) \geqslant t + \beta$ and that $g'(t) \geqslant 0$ for $t \geqslant t_0$. Furthermore, let conditions* (1.4; 1), (1.5) *and* (2.16) *hold. If*

$$\liminf_{t\to\infty} \int_t^{g(t)-\beta} q(s)\,\mathrm{d}s > \frac{\gamma^*}{\mathrm{e}}, \quad \beta,\ \gamma^* \text{ are positive constants,} \tag{2.17}$$

then

(i) *equation* $(1.1;-1)$ *is oscillatory provided* $0 \leqslant p < 1$, $\gamma^* = \frac{1}{M(1-p)}$, $\beta = 0$;

(ii) *equation* $(1.1;1)$ *is oscillatory provided* $p > 1$, $\gamma^* = \frac{p^2}{M(1-p)}$, $\beta = \sigma$.

Proof. Suppose that Eq. $(1.2;\delta)$ has a nonoscillatory solution $x(t)$ which is eventually positive i.e., there exists a t_0 such that

$$x(t) > 0, \quad x[t+\delta\sigma] > 0 \quad \text{and} \quad x[g(t)] > 0 \quad \text{for } t \geqslant t_0.$$

With y and z as defined in the proof of Theorem 2.1. we obtain (2.2) and then

$$z'(t) = q(t)f\big(x[g(t)]\big) \quad \text{for } t \geqslant t_0. \tag{2.18}$$

Use (2.16) in (2.18) to get

$$z'(t) \geqslant Mq(t)x[g(t)] \quad \text{for } t \geqslant t_0. \tag{2.19}$$

Now we consider two cases: (1) $\delta = -1$ and $0 \leqslant p < 1$; (2) $\delta = 1$ and $p > 1$. Proceeding as in the proof of Theorem 2.3 we get (2.11) and (2.13) respectively. Next we use (2.11) and (2.13) in (2.19) and obtain

$$\xi_i'(t) \geqslant \theta q(t)\xi_i[g(t)-\beta] \quad \text{for some } T^* \geqslant t_0. \tag{2.20}$$

where

$$\theta = \begin{cases} M(1-p), & \beta = 0, \qquad \text{if } i = 1; \\ M\dfrac{p-1}{p^2}, & \beta = \sigma \qquad \text{if } i = 2. \end{cases}$$

However, condition (2.17 implies that inequality (2.20) has no eventually positive solution (see analogous result in [10])), which is a contradiction. The proof of Theorem is now complete. □

Remark 2.2. 1. Theorems 2.3 and 2.4 are applicable to equations of the type $(1.1;\delta)$ where the argument g is of either advanced or of mixed type.

2. The results of this section can be extended to more general equations of the form considered in [6].

The following examples are illustrative.

Example 2.3. Consider the neutral superlinear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x(t) + px[t+2\pi\delta]\big) - \frac{1}{t}\big|x[t+\sin t+\beta]\big|^{\lambda} \operatorname{sgn} x[t+\sin t+\beta] = \cos t, \quad t \geqslant 2\pi \quad \text{and} \quad \lambda > 1, \tag{2.21;δ}$$

where $\delta = \pm 1$, p and β are nonnegative constants. We let $\eta(t) = \sin t$. For $\beta = 0$ or 2π we note that

$$\int\limits_{A(g,\beta)} q(s)\,\mathrm{d}s = \mathrm{d}s = \sum_{m=1}^{\infty} \int\limits_{2\pi m}^{(2m+1)\pi} \frac{1}{s}\,\mathrm{d}s = \infty.$$

We apply Theorem 2.3 to $(2.21;\delta)$ and conclude that

(i) equation $(2.21;\ -1)$ is oscillatory provided $0 \leqslant p < 1$ and $\beta = 0$;

(ii) equation $(2.21;\ 1)$ is oscillatory provided $p > 1$ and $\beta = 2\pi$.

Example 2.4. Consider the neutral linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) + px[t + 2\pi\delta]\right) - px\left[t + \frac{\alpha\pi}{2}\right] = \cos t, \quad t \geqslant 0 \tag{2.22;δ}$$

where $\delta = \pm 1$, p is a nonnegative constant and $\alpha \in \{1, 2, 5, 9, \ldots\}$. Here we take $\eta(t) = \sin t$ and apply Theorem 2.4 to $(2.22;\delta)$ to conclude that

(i) equation $(2.22;\ -1)$ is oscillatory if

$$0 \leqslant p < 1 \quad \text{and} \quad p(1-p)\frac{\alpha\pi}{2} > \frac{1}{\mathrm{e}}, \quad \alpha \in \{1, 5, 9, \ldots\};$$

(ii) equation $(2.22;\ 1)$ is oscillatory if

$$p > 1, \quad \left(\frac{p-1}{p}\right)\left(\frac{\alpha\pi}{2} - 2\pi\right) > \frac{1}{\mathrm{e}}, \quad \alpha \in \{5, 9, \ldots\}.$$

We note that $(2.22;\delta)$ has a oscillatory solution $x(t) = \sin t$.

3. Oscillation of Equation $(1.3;\delta)$

In this section we establish some oscillation criteria for second order neutral equation $(1.3;\delta)$.

Theorem 3.1. *Let condition* $(1.4;2)$ *hold. If*

$$\limsup_{t\to\infty} \frac{\eta(t)}{t} = \infty \quad \text{and} \quad \liminf_{t\to\infty} \frac{\eta(t)}{t} = -\infty \tag{3.1}$$

then $(1.3;\delta)$ *is oscillatory.*

Proof. To the contrary, suppose that $(1.3;\delta)$ has a nonoscillatory solution $x(t)$ which is such that

$$x(t) > 0, \quad x[t + \delta\sigma] > 0 \quad \text{and} \quad x[g(t)] > 0 \quad \text{for } t \geqslant t_0.$$

With y and z, as defined in Theorem 2.1, we have

$$z''(t) = -q(t)f\left(x[g(t)]\right) \leqslant 0 \quad \text{for } t \geqslant t_0, \tag{3.2}$$

and as shown in the proof of Theorem 2.1 we have $z(t) > 0$ for $t \geqslant t_0$. Hence, by Kiguradze's lemma [9], there exists a $t_1 \geqslant t_0$ such that $z'(t) > 0$ for $t \geqslant t_1$. Thus we have

$$z(t) > 0, \quad z'(t) > 0 \quad \text{and} \quad z''(t) \leqslant 0 \quad \text{for } t \geqslant t_0. \tag{3.3}$$

From (3.2) it is easy to verify that there exist a constant $M > 0$ and $t_2 \geqslant t_1$ such that

$$z(t) \leqslant Mt \quad \text{for } t \geqslant t_2. \tag{3.4}$$

Now,

$$z(t) + \eta(t) = y(t) = x(t) + px[t + \delta\sigma] > 0 \quad \text{for } t \geqslant T_2$$

or

$$\frac{z(t)}{t} > -\frac{\eta(t)}{t} \quad \text{for } t \geqslant t_2.$$

Taking limit superior on both sides of the above inequality, we get

$$\limsup_{t\to\infty} \frac{z(t)}{t} \geqslant \limsup_{t\to\infty} \Big(-\frac{\eta}{t} \Big) = -\liminf_{t\to\infty} \frac{\eta}{t} = \infty,$$

which contradicts (3.4). The proof is now complete. □

Now we study the oscillatory behavior of $(1.3;\delta)$ via comparison with a second order functional differential equation whose oscillatory character is known and which has been studied extensively in literature.

Theorem 3.2. *In addition to* (1.4; 2) *and* (1.5), *assume that* $f'(x) \geqslant 0$ *for* $x \neq 0$. *If the equation*

$$y''(t) + \gamma q(t) f\big(y[g^*(t)]\big) = 0, \tag{3.5}$$

is oscillatory, where $g^*(t) = \min\{t, g(t) - \beta\}$ *and is nondecreasing for* $t \geqslant t_0$ (γ, β *are constants, defined below), then*

(i) *equation* (1.3; -1) *is oscillatory provided* $0 \leqslant p < 1$, $\gamma = 1 - p$ *and* $\beta = 0$;
(ii) *equation* (1.3; 1) *is oscillatory provided* $p > 1$, $\gamma = \frac{p-1}{p^2}$ *and* $\beta = \sigma$.

Proof. To the contrary, suppose that $(1.3;\delta)$ has a nonoscillatory solution $x(t)$ which is such that

$$x(t) > 0, \quad x[t + \delta\sigma] > 0 \quad \text{and} \quad x[g(t)] > 0 \quad \text{for } t \geqslant t_0.$$

With y and z, as defined in Theorems 2.1 and 3.1, we have (3.2) i.e.,

$$z''(t) = -q(t)f\big(x[g(t)]\big) \leqslant 0 \quad \text{for } t \geqslant t_2.$$

Since $z(t)$ is an increasing function and $\eta(t)$ is periodic of period σ, we proceed as in the proof for the two cases considered in Theorem 2.3 and obtain (2.11) and (2.13). Using (2.11) and (2.13) in equation (3.2) we get

$$\xi_i''(t) + \gamma q(t)f\big(\xi_i[g(t) - \beta]\big) \leqslant 0 \quad \text{for } t \geqslant T^* \geqslant t_2,$$

or

$$\xi_i''(t) + \gamma q(t)f\big(\xi_i[g^*(t)]\big) \leqslant 0 \quad \text{for } t \geqslant T^* \geqslant t_2,$$

where

$$\gamma = \begin{cases} 1-p, \quad \beta = 0, & \text{if } i = 1; \\ \dfrac{p-1}{p^2}, \quad \beta = \sigma, & \text{if } i = 2. \end{cases}$$

As shown by Foster and Grimmer [1] the equation

$$\xi_i''(t) + \gamma q(t)f\big(\xi_i[g^*(t)]\big) = 0 \quad \text{for } t \geqslant T^* \geqslant t_2, \quad i = 1, 2,$$

has a positive nonoscillatory solution, which is a contradiction. Thus the proof of the Theorem is complete. □

The following examples are illustrative

Example 3.1. Consider the forced second order neutral differential equation

$$(3.6;\delta) \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(x(t) + px[t+\delta\sigma]\big) + \frac{(g(t))^{-\frac{\lambda}{2}}}{4}\Big(\frac{1}{t^{-3/2}} + \frac{1}{(t+\delta\sigma)^{-3/2}}\Big)$$
$$\times\big(|x[g(t)]|^{\lambda}\big) \times \operatorname{sgn} x[g(t)] = c\mathrm{e}^t \cos t, \quad \lambda > 0, \quad t > \pi,$$

where $\delta = \pm 1$, c, p and σ are non-negative constants, $g\colon [t_0, \infty) \to (0, \infty)$ is continuous with $\lim\limits_{t\to\infty} g(t) = \infty$. If $c = 2$ we take $\eta(t) = \mathrm{e}^t \sin t$. Thus, all the conditions of Theorem 3.1 are satisfied and hence $(3.6;\delta)$ is oscillatory. We note that if $c = 0$, $(3.6;\delta)$ has a non-oscillatory solution $x(t) = \sqrt{t}$.

Example 3.2. Consider the forced second order neutral differential equation
(3.7;δ)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(x(t) + px[t + 2\pi\delta]\big) + q(t)\big(|x[g(t)]|^{\lambda}\big)\operatorname{sgn} x[g(t)] = -\sin t, \quad t > 0, \quad \lambda > 0,$$

where $\delta = \pm 1$, p is a non-negative constant, q, $g\colon [t_0, \infty) \to \mathbb{R}$ are continuous, $q(t) \geqslant 0$ and not identically zero on any ray of the form $[t^*, \infty)$, $t^* \geqslant t_0$ and $\lim\limits_{t\to\infty} g(t) = \infty$.

We choose $\eta(t) = \sin t$ and apply Theorem 3.2 to conclude that $(3.7;\delta)$ is oscillatory if the second order equation

$$(*) \qquad y''(t) + \gamma q(t)\big(|y[h(t)]|^{\lambda}\big)\operatorname{sgn} y[h(t)] = 0, \quad t \geqslant 0, \quad \lambda > 0$$

is oscillatory. Here we have $h(t) = \min\{t, g(t) - \beta\}$, and $h'(t) > 0$ for $t > 0$, and

$$\gamma = \begin{cases} 1 - p, & \beta = 0, \qquad \text{if } \delta = -1, \quad 0 \leqslant p < 1; \\ \dfrac{p-1}{p^2}, & \beta = 2\pi, \qquad \text{if } \delta = 1, \quad p > 1. \end{cases}$$

According to results in [4] (specialized to $(*)$, for example, Theorem 5) $(3.7;\delta)$ is oscillatory if $p \in (0,1) \cup (1,\infty)$ and one of the following conditions is satisfied

(i) $\lambda > 1$ and $\int^{\infty} h(s)q(s)\,\mathrm{d}s = \infty$;

(ii) $\lambda = 1$ and there exists a differentiable function $\varrho\colon (t_0,\infty) \to (0,\infty)$ such that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\varrho(s)q(s) - \frac{\big(\varrho'(s)\big)^2}{4\varrho(s)h'(s)}\right] \mathrm{d}s = \infty;$$

(iii) $0 < \lambda < 1$ and $\int^{\infty} \big(h(s)\big)^{\lambda} q(s)\,\mathrm{d}s = \infty$.

From example 3.1 it is clear that the forcing term can generate oscillations, while, in example 3.2 we note that the periodic forcing term can preserve oscillations.

Remark 3.1. 1. It is easy to verify that all of our results remain valid when $p = 0$. Moreover, the conclusions of Theorems 2.3, 2.4 and 3.2 remain valid even when $e(t) \equiv 0$.

2. Theorems 2.1, 2.2 as well as other results of section 3 are applicable to equations of the type $(1.i;\delta)$, $i = 1, 2, 3$ for any type of deviating argument g, retarded, advanced or of mixed type.

3. The forcing term considered in this paper need not be "small" as is the case in [7], [8] and the references cited therein.

4. The results of this paper are extendable to higher order neutral differential equations of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(x(t) + px[t + \delta\sigma]\big) \pm q(t)f\big(x[g(t)]\big) = e(t), \quad n \geqslant 3.$$

References

[1] *K. E. Foster and R. C. Grimmer*: Nonoscillatory solutions of higher order delay differential equations. J. Math. Anal. Appl. *77* (1980), 150–164.

[2] *S. R. Grace and B. S. Lalli*: Oscillation of nonlinear second order neutral delay differential equations. Rad. Mat. *3* (1987), 77–84.

[3] *S. R. Grace and B. S. Lalli*: Oscillation and asymptotic behavior of certain second order neutral differential equations. Rad. Mat. *5* (1989), 121–126.

[4] *S. R. Grace and B. S. Lalli*: Oscillation theorems for second order functional differential equations with damping. Bull. Instit. Math. Academia Sinica *13* (1985), 183–292.

[5] *M. K. Grammatikopoulos, G. Ladas and A. Meimaridou*: Oscillation and asymptotic behavior of higher order neutral differential equations with variable coefficients. Chines Ann. Math. Ser. B *9* (1988), 322–338.

[6] *J. Jaroš and T. Kusano*: Oscillation properties of first order nonlinear functional differential equations of neutral type. Diff. and Integral Equations *4* (1991), 425–436.

[7] *A. G. Kartsatos*: On the maintenance of oscillations of n^{th} order equations under the effect of small forcing term. J. Diff. Equations *10* (1971), 355–363.

[8] *A. G. Kartsatos*: Maintenance of oscillations under the effect of a periodic forcing term. Proc. Amer. Math. Soc. *33* (1972), 377–383.

[9] *I. T. Kiguradze*: On the oscillations of equation $\frac{d^m u}{dt^m} + a(t)|u|^n \operatorname{sgn} u = 0$. Mat. Sb. *65* (1964), 172–187. (In Russian.)

[10] *R. G. Koplatadze and T. A. Chanturija*: On oscillatory and monotone solutions of first order differential equations with deviating arguments. Differential'nye Uravnenija *18* (1982), 1463–1465. (In Russian.)

[11] *G. Ladas and Y. G. Sficas*: Oscillations of higher order neutral equations. J. Austral. Math. Soc., Ser B *27* (1986), 502–511.

[12] *G. Ladas and Y. G. Sficas*: Oscillations of neutral delay differential equations. Canad. Math. Bull. *29* (1986), 435–445.

Authors' addresses: *S. R. Grace*, Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12000, Egypt; *B. S. Lalli*, Department of Mathematics, University of Saskatchewan, Saskatoon, S7N 0W0, Canada.