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ON SOME PROPERTIES OF SETS WITH POSITIVE MEASURE

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In this paper E^1 denotes the 1-dimensional Euclidean space, $A_1, A_2, \ldots, A_n, A_1^*, A_2^*, \ldots, A_n^*$ are subsets of E^1; $\mathscr{L}$ denotes the σ-algebra of Lebesgue measurable sets in E^1 and the Lebesgue measure of any $A \in \mathscr{L}$ is denoted by $m(A)$.

We note that $m(\alpha A) = |\alpha| m(A)$ for any $\alpha \in E^1$ where the set αA is the collection of all real numbers of the form $\alpha x (x \in A)$. By $R[A : B]$ we have meant the collection of all real numbers of the form $\frac{x}{y}$ or $\frac{y}{x}$ where $x \in A$ and $y \in B$. Many papers have been devoted to the study of ratio sets in E^1. The articles [1], [4] are representations of works in this area. In our paper we intend to study some properties of sets with positive measure using primarily properties of continuous functions and Lebesgue density theorem.

Theorem 1. *Let $A_i \in \mathscr{L}$ with $m(A_i) > 0$ and for each i $(1 \leqslant i \leqslant n)$,*

$$A_i^* = \{x \in E^1 \setminus \{0\} \colon A_i \text{ has metric density 1 at } x\}.$$

Then for every fixed choice of $a_i \in A_i^*$, there exists $\delta (> 0)$ and a real number η s.t.

$$m\Big(\frac{A_1}{x_1} \cap \frac{A_2}{x_2} \cap \ldots \cap \frac{A_n}{x_n}\Big) > \eta > 0 \text{ whenever } |x_i - a_i| < \delta.$$

The proof of the above theorem is done with the help of the following lemmas.

Lemma 1. *If A, B and $C (\in \mathscr{L})$, then*

$$\big|m(A \cap C) - m(A \cap B)\big| \leqslant m(B \Delta C),$$

where ΔC is the symmetric difference of B and C.

It follows by the subadditivity of m [2].

Lemma 2. *If A, B ($\in \mathscr{L}$) and have finite measures, then the function*

$$w(t) = m(A \cap \alpha e^t B)$$

(for $t \in E^1$) is continuous for every $t \in E^1$, whatever be the choice of α (in E^1).

Proof of the Lemma. For $\alpha = 0$, the proof is obvious. So we start proving the lemma with the supposition $\alpha \neq 0$. We assume that both A and B are compact.

Let $\varepsilon > 0$ and $t \in E^1$. We set $B_t = \alpha e^t B$.

Then there exists an open set $G \supseteq B_t$ s.t. $m(G \setminus B_t) < \frac{\varepsilon}{2}$. We may take G as bounded and put $M = \operatorname{Sup}\{|x|: x \in G\}$. Let $d = \operatorname{dist}(B_t, G')$ where G' is the complement of G in E^1. Then $d > 0$.

We shall now show that for every real number h satisfying $1 \leqslant e^h < 1 + \frac{d}{M+1}$, we have $e^h B_t \subseteq G$. When $e^h = 1$, it clearly follows that $e^h B_t \subseteq G$. Now if possible, there exists some $h' \in E^1$ satisfying $1 < e^{h'} < 1 + \frac{d}{M+1}$ but for which $e^{h'} B_t \nsubseteq G$. This would however mean that there exists at least an $x_0 \in B_t$ s.t. $e^{h'} x_0 \notin G$ and hence $|e^{h'} x_0 - x_0| > d$.

But by choice of h' (in E^1), $|e^{h'} x_0 - x_0| = (e^{h'} - 1)|x_0| < \frac{d}{M+1} M < d$ and hence a contradiction. Hence for every real number h which satisfies $1 \leqslant e^h < 1 + \frac{d}{M+1}$ we have $e^h B_t \subseteq G$. But then

$$\begin{aligned} m(G \setminus e^h B_t) &= m(G \setminus \alpha e^{t+h} B) \\ &= m(G) - e^h . m(\alpha e^t B) \\ &\leqslant m(G) - m(B_t) = m(G \setminus B_t) < \frac{\varepsilon}{2}. \end{aligned}$$

Hence for every $h \in E^1$ and satisfying

$$1 \leqslant e^h < 1 + \frac{d}{M+1},$$

we have

$$\begin{aligned} |w(t+h) - w(t)| &= |m(A \cap \alpha e^{t+h} B) - m(A \cap \alpha e^t B)| \\ &= |m(A \cap e^h B_t) - m(A \cap B_t)| \\ &\leqslant m(e^h B_t \Delta B_t) \quad \text{[By Lemma 1]} \\ &< \varepsilon \end{aligned}$$

Since the function τ defined by $\tau(x) = e^x$ (for $x \in E^1$) is a continuous function of x for every $x \in E^1$, there exists $\delta > 0$ s.t.

$$\begin{aligned} h \in [0, \delta) &\Longrightarrow 1 \leqslant e^h < 1 + \frac{d}{M+1} \\ &\Longrightarrow |w(t+h) - w(t)| < \varepsilon. \end{aligned}$$

Since ε and t are arbitrary, it follows that the function w is continuous from the right at every $t \in E^1$.

On similar lines as above it may be verified that the function $\tilde{w}$ defined by

$$\tilde{w}(t) = m\Big(B \cap \frac{e^{-t}}{\alpha} A\Big) \quad (\text{for } t \in E^1)\ (\alpha \neq 0)$$

is continuous from the left at every $t \in E^1$.

Since the function φ defined by

$$\varphi(t) = |\alpha| e^t \quad (\text{for } t \in E^1)$$

is continuous on E^1 and $w(t) = \varphi(t)\tilde{w}(t)$ [for every $t \in E^1$], it follows that the function w is continuous from the left at every point $t \in E^1$.

Hence the function w becomes continuous on E^1 when it is defined in terms of compact sets A and B.

Now consider the general case, i.e., A, B ($\in \mathscr{L}$). Let us consider the strict monotonically decreasing sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$.

For each natural number n, we fix compact sets $A_n \subseteq A$ and $B_n \subseteq B$ with $m(A \setminus A_n) < \frac{1}{n}$, $m(B \setminus B_n) < \frac{1}{n}$.

Let $s \in E^1$ and we choose a non-degenerate closed interval $[a, b]$ with s as one of its interior points.

We consider the functions

$$w_n \colon [a, b] \to E^1 \text{ defined by}$$
$$w_n(t) = m(A_n \cap \alpha e^t B_n) \text{ for } n = 1, 2, \ldots.$$

Clearly each w_n is a continuous function on $[a, b]$ and

$$\begin{aligned}
|w(t) - w_n(t)| &= |m(A \cap \alpha e^t B) - m(A_n \cap \alpha e^t B_n)| \\
&= |m(A \cap \alpha e^t B) - m(A \cap \alpha e^t B_n) + m(A \cap \alpha e^t B_n) - m(A_n \cap \alpha e^t B_n)| \\
&\leqslant |m(A \cap \alpha e^t B) - m(A \cap \alpha e^t B_n)| + |m(A \cap \alpha e^t B_n) - m(A_n \cap \alpha e^t B_n)| \\
&\leqslant m(\alpha e^t B \setminus \alpha e^t B_n) + m(A \setminus A_n) \\
&\qquad [\text{By Lemma 1 and since } A_n \subseteq A,\ B_n \subseteq B] \\
&< |\alpha| e^b m(B \setminus B_n) + m(A \setminus A_n) < \frac{|\alpha| e^b + 1}{n} \quad (\text{for all } t \in [a, b])
\end{aligned}$$

This establishes that w is the uniform limit (on $[a, b]$) of the sequence $\{w_n\}_{n=1}^{\infty}$ of continuous functions. Hence w is continuous on $[a, b]$ and hence continuous at s.

Hence the Lemma is proved. □

P r o o f o f t h e m a i n t h e o r e m. Let us choose and fix $a_i \in A_i^*$ $(1 \leqslant i \leqslant n)$. We first show that $a_1 \in (\frac{A_i}{w_i})^*$ where $w_i = \frac{a_i}{a_1}$ $(1 \leqslant i \leqslant n)$.

For each r (> 0), we have

$$\frac{m\left(\frac{A_i}{w_i} \cap \Delta_r\right)}{r} = \frac{m(A_i \cap w_i \Delta_r)}{|w_i| r}$$

where $\Delta_r = \left[a_1 - \frac{r}{2}, a_1 + \frac{r}{2}\right]$. Then

$$\operatorname*{Lt}_{r \to 0+} \frac{m\left(\frac{A_i}{w_1} \cap \Delta_r\right)}{r} = \operatorname*{Lt}_{r \to 0+} \frac{m(A_i \cap w_i \Delta_r)}{|w_i| r} = 1.$$

Hence $a_1 \in \left(\frac{A_i}{w_i}\right)^*$ $(i = 1, 2, \ldots, n)$ which implies that $a_1 \in \left(A_1 \cap \frac{A_2}{w_2} \ldots \cap \frac{A_n}{w_n}\right)^*$. [This follows from a repeated application of this well-known result—if p be a point of density common to both the sets $A \in \mathscr{L}$ and $B \in \mathscr{L}$ then p is also a point of density of $A \cap B$. For proof see [3].]

We now choose and fix $0 < \varepsilon_1 < \frac{1}{n}$.

Then there exists a non-degenerate closed interval Δ of length r_0 (say) having its mid-point as a_1 s.t.

$$m\left(A_1 \cap \frac{A_2}{w_2} \ldots \cap \frac{A_n}{w_n} \cap \Delta\right) > (1 - \varepsilon_1) r_0$$
$$\Longrightarrow m\left(\frac{A_1}{a_1} \cap \frac{A_2}{a_2} \ldots \cap \frac{A_n}{a_n} \cap \tilde{\Delta}\right) > (1 - \varepsilon_1) b r_0$$

where $\tilde{\Delta} = \frac{1}{|a_1|} \Delta$ and $b = \frac{1}{|a_1|}$. Let us now define the functions

$$\psi_i(t) = m\left(\mathrm{e}^t \frac{A_i}{a_i} \cap \tilde{\Delta}\right), \quad (\text{for } t \in E^1) \quad (i = 1, 2, \ldots, n).$$

By Lemma (2) each ψ_i is continuous function (on E^1).

Since for each $i = 1, 2, \ldots, n$,

$$\psi_i(0) = m\left(\frac{A_i}{a_i} \cap \tilde{\Delta}\right) > (1 - \varepsilon_1) b r_0 > 0, \text{ there exists } \delta_i \ (> 0) \text{ s.t. } t \in (-\delta_i, \delta_i)$$
$$\Longrightarrow \psi_i(t) > (1 - \varepsilon_1) b r_0.$$

Let $\delta_0 = \operatorname{Min}\{\delta_i,\ i = 1, 2, \ldots, n\}$. Then $\psi_i(t) > (1 - \varepsilon_1) b r_0$ whenever $t \in (-\delta_0, \delta_0)$ $(i = 1, 2, \ldots, n)$. Hence for each i $(= 1, 2, \ldots, n)$, there exists intervals J_i with a_i as interior points and diameter $|a_i|$ $(\mathrm{e}^{\delta_0} - \mathrm{e}^{-\delta_0})$ s.t.

$$m\left(\frac{A_i}{x_i} \cap \tilde{\Delta}\right) > (1 - \varepsilon_1) b r_0 \quad \text{whenever} \quad x_i \in J_i.$$

We may further replace the intervals J_i by intervals I_i having their centres at a_i and all having the same diameter 2δ (say) s.t.

$$m\Big(\frac{A_i}{x_i}\cap\tilde{\Delta}\Big) > (1-\varepsilon_1)br_0 \quad \text{whenever} \quad x_i\in I_i$$

i.e. whenever $|x_i - a_i| < \delta$. For each i we choose $x_i \in (a_i-\delta, a_i+\delta)$ and fix it.

We set $E_i(x_i) = \frac{A_i}{x_i}\cap\tilde{\Delta}$ and let $CE_i(x_i)$ denote the complement of $E_i(x_i)$ in $\tilde{\Delta}$. Then

$$\bigcap_{i=1}^{n} E_i(x_i) = \tilde{\Delta}\setminus\bigcup_{i=1}^{n} CE_i(x_i),$$

and

$$m\big(CE_i(x_i)\big) = m(\tilde{\Delta}) - m\big(E_i(x_i)\big) < br_0 - (1-\varepsilon_1)br_0 = \varepsilon_1 br_0.$$

Hence

$$\begin{aligned} m\Big(\bigcap_{i=1}^{n} E_i(x_i)\Big) &\geqslant m(\tilde{\Delta}) - \sum_{i=1}^{n} m\big(CE_i(x_i)\big)\\ &> br_0 - \sum_{i=1}^{n}\varepsilon_1\cdot br_0\\ &= br_0 - n\varepsilon_1 br_0 = (1-n\varepsilon_1)br_0 > 0. \end{aligned}$$

Put $\eta = (1-n\varepsilon)br_0$. Then

$$m\Big(\frac{A_1}{x_1}\cap\ldots\frac{A_n}{x_n}\Big) \geqslant m\Big(\bigcap_{i=1}^{n} E_i(x_i)\Big) > \eta$$

whenever $|x_i - a_i| < \delta$.

This complete the proof of Theorem 1. □

Bose Majumdar [1] proved that $R[A_1 : A_2]$ contains an interval when A_1, A_2 ($\in\mathscr{L}$) with $m(A_1) > 0$ and $m(A_2) > 0$. From our text theorem we establish as a consequence a new result more stronger than that of Bose Majumdar.

Theorem 2. *Let A_1, A_2 ($\in\mathscr{L}$) with $m(A_1)$, $m(A_2)$ (>0). Then $R[A_1^* : A_2^*]$ is an open set.*

P r o o f. Here it suffices to prove that to each $d\in R[A_1^* : A_2^*]$ there corresponds a positive number δ s.t.

$$m\Big(A_1^*\cap\frac{A_2^*}{y}\Big) > 0$$

whenever $y(\neq 0) \in E^1$ and $|y - d| < \delta$.

Now let $d \in R[A_1^* : A_2^*]$. Then either $d = \frac{a_1}{a_2}$ or $\frac{a_2}{a_1}$ where $a_1 \in A_i^*$ $(i = 1, 2)$. Without loss generality we may suppose that $d = \frac{a_1}{a_2}$. By Lebesgue density theorem and Lemma 1 we have

$$m\Big(\frac{A_1^*}{y} \cap A_2^*\Big) = m\Big(\frac{A_1}{y} \cap A_2\Big) \quad (y \neq 0).$$

By the above theorem there exists δ, η (> 0) s.t.

$$m\Big(\frac{A_1}{a_2 y} \cap \frac{A_2}{a_2}\Big) > \eta \quad \text{whenever} \quad \Big|y - \frac{a_1}{a_2}\Big| = |y - d| < \delta$$

(if we put $x_1 = a_2 y$, $x_2 = a_2$)

$$\Longrightarrow m\Big(\frac{A_1^*}{y} \cap A_2^*\Big) > \eta |a_2| > 0 \quad \text{whenever} \quad |y - d| < \delta.$$

This proves that $R[A_1^* : A_2^*]$ is an open set. □

We now show that there exist measurable sets $\overline{A}_1 \subseteq A_1$, $\overline{A}_2 \subseteq A_2$ (which should not be confused with the closures of A_1 and A_2) such that $m(A_1 \setminus \overline{A}_1) = 0$, $m(A_2 \setminus \overline{A}_2) = 0$ and for which $R[\overline{A}_1 : \overline{A}_2]$ is an open set.

Note. If we set $\overline{A}_i = A_i \cap A_i^*$ $(i = 1, 2)$, then $m(A_i \setminus (\overline{A}_i)) = 0$ $(i = 1, 2)$, by Lebesgue density theorem. Since then the above proof could be applied for $(\overline{A}_1)^*$ and $(\overline{A}_2)^*$ and $(\overline{A}_i)^* = \overline{A}_i$ $(i = 1, 2)$, it follows that $R[\overline{A}_1 : \overline{A}_2]$ is an open set.

Theorem 3. *Let $A_i \in \mathscr{L}$ $(i = 1, 2, \ldots, n)$ with $m(A_i) > 0$, $a_i \in A_i^*$ and $\{\alpha_k^{(i)}\}_{k=1}^{\infty}$ $(i \neq 1)$ are sequences of non-zero reals with $\operatorname*{Lt}_{k \to \infty} \alpha_k^{(i)} = \frac{a_i}{a_1}$. The there exists $\lambda_1 > 0$ s.t. the set $M_1 = \{x \in A_1 ;\, x \cdot \alpha_k^{(i)} \in A_i$ for infinitely many k, $2 \leqslant i \leqslant n\} \in \mathscr{L}$ with $m(M_1) \geqslant \lambda_1$.*

Similarly there exists $\lambda_j > 0$ s.t. $m(M_j) \geqslant \lambda_j$ where M_j (for $j = 2, \ldots, n$) are similarly defined with the sequences $\{\alpha_k^{(i)}\}_{k=1}^{\infty}$ such that $\operatorname*{Lt}_{k \to \infty} \alpha_k^{(i)} = \frac{a_i}{a_j}$ $(i \neq j)$.

Proof. By the above theorem there exists δ, η (> 0) s.t.

$$m\Big(\frac{A_1}{a_1} \cap \frac{A_2}{x_2} \cap \ldots \cap \frac{A_n}{x_n}\Big) > \eta > 0 \text{ whenever } |x_i - a_i| < \delta.$$

For each $i = 2, \ldots, n$ we set $\beta_k^{(i)} = a_1 \alpha_k^{(i)}$ $(k = 1, 2, \ldots)$. Then $\{\beta_k^{(i)}\}_{k=1}^{\infty}$ (for $i = 2, \ldots, n$) are real sequences with $\operatorname*{Lt}_{k \to \infty} \beta_k^{(i)} = a_i$.

The convergence of the sequences $\{\beta_k^{(i)}\}_{k=1}^{\infty}$ imply that there exists a natural number m s.t. for $i = 2, \ldots, n$, $|\beta_k^{(i)} - a_i| < \delta$ (for all $k \geqslant m$) and hence

$$m\Big(\frac{A_1}{a_1} \cap \frac{A_2}{\beta_k^{(2)}} \ldots \cap \frac{A_n}{\beta_k^{(n)}}\Big) > \eta \quad \text{(for all } k \geqslant m)$$
$$\Longrightarrow m\Big(A_1 \cap \frac{A_2}{\alpha_k^{(2)}} \cap \ldots \frac{A_n}{\alpha_k^{(n)}}\Big) > \eta|a_1| > 0 \quad \text{(for all } k \geqslant m).$$

We set

$$E_k = A_1 \cap \frac{A_2}{\alpha_k^{(2)}} \cap \ldots \cap \frac{A_n}{\alpha_k^{(n)}} \quad (k = 1, 2, \ldots)$$

and

$$P = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k = \bigcap_{j=1}^{\infty} D_j,$$

where $D_j = \bigcup\limits_{k=j}^{\infty} E_k$ $(j = 1, 2, \ldots)$.

But $m(D_j) > \eta|a_1|$ and D_j forms a monotonically decreasing sequence of bounded sets in E^1. Therefore, $m(P) \geqslant \eta|a_1| > 0$. Hence, the set

$$M_1 = \{x \in A_1\,;\, x \cdot \alpha_k^{(i)} \in A_i \text{ for infinitely many } k, i \neq 1\} \in \mathscr{L},$$

with $m(M_1) \geqslant \lambda_1$, where $\lambda = \eta|a_1|$. If we set $\lambda_j = \eta|a_j|$, then $M_j \in \mathscr{L}$ and $m(M_j) \geqslant \lambda_j$ where M_j for $j = 2, \ldots, n$ are similarly defined i.e.,

$$M_j = \{x \in A_j\,;\, x \cdot \alpha_k^{(i)} \in A_i \text{ for infinitely may } k, i \neq j\},$$

with $\operatorname*{Lt}\limits_{k\to\infty} \alpha_k^{(i)} = \frac{a_i}{a_j}$ $(i \neq j)$. □

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References

[1] *Bose Majumdar, N.C.*: On some properties of sets with positive measure. Annali dell' Universita di Ferrara *X. N1, p1* (1962), 1–11.

[2] *Ganguly, D.K. and Majumdar, M*: Some results on mid-point sets. Journal of Pure Mathematics *7* (1990), 27–33.

[3] *Oxtoby, J.C.*: Measure and category. Springer Verlag, 1980.

[4] *Ray, K.C.*: On some properties of ratio sets. Seminario mathematics del Universita di Ferrara, Sci *VI (NS)* (1962–65), 45–49.

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