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A NOTE ON COFLAT ABELIAN GROUPS

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1. INTRODUCTION

Often the approach to studying abelian groups is to view them as modules over their endomorphism rings. This approach was initiated by J. Reid and continued by several others. (For example, [A], [R], [AP], [VW], [V]).

One natural problem is to describe the injective hull of A as a module over its endomorphism ring $E(A)$. In particular, when does the injective hull of the $E(A)$-module A coincide with the divisible hull of the abelian group A? A description of the $E(A)$-injective hull of A was given in [VW] for torsion-free groups of finite rank and an answer to the question was given in [V] under further restrictions on A.

Our purpose is to address this question further. This leads to the following property which is dual to flatness. A torsion-free group A is called coflat if, for every n, whenever B is a pure subgroup of A^n which contains an epimorphic image of A^m as an essential subgroup for some m, then A^n/B is a subgroup of a direct sum of copies of A.

In Section 2, we explore the coflat property and give various characterizations of coflatness in Theorem 2.4. These results are applied to finite rank coflat groups in Section 3. In particular we show that a finite rank group A is coflat if and only if the $E(A)$-injective and $\mathbb{Z}$-divisible hulls of A coincide.

Section 4 discusses the relationship between the coflatness of A and ring-theoretic properties of $E(A)$. We show that in order for A to be coflat, $E(A)$ must be of a certain genre, so we introduce the concept of a coflat ring. A ring R is called coflat if whenever M is an R-module with M^+ torsion-free and $0 \to V \to \bigoplus_n R \to M \to 0$ is exact with V containing a finitely generated module U with V/U torsion, then M is a submodule of a finitely generated projective module. We show that when A is faithfully flat as an $E(A)$-module, A is coflat if and only if $E(A)$ is a coflat ring. This is in sharp contrast to the case when A is flat over $E(A)$, since for every cotorsion-free reduced ring E there is a faithfully flat group A with $E = E(A)$. Furthermore, our results allow us to give examples of coflat groups and non-coflat groups.

We write $R_B(A)$ for $\bigcap\{\ker f \mid f\colon A \to B\}$. The symbols H_A and T_A denote the functors $H_A(G) = \operatorname{Hom}(A, G)$ and $T_A(M) = M \otimes_{E(A)} A$ respectively. Associated with these functors is the evaluation map $\theta_G\colon T_A H_A(G) \to G$. The class $\mathscr{C}_A$ of A-solvable groups consists of the abelian groups G for which θ_G is an isomorphism. The natural map from a right $E(A)$-module M into $H_A T_A(M)$ will be denoted by φ_M.

2. Coflat groups of arbitrary rank

Our description of coflat group requires the following discussion of finitely A-generated subgroups of A^I, i.e. subgroups which are images of A^n for some n. We say that a submodule U of the left $E(A)$-module $\operatorname{Hom}(A^n, A)$ is an annihilator if $U = \{f \in \operatorname{Hom}(A^n, A) \mid f(X) = 0\}$ for some subset X of A^n.

Theorem 2.1. *The following conditions are equivalent for a torsion-free abelian group A:*

(a) $\operatorname{Hom}(A^n, A)$ *has the ACC for annihilators for all* $n < \omega$.

(b) *For every index-set I and every finitely A-generated subgroup U of A^I, there is a finite subset J of I such that $\ker \pi_J \cap U = 0$ where $\pi_j\colon A^I \to A^J$ is the projection whose kernel is $A^{I\setminus J}$.*

Proof. We will first show that (a) $\Rightarrow$ (b).

Let U be a finitely A-generated subgroup of A^I for some index-set I. There are $m < \omega$ and an epimorphism $\varphi\colon A^m \to U$. Assume, $\ker \pi_j \cap U \neq 0$ for all finite subsets J of I. Let $j_0 \in I$ be arbitrary. If we have found $J_n = \{j_0, \dots, j_n\} \subset I$, then $U \cap \ker \pi_{J_n} \neq 0$ allows us to choose an index $j_{n+1} \in I \setminus J_n$ and $u_{n+1} \in U \cap \ker \pi_{J_n}$ with $\pi_{j_{n+1}}(u_{n+1}) \neq 0$.

Let X_n be the kernel of the map $\pi_{J_n}\varphi$ and $U_n = \operatorname{ann}(X_n)$, an annihilator in $\operatorname{Hom}(A^m, A)$. Because of $J_n \subset J_{n+1}$, we have $X_{n+1} \subset X_n$ and $U_n \subset U_{n+1}$. Since $\operatorname{Hom}(A^m, A)$ has the ACC for annihilators, there is $k < \omega$ with $U_n = U_k$ for all $n \geqslant k$. If $x \in A^m$ satisfies $\varphi(x) = u_{k+1}$, then $\pi_{j_{k+1}}\varphi(x) \neq 0$. Since $u_{k+1} \in \ker \pi_{J_k}$, we have $x \in X_k$. Therefore, $\pi_{j_{k+1}}\varphi \notin U_k = U_{k+1}$. On the other hand, let $z \in X_{k+1}$. Then $\pi_{J_{k+1}}\varphi(z) = 0$ implies $\pi_{j_{k+1}}\varphi(z) = 0$. Hence, $\pi_{j_{k+1}}\varphi \in U_{k+1}$, which results in a contradiction.

Conversely, suppose that the groups A^I have the described property for their finitely A-generated subgroups. Let $\{U_n\}_{n<\omega}$ be an ascending chain of annihilator submodules of $\operatorname{Hom}(A^m, A)$ where $m < \omega$. For each $n < \omega$, choose $f_n \in U_{n+1} \setminus U_n$

and define a map $\alpha\colon A^m \to A^\omega$ by $\alpha(x) = \big(f_n(x)\big)_{n<\omega}$ for all $x \in A^m$. Since $\alpha(A^m)$ is a finitely A-generated subgroup of A^ω, there exists a finite subset, $J \subset \omega$ such that $\alpha(A^m) \cap \ker \pi_J = 0$. Let i be the largest element of J.

Write $U_n = \operatorname{ann}(X_n)$, and choose $x \in X_i$ with $f_{i+1}(x) \neq 0$. This x exists because of $f_{i+1} \in U_{i+1} \setminus U_i$. For $n \leqslant i$, we have $f_n \in U_n \subset U_i$ and $f_n(x) = 0$. Therefore, $\alpha(x)$ is a non-zero element of $\alpha(A^m)$ which is contained in $\pi_{n-i}A \subset \ker \pi_J$, a contradiction. □

Corollary 2.2. *If $E(A)$ has finite rank as an abelian group or is left Noetherian, then $\operatorname{Hom}(A^n, A)$ has the ascending chain condition for annihilators.*

P r o o f. Observe that $\operatorname{Hom}(A^n, A)$ is a finitely generated free left $E(A)$-module, and that annihilators are pure subgroups of $\operatorname{Hom}(A^n, A)$. □

A partial characterization of the groups A such that $\operatorname{Hom}(A^n, A)$ has the ACC for annihilators is obtained in

Theorem 2.3. *The following conditions are equivalent for a torsion-free abelian group A which is faithfully flat as an $E(A)$-module and has a strongly non-singular endomorphism ring:*
(a) *$E(A)$ has finite Goldie-dimension as a right $E(A)$-module.*
(b) *The module $\operatorname{Hom}(A^n, A)$ has the ACC for annihilators for all $n < \omega$.*

P r o o f. (a) $\Rightarrow$ (b): Suppose that (b) fails. By Theorem 2.1, there exists $m < \omega$ such that we can find an infinite sequence $0 < \ell_1 < \ldots < \ell_n < \ldots < \omega$ and maps $\beta_n \in \operatorname{Hom}(A^m, A^{\ell_n})$ with $\ker \beta_{n+1} \subsetneq \ker \beta_n$. To simplify our notation, write $U_n = \ker \beta_n$. Since A is flat as an $E(A)$-module, U_n is A-solvable because $\mathscr{C}_A$ is A-closed [A1]. Moreover, we have an exact sequence $0 \to H_A(U_n) \to H_A(A^m) \to H_A\big(\beta_n(A^m)\big)$ where $H_A\big(\beta_n(A^m)\big) \subseteq H_A(A^{\ell_n})$ is a non-singular right $E(A)$-module. Thus, $H_A(A^m)/H_A(U_n)$ is non-singular for all $n < \omega$. In particular, $H_A(U_{n+1})$ is not essential in $H_A(U_n)$ since otherwise $H_A(U_n)/H_A(U_{n+1})$ would be a singular submodule of $H_A(A^m)/H_A(U_{n+1})$, which is isomorphic to a submodule of $E(A)^{\ell_n}$. But this is only possible if $H_A(U_n) = H_A(U_{n+1})$. Since U_n and U_{n+1} are A-solvable, this would yield $U_n = U_{n+1}$, which contradicts $U_{n+1} \subsetneq U_n$. Therefore, we can choose a non-zero submodule W_n of $H_A(U_n)$ with $W_n \cap H_A(U_{n+1}) = 0$. Then $\bigoplus_{n<\omega} W_n$ is an infinite direct sum of non-zero submodules of $H_A(A^m) \cong \bigoplus_m E(A)$, which has finite Goldie dimension by (a), contradiction.

(b) $\Rightarrow$ (a): Suppose that $E(A)$ has infinite right Goldie dimension, and let $U_0 \oplus \ldots \oplus U_n \oplus \ldots$ be an infinite direct sum of non-zero right ideals of $E(A)$. Denote the $\mathscr{S}$-closure of $\bigoplus_{i \geqslant n} U_n$ in $E(A)$ by V_n ([G]). We have $V_{n+1} \subsetneq V_n$ since $V_{n+1} = V_n$

would imply $U_n \subset V_{n+1}$. On the other hand, $\bigoplus_{i\geqslant n+1} U_i$ is essential in V_{n+1}. Thus, $U_n \cap \bigoplus_{i\geqslant n+1} U_i \neq 0$, which is not possible.

Since $E(A)$ is strongly non-singular, we obtain that $E(A)/V_n \subsetneqq E(A)^{\ell_n}$ for some suitable $\ell_n < \omega$. For $n > 0$, we have that $T_A\big(E(A)/V_n\big)$ is a non-zero subgroup of $A^{\ell_n} \cong T_A\big(E(A)^{\ell_n}\big)$ since A is faithfully flat and $E(A)/V_n \neq 0$. In particular, $V_{n+1}A$ is a proper subgroup of $V_nA \cong T_A(V_n)$ again by the faithful flatness of A. For $n > 0$, there is a map $\alpha_n \colon A \to A^{\ell_n}$ with $\ker\alpha_n = V_nA$. Define $\alpha\colon A \to \prod\limits_{n>0} A^{\ell_n}$ by $\alpha(a) = \big(\alpha_n(a)\big)_{n>0}$.

By (b), there is a $k < \omega$ such that the projection $\pi_k\colon \prod\limits_{n>0} A^{\ell_n} \to A^{\ell_1} \oplus \ldots \oplus A^{\ell_k}$ with $\ker\pi_k = \prod\limits_{n>k} A^{\ell_n}$ satisfies $\ker\pi_k \cap \alpha(A) = 0$. Let $x \in V_kA \setminus V_{k+1}A$. Then $\alpha_{k+1}(x) \neq 0$, but $\alpha_i(x) = 0$ for all $i \leqslant k$. Hence, $\alpha(x)$ is a non-zero element of $\prod\limits_{n>k} A^{\ell_n}$, which is not possible. □

We now apply the results of the last theorems to obtain our main result.

Theorem 2.4. *The following conditions are equivalent for a torsion-free abelian group A such that $\mathrm{Hom}(A^m, A)$ has the ACC for annihilators for all $n < \omega$:*

(a) *A is coflat.*

(b) *If $n < \omega$ and $f \in \mathrm{Hom}(A, A^n)$, then $R_A\big(A^n/[f(A)]_*\big) = 0$.*

(c) *$\mathrm{Hom}(QA, Q)$ is a flat $QE(A)$-module.*

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c): By [Rn, Exercise 3.39], it suffices to show that, whenever $\sum\limits_{i=1}^{n} \alpha_i f_i = 0$ for some $\alpha_1, \ldots, \alpha_n \in \mathrm{Hom}(QA, Q)$ and $f_1, \ldots, f_n \in QE(A)$, there exists $g_{ij} \in QE(A)$ and $\beta_j \in \mathrm{Hom}(QA, Q)$ for $j = 1, \ldots, m$ and $i = 1, \ldots, n$ with $\sum\limits_{j=1}^{m} \beta_j g_{ij} = \alpha_i$ and $\sum\limits_{i=1}^{n} g_{ij} f_i = 0$.

Choose a non-zero integer s with $sf_i \in E(A)$, and define $f\colon A \to A^n$ by $f(a) = \big(sf_1(a), \ldots, sf_n(a)\big)$. By (b), there exists an index-set I and a monomorphism $\varphi\colon A^n/[f(A)]_* \to A^I$. Because of Theorem 2.1, we may assume that $I = m$ for some $m < \omega$ since $\mathrm{im}\,\varphi$ is a finitely A-generated subgroup of A^I.

Let $\nu_i\colon A \to A^n$ be the embedding into the ith coordinate; and $\pi_j\colon A^m \to A$ be the projection onto the jth coordinate. Denote the projection $A^n \to A^n/[f(A)]_*$ by ε, and set $g_{ij} = \pi_j\varphi\varepsilon\nu_i$. For $a \in A$, we have $\sum\limits_{i=1}^{n} g_{ij} s f_i(a) = \sum\limits_{i=1}^{n} \pi_j\varphi\varepsilon\nu_i s f_i(a) = \pi_j\varphi\varepsilon f(a) = 0$. Thus, $\sum\limits_{i=1}^{n} g_{ij}(sf_i) = 0$ in $E(A)$ and the same holds for $\sum\limits_{i=1}^{n} g_{ij} f_i = \frac{1}{s}\sum\limits_{i=1}^{n} g_{ij}(sf_i)$ in $QE(A)$.

It remains to construct $\beta_1, \dots, \beta_m$. Define a map $\alpha\colon A^n \to Q$ by $\alpha(a_1, \dots, a_n) = \sum_{i=1}^{n} \alpha_i(a_i)$. Then $af = 0$, and α induces a map $\bar{\alpha}\colon A^n/[f(A)]_* \to Q$ defined by $\bar{\alpha}(x + [f(A)]_*) = \alpha(x)$. Since Q is injective, there is $\beta\colon A^m \to Q$ with $\beta\varphi = \bar{\alpha}$. Set $\beta_j = \beta$ restricted to the jth component of A^m. Since Q is injective, we may regard β_j as a map $\beta_j\colon QA \to Q$. If $x = (x_1, \dots, x_m) \in A^m$, then $\beta(x) = \sum_{j=1}^{m} \beta_j\pi_j(x)$. Let $a \in A$. Then $\varepsilon\nu_i(a) \in A^n/[f(A)]_*$ and $\bar{\alpha}\varepsilon\nu_i(a) = \alpha_i(a)$. On the other hand, $\bar{\alpha}\varepsilon\nu_i(a) = \beta\varphi\varepsilon\nu_i(a) = \sum_{j=1}^{m} \beta_j g_{ij}(a)$. Thus $\left(\alpha_i - \sum_{j=1}^{m} \beta_j g_{ij}\right)\big|_A = 0$. Since QA/A is torsion, we have $\alpha_i = \sum_{j=1}^{m} \beta_j g_{ij}$.

(c) $\Rightarrow$ (a): Let $m < \omega$ and $B = A^m$. We show in the first step that $M = \operatorname{Hom}(QB, Q)$ is a flat $QE(B)$-module. For this, we compute the character module $\operatorname{Hom}_{\mathbb{Z}}(M, Q/\mathbb{Z})$ and show that it is injective. We have

$$M \cong \operatorname{Hom}_{\mathbb{Z}}\left(\operatorname{Hom}_{QE(A)}(QE(A)^m, QA), Q\right) \cong \operatorname{Hom}_{\mathbb{Z}}(QA, Q) \otimes_{QE(A)} QE(A)^m.$$

Hence,

$$\begin{aligned}\operatorname{Hom}_{\mathbb{Z}}(M, Q/\mathbb{Z}) &\cong \operatorname{Hom}_{\mathbb{Z}}\left(\operatorname{Hom}_{\mathbb{Z}}(QA, Q) \otimes_{QE(A)} QE(A)^m, Q/\mathbb{Z}\right) \\ &\cong \operatorname{Hom}_{QE(A)}\left(QE(A)^m, \operatorname{Hom}_{\mathbb{Z}}\left(\operatorname{Hom}_{\mathbb{Z}}(QA, Q), Q/\mathbb{Z}\right)\right)\end{aligned}$$

in which $\operatorname{Hom}_{\mathbb{Z}}(QA, Q)$, is a flat $QE(A)$-module whose character module is injective. Since $\operatorname{Hom}_{QE(A)}\left(QE(A)^m, -\right)$ is a category equivalence between ${}_{QE(A)}\mathcal{M}$ and ${}_{QE(B)}\mathcal{M}$ which preserves injectives, we have that $\operatorname{Hom}_{\mathbb{Z}}(M, Q/\mathbb{Z})$ is injective.

Suppose $f \in \operatorname{Hom}(B, B^n)$ and let $\bar{x}$ be a non-zero element of $R_B\left(B^n/[f(B)]_*\right)$. Set $C = B^n/[f(B)]_*$ and choose a pure corank-1 subgroup K of C with $\bar{x} \notin K$. Let $\varepsilon\colon C/K \to Q$ be a monomorphism, and $\varphi\colon B^n \to C/K$ the factor map. As before $\nu_i\colon B \to B^n$ is the embedding into the ith coordinate. Set $\alpha_i = \varepsilon\varphi\nu_i$. Since Q is injective, α_i extends to a map in $\operatorname{Hom}(QB, Q)$. Denote the ith component map of f by $f_i \in E(B)$. We have $\sum_{i=1}^{n} \alpha_i f_i(b) = \sum \varepsilon\varphi_\nu f_i(b) = \varepsilon\varphi f(b) = 0$ for all $b \in B$. Since QB/B is torsion, $\sum_{i=1}^{n} \alpha_i f_i = 0$. Let $x = (x_1, \dots, x_n) \in A^n$ with $\varphi(x) = \bar{x} + K = 0$. Then $\sum_{i=1}^{n} \alpha_i(x_i) = \sum \varepsilon\varphi\nu_i(x_i) = \varepsilon\varphi(x) \neq 0$ since ε is one-to-one.

Since $\operatorname{Hom}(QB, Q)$ is a flat $QE(B)$-module, there are $g_{ij} \in QE(B)$ and $\beta_j \in \operatorname{Hom}(QB, Q)$ with $\sum_{i=1}^{n} g_{ij} f_i = 0$ and $\sum_{j=1}^{m} \beta_j g_{ij} = \alpha_i$. There is a non-zero integer s with $sg_{ij} \in E(B)$. Define $g_j\colon B^n \to B^n$ by $g_j(b_1, \dots, b_n) = \left(sg_{1j}(b_1), \dots, sg_{nj}(b_n)\right)$, and $\sigma\colon B^n \to B$ by $\sigma(b_1, \ldots, b_n) = \sum_{i=1}^{n} b_i$. We obtain,

$$\sigma g_j f(b) = \sum_{i=1}^{n} \sigma g_j(0, \ldots, f_i(b), 0, \ldots) = \sum^{n} s g_{ij} f_i(b) = 0.$$

Thus, σg_j induces a map $h_j\colon B^n/[f(B)]_* \to B$ for each j.

Define $h\colon C \to B^m$ by $h(y) = \big(h_1(y), \ldots, h_m(y)\big)$ for $y \in C$, and $\beta\colon (QB)^m \to Q^m$ by $\beta = (\beta_1, \ldots, \beta_m)$. If $\gamma\colon B^m \to B$ is the summation map, then

$$\begin{aligned}\beta h(\bar{x}) &= \beta\big(h_1(\bar{x}), \ldots, h_m(\bar{x})\big) = \beta\big(\sigma g_1(x), \ldots, \sigma g_m(x)\big)\\ &= \beta\Big(\sum_{i=1}^{n} s g_{i1}(x_i), \ldots, \sum_{i=1}^{n} s g_{im}(x_i)\Big)\\ &= \Big(\beta_1 \sum_{i=1}^{n} s g_{i1}(x_i), \ldots, \beta_m \textstyle\sum_{i=1}^{n} s g_{im}(x_i)\Big).\end{aligned}$$

Hence, $\gamma\beta h(\bar{x}) = \sum_{i=1}^{n} \big(\sum_{j=1}^{m} \beta_j s g_{ij}\big)(x_i) = s \sum_{i=1}^{n} \alpha_i(x) \neq 0$ since $\sum_{i=1}^{n} \alpha_i(x_i) \neq 0$ and Q is torsion-free. Therefore $h(\bar{x}) \neq 0$, and $\bar{x} \notin R_B(C)$, a contradiction.

Finally, let $f \in \operatorname{Hom}(A^m, A^n)$, and view f as an element $\operatorname{Hom}(B, B^n)$. There exists an index-set I with $A^n/[f(A^m)]_* \subseteq B^n/[f(B)]_* \subseteq B^I = (A^m)^I$. Thus, $A^n/[f(A^m)]_*$ is a finitely A-generated subgroup of $(A^n)^I$, and there exists a $k < \omega$ with $A^n/[f(A^m)]_* \underset{\sim}{\subseteq} A^k$ by Theorem 2.1 since $\operatorname{Hom}(A^n, A)$ has the ACC for annihilators. □

3. Coflat groups of finite rank

We will explore various equivalent characterizations of coflat finite rank groups. The first gives a description of the groups A such that QA is injective as an $E(A)$-module. We note that if A has finite rank, then $QA^* = \operatorname{Hom}(QA, Q)$ carries a natural right $QE(A)$ structure and $QA^{**} \cong_{\text{nat}} QA$.

Proposition 3.1. *Let A have finite rank. Then A is coflat if and only if QA is the injective hull of A as an $E(A)$-module.*

Proof. Assume that A is coflat. Then by Corollary 2.2 and Theorem 2.4, QA^* is projective since $QE(A)$ is Artinian. For an ideal I of $QE(A)$, QA^* is projective with respect to $0 \to \big(QE(A)/I\big)^* \to QE(A)^* \to I^* \to 0$. Consequently, $QA^{**} \cong QA$ is injective with respect to

$$\begin{array}{ccccc} 0 & \longrightarrow & I^{**} & \longrightarrow & QE(A)^{**} \\ & & \uparrow\iota & & \uparrow\iota \\ 0 & \longrightarrow & I & \longrightarrow & QE(A). \end{array}$$

This implies that QA is injective over $QE(A)$ by Baer's injective test lemma. Thus QA is injective as a $QE(A)$-module. This is equivalent to QA being injective as an $E(A)$-module.

If QA is injective as an $E(A)$-module, then QA is injective over $QE(A)$. Let $QE(A)^k \to QA^*$ be a resolution of QA^*. Then $0 \to QA^{**} \to \left(QE(A)^*\right)^k$ is split so $QA^* \cong QA^{***}$ is a summand of a free module. By Corollary 2.2 and Theorem 2.4, A is coflat.

In [RW] the authors consider a class $\mathscr{C}$ of modules described by a term T. They form the class $\xi(\mathscr{C})$ of all exact sequences $\varepsilon\colon 0 \to U \to V \to W \to 0$ in the module category, relative to which each $X \in \mathscr{C}$ is projective. They call a module M a co-T module, if M is injective with respect to each ε in $\xi(\mathscr{C})$. The sequences ε in $\xi(\mathscr{C})$ are called proper (with respect to $\mathscr{C}$).

For example, if $\mathscr{C}$ is the class of all flat modules, then the co-flat modules are the modules injective with respect to each ε in $\xi(\mathscr{C})$. □

Corollary 3.2. *Let A have finite rank. Then, in the category of all left $E(A)$-modules, QA is co-flat if and only if A is a coflat group.*

Proof. If A is coflat then QA is injective as an $E(A)$-module so it is certainly co-flat. Conversely, it is clear that QA is co-flat in the category of all left $QE(A)$-modules. We will show that any sequence $0 \to U \to V \to W \to 0$ with V a finitely generated $QE(A)$-module, is proper (with respect to the flat $QE(A)$-modules).

Let F be a flat $QE(A)$-module and $\alpha\colon F \to W$. Since F^* is injective (Theorem 3.44 in [Rn]) the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & W^* & \longrightarrow & V^* \\ & & {\scriptstyle\alpha^*}\downarrow & & \\ & & F^* & & \end{array}$$

can be completed. Let $\bar{\alpha}\colon V^* \to F^*$ make the diagram commute.

By the contravariance and naturality of $\mathrm{Hom}_Q(\cdot, Q)$, the diagram

$$\begin{array}{ccccc} & & F \subset F^{**} & & \\ & \beta & \downarrow{\scriptstyle\alpha} & & \\ V & \longrightarrow & W & \longrightarrow & 0 \\ {\scriptstyle\iota}\downarrow & & {\scriptstyle\iota}\downarrow & & \\ V^{**} & \longrightarrow & W^{**} & \longrightarrow & 0 \end{array}$$

is commutative where $\beta = \bar{\alpha}^*\big|_F$. Hence any sequence $0 \to U \to V \to W \to 0$ with V finitely generated is proper. In particular, $0 \to I \to QE(A)$ is proper for any left ideal I of $QE(A)$, and QA is injective by Baer's criterion. □

When A has finite rank, there is an $X \leqslant Q$ of least type such that A is isomorphic to a subgroup of X^r for $r = \operatorname{rank} A$. The type of X is the outer type of A, $OT(A)$ (§1 in [A]).

Lemma 3.3. *Let A have finite rank and $X \leqslant Q$ have type $OT(A)$. Then $G = A \oplus X$ is coflat.*

Proof. If $f \in \operatorname{Hom}(G^n, G^m)$ then $OT\big(G^m/[f(G^n)]_*\big) \leqslant OT(G^m) = OT(A) = \operatorname{type} X$. Therefore, if $r = \operatorname{rank} A$, the group $G^m/\langle f(G^n)\rangle_*$ is isomorphic to a subgroup of $X^{m(r+1)} \leqslant G^{m(r+1)}$. □

Although Lemma 3.3 points out the complexity of coflat groups, we can determine the almost completely decomposable groups for which the divisible and E-injective hulls coincide. This description is equivalent to the one given in [VW]:

Theorem 3.4. *Assume that A is quasi-isomorphic to $A_1 \oplus \ldots \oplus A_r$ where each A_i is a rank-1 group of type τ_i. Then QA is the E-injective hull of A if and only if for every i, j and k, if $\tau_i \wedge \tau_j \geqslant \tau_k$, then for some m, $\tau_m \geqslant \tau_i \vee \tau_j$.*

Proof. Assume QA is injective as an E-module and $Z \leqslant A_k \leqslant Q$ for all ℓ. If $\tau_i \wedge \tau_j \geqslant \tau_k$, then there is an integer $s \neq 0$ such that $sA_k \subset A_i \cap A_j$. Define $f\colon A_k \to A_i \oplus A_j$ by $f(a) = (sa, sa)$. Now type $(A_i \oplus A_j)/[f(A_k)]_* = \tau_i \vee \tau_j$, and since A is coflat, $A_i \oplus A_j/[f(A_k)]_*$ embeds in A, so consequently $\tau_m \geqslant \tau_i \vee \tau_j$ for some m.

Conversely, let $\mu_1, \ldots, \mu_n$ be the maximal elements in $T = \{\tau_i \mid i = 1, 2, \ldots, r\}$. Because of the condition on the types, connected components in the graph of T have a unique maximal element. Therefore, $B_i = \bigoplus\{A_j \mid \tau_j \leqslant \mu_i\}$ is fully invariant in A and A is quasi-isomorphic to $B_1 \oplus \ldots \oplus B_n$. Since $E(A)$ is quasi-isomorphic to $E(B_1) \times \ldots \times E(B_n)$, it suffices to show that QB_i is injective over $E(B_i)$. But $B_i = C_i \oplus X_i$ where X_i has type μ_i and $C_i = 0$ or $OT(C_i) \leqslant \mu_i$. So B_i is coflat by Lemma 3.3. □

4. Endomorphism rings of coflat abelian groups

A ring R whose additive groups is torsion-free is *coflat* if every module M which admits an exact sequence $0 \to V \to \bigoplus_n R \to M \to 0$ in which V is the $\mathbb{Z}$-purification of a finitely generated submodule of $\bigoplus_n R$, is contained in a finitely generated free module.

Theorem 4.1. *Let A be a torsion-free abelian group which is faithfully flat as an $E(A)$-module. Then, A is coflat if and only if $E(A)$ is a coflat ring.*

Proof. Let $E(A)$ be a coflat ring and consider an exact sequence $0 \to V \xrightarrow{i_V} \bigoplus_n A \xrightarrow{\beta} G \to 0$ of torsion-free abelian groups, in which V contains a finitely A generated subgroup U such that V/U is torsion, and i_V is the inclusion map. Denote the inclusion $U \subset V$ by i_U. Then, $i_V i_U$ is the inclusion $U \leqslant \bigoplus_n A$.

We may assume $H_A(V) = \operatorname{im} H_A(i_V)$ and $H_A(U) = \operatorname{im} H_A(i_U) \subset H_A(V)$. Let W be the $\mathbb{Z}$-purification of $H_A(U)$ in $H_A\big(\bigoplus_n A\big)$ and denote the embedding $W \subset H_A\big(\bigoplus_n A\big)$ by ε. Since $H_A(V)$ is pure in $H_A\big(\bigoplus_n A\big)$, we obtain that $W \subset H_A(V)$. Define a map $\theta_1 : T_A(W) \to V$ by $\theta_1(\alpha \otimes a) = \alpha(a)$ for $\alpha \in W$ and $a \in A$. Moreover, $H_A(i_U) \colon H_A(U) \to W$. For $\sigma \in H_A(U)$ and $a \in A$, we obtain $i_U\theta_U(\sigma \otimes a) = \sigma(a) = \theta_1 T_A H_A(i_U)$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & T_A(W) & \xrightarrow{T_A(\varepsilon)} & T_A H_A\big(\bigoplus_n A\big) & \xrightarrow{T_A(\pi)} & T_A\Big(H_A\big(\bigoplus_n A\big)/W\Big) & \longrightarrow 0 \\
 & \downarrow \theta_1 & & \iota\downarrow \theta_{\bigoplus_n A} & & \downarrow \theta_2 & \\
0 \longrightarrow & V & \xrightarrow{i_V} & \bigoplus_n A & \xrightarrow{\beta} & G & \longrightarrow 0
\end{array}$$

in which π is the projection $H_A\big(\bigoplus_n A\big) \to H_A\big(\bigoplus_n A\big)/W$. The map θ_2 making the diagram commute exists once we have established that the first square commutes. Let $\alpha \in W$ and $a \in A$. Then $\theta_{\bigoplus_n A} T_A(\varepsilon)(\alpha \otimes a) = (\varepsilon\alpha)(a) = \alpha(a)$ and $i_V\theta_1(\alpha \otimes a) = \alpha(a)$. Diagram chasing yields that θ_2 is onto.

Let $x \in \ker \theta_2$. Choose $y \in T_A H_A\big(\bigoplus_n A\big)$ with $T_A(\pi)(y) = x$. Since $0 = \theta_2 T_A(\pi)(y) = \beta\theta_{\bigoplus_n A}(y)$, we have that $\theta_{\bigoplus_n A}(y) = i_V(z)$ for some $z \in V$. There is a non-zero integer m such that $mz \in \operatorname{im} \theta_v$. Hence, we can find $u \in T_A H_A(U)$ with $mz = i_U\theta_U(u) = \theta_1 T_A H_A(i_U)(u)$ as shown before. Set $w = T_A H_A(i_U)(u)$. Then, $\theta_{\bigoplus_n A} T_A(\varepsilon)(w) = i_V\theta_1(w) = i_V(mz) = \theta_{\bigoplus_n A}(my)$. Since $\theta_{\bigoplus_n A}$ is one-to-one, we obtain $my = T_A(\varepsilon)(w)$. Hence, $mx = T_A(\pi)(my) = T_A(\pi)T_A(\varepsilon)(w) = 0$. Consequently, $\ker \theta_2$ is contained in the torsion-subgroup of $T_A\big(H_A\big(\bigoplus_n A\big)/W\big)$ which is zero since A is flat and $H_A\big(\bigoplus_n A\big)/W$ is torsion-free. Therefore, θ_2 is an isomorphism, and it suffices to show that $T_A\big(H_A\big(\bigoplus_n A\big)\big)/W$ is contained in a finitely generated free module since A is flat.

By the fact that $E(A)$ is coflat ring, this holds once we have shown that $H_A(U)$ is finitely generated. There exists an exact sequence $\bigoplus_m A \xrightarrow{\delta} U \to 0$ since U is finitely A-generated. Because of $U \subset \bigoplus_n A$, the group U is A-solvable, and $H_A(\delta)$ is onto since A is faithfully flat.

Conversely, suppose that A is coflat and consider an exact sequence $0 \to U \to \bigoplus_n E(A) \to M \to 0$ such that M^+ is torsion-free, and U contains a finitely generated V with U/V torsion. There exist exact sequences $0 \to T_A(U) \to T_A\big(\bigoplus_n E(A)\big) \to T_A(M) \to 0$ and $0 \to T_A(V) \to T_A(U) \to T_A(U/V) \to 0$ of torsion-free groups in which $T_A(V)$ is finitely A-generated and $T_A(U/V)$ is torsion. Thus, $T_A(U)$ is the A-purification of a finitely A-generated subgroup of $T_A\big(\bigoplus_n E(A)\big)$. Since A is coflat, there is a monomorphism $\alpha : T_A(M) \to \bigoplus_m A$. Hence, $H_A T_A(M)$ is contained in a finitely generated free module.

Let $\bigoplus_I E(A) \xrightarrow{\alpha} U \to 0$ be exact. Since $U \subset \bigoplus_n E(A)$, we have that $T_A(U) \subset T_A(\bigoplus_n E(A))$ is A-solvable. Hence $H_A T_A(\alpha)$ is onto, and

$$\begin{array}{ccccc} H_A T_A(\bigoplus_I E(A)) & \xrightarrow{H_A T_A(\alpha)} & H_A T_A(U) & \longrightarrow & 0 \\ \uparrow l & & \varphi_U \uparrow & & \\ \bigoplus_I E(A) & \xrightarrow{\alpha} & U & \longrightarrow & 0 \end{array}$$

yields that φ_U is onto. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_A T_A(U) & \longrightarrow & H_A T_A(\bigoplus_u E(A)) & \longrightarrow & H_A T_A(M) & \longrightarrow & 0 \\ & & \uparrow \varphi_U & & l \uparrow \varphi_{\bigoplus_n E(A)} & & \uparrow \varphi_M & & \\ 0 & \longrightarrow & U & \longrightarrow & \bigoplus_n E(A) & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

whose rows are exact since $T_A(M) \subset \bigoplus_m A$ is A-solvable. Thus, φ_U is an isomorphism and the same holds for φ_M. Thus $M \cong H_A T_A(M) \underset{\sim}{\subseteq} \bigoplus_m E(A)$. □

Theorem 4.2. *A torsion-free ring R is coflat (iff QR is coflat) iff every finitely related QR-module is isomorphic to a submodule of a free module.*

Proof. Let R be coflat, and consider an exact sequence $0 \to U \to \bigoplus_n QR \to M \to 0$ of finitely generated QR-modules. Choose a finitely generated R-submodule W of U such that V/W is torsion; and set $V = U \cap \bigoplus_n R$. Then, $\bigoplus_n R/V \cong \langle \bigoplus_n R, U \rangle / U \cong M$ is torsion-free, and $U/V \cong \langle U, \bigoplus_n R \rangle / (\bigoplus_n R)$ is torsion. Thus, $\langle W, V \rangle / V$ is bounded, and we may assume $W \subset V$. In particular, $QW = QV = U$. Let W_* be the $\mathbb{Z}$-purification of W in $\bigoplus_n R$. Then, $W_* \subset V$, and $\bigoplus_n R/W_*$ is a submodule of a free R-module F. Hence $Q(\bigoplus_n R)/QW_* \cong Q(\bigoplus_n R/W_*) \subset QF$, a free QR-module. On the other hand, $QW_* = QV = U$ yields $M \cong Q(\bigoplus_n R)/QW_* \subset QF$.

Conversely, suppose the latter condition holds, and consider a pure exact sequence $0 \to V \to \bigoplus_n R \to M \to 0$ of R-modules in which V contains a finitely generated submodule U with V/U torsion. Then $QV = QU$ is a finitely generated QR-module, and there is a free QR-module F such that $QM \cong Q(\bigoplus_n R)/QV \underset{\sim}{\subseteq} F$. Since $M \underset{\sim}{\subseteq} QM$ is finitely generated, there is a finitely generated free R-submodule P of F and a non-zero integer m such that $mM \subset P$. Since $M \cong mM$, we have that M is a submodule of a free R-module. □

Closely related to the notion of coflat is the following concept. An abelian group A is *strongly coflat* if every subgroup U of A^n, where $n < \omega$ and $S_A(U) = U$, satisfies A^n/U_* is a subgroup of an A-projective group of finite A-rank. Similarly, a ring R is strongly coflat if every finitely generated R-module which is torsion-free as abelian group is contained in a free module.

Using the same methods as in the proof of the previous results, we obtain

Corollary 4.3. *The following conditions are equivalent for a torsion-free abelian group A which is faithfully as an $E(A)$-module:*
(a) *A is strongly coflat.*
(b) *$E(A)$ is strongly coflat.*
(c) *Finitely generated $QE(A)$-modules are submodules of free modules.*

With this we obtain

Corollary 4.4. *The following are equivalent for a torsion-free group A which is faithfully flat as an $E(A)$-module:*
(a) *A is strongly coflat, and $E(A)$ is non-singular.*
(b) *$E(A)$ is non-singular, finite dimensional ring, and A is coflat.*
(c) *QA is semi-simple Artinian.*

Proof. (a) $\Rightarrow$ (c): Since $E(A)$ is non-singular, the same holds for $QE(A)$. Suppose $0 \neq I$ is an essential right ideal of $QE(A)$. Then $QE(A)/I$ is a submodule of a free module, and hence $QE(A)/I$ is non-singular. This results in a contradiction unless $I = QE(A)$. Thus, $QE(A)$ is semi-simple Artinian.

(c) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a): Let U be a submodule of $\bigoplus_n QE(A)$. Then, U contains a finitely generated submodule V which is essential since $QE(A)$ has finite Goldie-dimension. By (b), $\bigoplus_n QE(A)/U$ is non-singular. Suppose $V \neq U$. Then $0 \neq V/U$ is a singular submodule of the non-singular module $\bigoplus_n QE(A)/U$, a contradiction. □

Corollary 4.5. *The following conditions are equivalent for a torsion-free abelian group A which is faithfully flat as an $E(A)$-module and has an integral domain as its endomorphism ring:*
(a) *A is coflat.*
(b) *A is strongly coflat.*
(c) *$QE(A)$ is a field.*

Example 4.6. Let A be faithfully flat with $E(A) \cong \mathbb{Z}[x]$. Then A is not coflat.

Example 4.7. A generalized rank-1 group A is coflat if and only if $QE(A)$ is semi-simple Artinian.

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