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SOME CONSTRUCTIONS OF λ-MINIMAL GRAPHS

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1. INTRODUCTION

Let G be a simple undirected graph, $V(G)$ the set of vertices, n the order of $V(G)$ and $E(G)$ the set of edges of G. We will denote by $N(x)$ the set of the neighbors of a given vertex x of G; when no confusion arises we will use the same symbol to denote the subgraph of G induced by the neighbors of x.

We recall some definitions given by Harary and others in [4]. A *k-coloring* of G is a mapping f from $V(G)$ to the k-set $\{1, 2, \ldots, k\}$. The color of an edge $e = \{u, v\}$ of G induced by f is $f(e) = \{f(u), f(v)\}$ and f is a *line distinguishing coloring* of G when $f(e_1) \neq f(e_2)$ for any two distinct edges e_1 and e_2 of G. The *line-distinguishing chromatic number* of G, denoted $\lambda(G)$, is the minimum number k such that G has a line-distinguishing k-coloring. G is called *λ-minimal* if $\lambda(G - e) = \lambda(G) - 1$ for each edge e of G. We will say briefly that G is *r-minimal* instead of G is *λ-minimal* and $\lambda(G) = r$. Let us say that an edge is *hated* when it is contained in at least one triangle.

In [4] the authors asked characterizations of λ-minimal graphs. We have constructed in [6] the n-minimal graphs of maximum degree $n - 1$ or $n - 2$. Here we construct the triangulated n-minimal graphs, the $(n-1)$-minimal graphs having maximum degree $n - 1$ or $n - 2$ and the triangulated $(n - 1)$-minimal graphs with at least one nonhated edge. Moreover we give a conjecture on the remaining triangulated $(n - 1)$-minimal graphs of diameter 3.

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2. Triangulated n-minimal Graphs.

We recall that a graph G is *triangulated* if it contains no induced cycle of length greater than 3.

The following Proposition, proved in [6], is useful to recognize n-minimal graphs.

Proposition 2.1. *A graph G is n-minimal if and only if any two distinct vertices of G have a common neighbor and for every edge e of G there is an edge e' of G which is adjacent to e such that the common vertex is the unique common neighbor of the other end of e' with the other end of e.*

Example 2.1. We have proved in [6] that a graph G with a vertex z of degree $n-1$ is n-minimal if and only if $N(z)$ is an union of stars. We call these graphs *hated stars* because they can be obtained by adding at least one hat on each edge of a star. It is easily seen that any hated star is also triangulated.

Example 2.2. We call *hated triangle* any graph obtained by adding at least one hat on each edge of a triangle. It is easily seen that these graphs are n-minimal and triangulated.

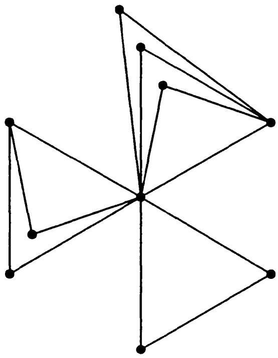

Figure 1.

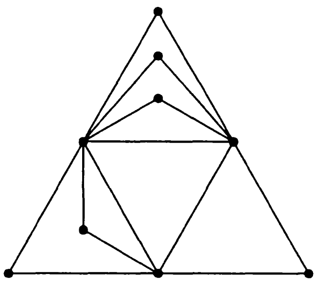

Figure 2.

We will prove that there are not triangulated n-minimal graphs other than these ones.

Proposition 2.2. *The maximum order of a clique in a triangulated n-minimal graph G is 3.*[1]

Proof. G certainly contains some clique of order 3. We suppose by contradiction that G contains a clique K of order 4.

[1] By passing, we note that the chromatic number of any triangulated n-minimal graph is 3.

Let l be any edge of K. By Prop. 2.1 there is an edge l' of G which is adjacent to l such that the common vertex x is the unique common neighbor of the other end x' of l' with the other end y of l. Since K is a clique of order 4, $x' \notin K$ and

(1) $$N(x') \cap K \subseteq \{x, y\}.$$

We claim that

(2) $$N(x') \cap K = \{x\}.$$

Let m be the edge of K which is not adjacent to l. By Prop. 2.1, there is an edge m' of G which is adjacent to m such that the common vertex q is the unique common neighbor of the other end q' of m' with the other end r of m. Since K is a clique of order 4, $q' \notin K$ and

(3) $$N(q') \cap K \subseteq \{q, r\}.$$

We note that $x' \neq q'$. By Prop. 2.1 x' and q' have a common neighbor w and, by (1) and (3), $w \notin K$. We consider the cycle $xx'wq'q$ and obtain, by the assumption G triangulated (see Fig. 3), that

(4) $$x \text{ adj } w \text{ adj } q.$$

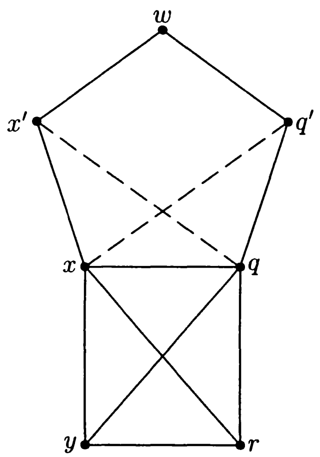

Figure 3.

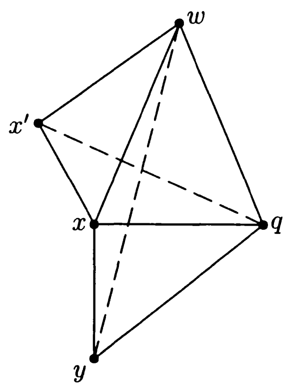

Figure 4.

Now we consider the path $x'wqy$ and obtain, by the assumption G triangulated (see Fig. 4), that x' non adj y. This, together with (1), proves our claim. By symmetry we also obtain

$$N(q') \cap K = \{q\}. \tag{5}$$

Now we consider the edge $n = \{r, y\}$. By Prop. 2.1 there is an edge n' of G which is adjacent to n such that the common vertex is the unique common neighbor of the other end of n' with the other end of n. By symmetry we can suppose that $n' = \{y', y\}$. We have $N(y') \cap N(r) = \{y\}$, $y' \notin K$ and

$$N(y') \cap K = \{y\}. \tag{6}$$

We note that $x' \neq y'$. By Prop. 2.1, x' and y' have a common neighbor z and, by (2) and (6), $z \notin K$. Finally we consider the cycle $xx'zy'y$ and obtain the desired contradiction with the assumption G triangulated (see Fig. 5). □

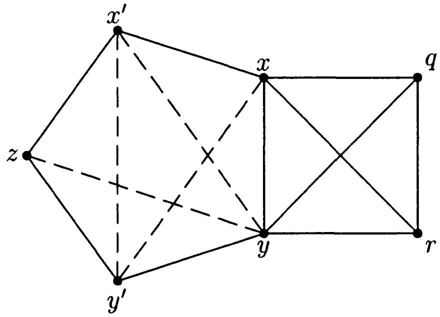

Figure 5.

A vertex x of a graph G is called *simplicial* if $N(x)$ is a clique. Dirac [2] proved that a triangulated graph always has a simplicial vertex. Thus we have the

Corollary 2.3. *Any simplicial vertex of a triangulated n-minimal graph G has degree two.*

We now are ready to prove the

Theorem 2.4. *A graph G is triangulated and n-minimal iff it is either a hated star or a hated triangle, according to G has maximum degree $n-1$ or less.*

Proof. We only prove the nontrivial direction. We already know that an n-minimal graph of degree $n-1$ is a hated star, so we have to show that any triangulated n-minimal graph G with maximum degree smaller that $n-1$ is a hated triangle.

Let u be a simplicial vertex of G. By Corollary 2.3 $N(u)$ has exactly two vertices x_1, x_2. The set of vertices having distance 2 from u splits into the subset A of the vertices which are adjacent to x_1 but x_2, the subset B of the vertices adjacent to both x_1 and x_2 and the subset C of the vertices adjacent to x_2 but x_1. A and C are not empty, otherwise G should have maximum degree $n-1$.

Step 1: no vertex of A is adjacent to a vertex of C, otherwise we should obtain an induced cycle of length 4.

Step 2: any two vertices $b_1, b_2 \in B$ cannot be adjacent, otherwise G should contain a clique of order 4.

Step 3: every vertex of A or C is adjacent to exactly one vertex of B. Indeed it is clear that any vertex $a \in A$ is adjacent to at least one vertex of B, for example a common neighbor of a and any $c \in C$. On the other hand, if a is adjacent to two vertices $b_1, b_2 \in B$, then $ab_1x_2b_2$ should be, by Step 2, a cycle of length 4.

Step 4: there is a vertex $b' \in B$ which is adjacent to every vertex of A and every vertex of C. Indeed, if b' is a common neighbor of $a_1 \in A$ and $c_1 \in C$, then any $c_i \in C$ must be adjacent to b' otherwise a_1 and c_i could not have, by Step 3, common neighbors. Analogously any $a_i \in A$ must be adjacent to b'.

Step 5. Finally we note that there is an hated triangle H, based on the triangle $x_1b'x_{2'}$ which is a spanning subgraph of G. Since G is n-minimal, $G = H$. □

3. The $(n-1)$-minimal Graphs of Maximum Degree $n-1$ or $n-2$.

The most elementary examples of $(n-1)$-minimal graphs are the stars (with $n \geqslant 3$). On the other hand, if G is any $(n-1)$-minimal graph of maximum degree $n-1$ (with $n \geqslant 3$), then it has a spanning subgraph which is a star and hence G has to be a star. Thus we have the

Proposition 3.1. *Let G be any graph of maximum degree $n-1$ (with $n \geqslant 3$). Then G is $(n-1)$-minimal iff it is a star.*

In order to construct the $(n-1)$-minimal graphs of maximum degree $n-2$ it will be useful the

Lemma 3.2. *Let u be a vertex of degree d of a graph G. If $\lambda(G) = d$, then there is an end-vertex adjacent to u.*

Proof. We consider a line-distinguishing d-coloring of G and note that the colors of the edges of G incident to u are $\{i,1\}, \ldots, \{i\}, \ldots, \{i,d\}$, where i $(1 \leqslant i \leqslant d)$ is the color of u; so the neighbor of u having color i is an end-vertex of G. □

Theorem 3.3. *Let G be any graph with $\lambda(G) = n-1$, u a vertex of degree $n-2$ and w the vertex not adjacent to u. Then G is $(n-1)$-minimal if and only if $N(w)$ consists of pairwise nonadjacent vertices $v_1, \ldots, v_p$ and $N(u)$ is the union of $N(w)$ with some stars $S_1, \ldots, S_q$ (p, $q \geqslant 0$ but $p+q > 0$).*

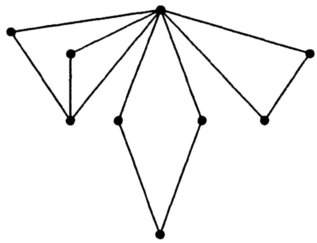

Figure 6.

Proof. We only prove the nontrivial direction.

First of all we claim that none of the neighbors of u is an end-vertex. Otherwise, if x is an end-vertex adjacent to u, we argue as follows. We color u and x with the same color 1 and the other vertices with 2, 3, ..., $n-1$, where $n-1$ is the color of w. This is a line-distinguishing $(n-1)$-coloring of G. So, if we replace the color $n-1$ of w with another color $1 < i < n-1$, we have not a line-distinguishing coloring of G. This means that w has a common neighbor with every vertex but x and u. Then it is not hard to see that any two distinct vertices of $G - x$ have a common neighbor. Thus $\lambda(G - ux) = n - 1$, which is in contradiction with the assumption G λ-minimal. This proves our claim; so we can say that the subgraph U of G induced by the neighbors of u which are not neighbors of w has not isolated vertices. Then it follows that U has a spanning subgraph S which is isomorphic to an union of stars. Finally we note that the edges of G incident to u, the edges of S and the edges of G which are incident to w but are not incident to some edge of S form a spanning subgraph H of G isomorphic to that described in the statement. Since G is $(n-1)$-minimal, $G = H$. □

We note that these graphs are exactly those obtained from hated stars by deleting one of the edges incident to the centre.[2]

[2] We point out that certain extremal graphs are $(n-1)$-minimal. Erdös and others studied in [3] the minimum number $F_d(n,k)$ of edges of a graph having n vertices, maximum degree k and diameter d. They proved that $F_2(n, n-2) = 2n-4$ and gave as examples the graphs of our Th. 3.3, with $q = 0$. Further, they proved that $F_2(n,k) = 2n-4$ for $(2n-2)/3 \leqslant k \leqslant n-5$ and gave as examples the graphs obtained by deleting an edge (with at least two hats) from the central triangle of an hated triangle. These graphs are $(n-1)$-minimal.

4. Triangulated $(n-1)$-minimal Graphs.

The following remark will be useful to recognize the graphs having line-distinguishing chromatic number $n-1$.

Proposition 4.1. *Let G be any graph. Then $\lambda(G) = n - 1$ iff the following conditions hold:*

i) *the maximum number of vertices of G having pairwise no common neighbor is two;*

ii) *if u, v, x, y are distinct vertices of G such that u and v, as well as x and y, have no common neighbor, then*

$$(u \text{ adj } x \text{ and } v \text{ adj } y) \text{ or } (u \text{ adj } y \text{ and } v \text{ adj } x).$$

Proof. We only prove the "only if" part of the statement.

i) Obviously there are two vertices of G having no common neighbor. On the other hand, if there are three vertices of G having pairwise no common neighbor, then we color them with the color 1 and the other vertices of G with the colors $2, \ldots, n-2$ and obtain a line-distinguishing coloring of G, in contradiction with $\lambda(G) = n - 1$.

ii) We color the vertices u, v with the color 1, the vertices x, y with the color 2, the remaining vertices of G with the colors $3, \ldots, n-2$. This cannot be a line-distinguishing coloring of G, so there are two distinct edges of G having the same color. This color must be $\{1, 2\}$. □

Examples of triangulated $(n-1)$-minimal graphs are:

1) any star graph;

2) any graph having a vertex u with is adjacent to every other but a vertex w, such that $N(w)$ is the trivial graph and $N(u)$ is the union of $N(w)$ with some stars (see Th. 3.3);

3) any graph obtained from two complete graphs H and K (each of order at least 3) and from a set L of pairwise non adjacent vertices by joining one vertex $x \in H$ with one vertex $y \in K$ and each vertex of L with both x and y;

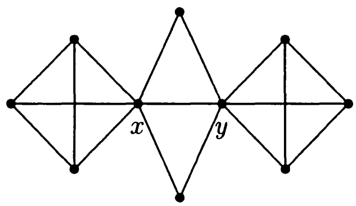

Figure 7.

4) any union of two nontrivial complete graphs. We note that, conversely, if an $(n-1)$-minimal graph G is the union of two nontrivial graphs H and K, then H and K are complete graphs. Indeed, if u, $x \in H$ and v, $y \in K$, then u has no neighbor in common with v and x has no neighbor in common with y; then, by Proposition 4.1, we have u adj x and v adj y.

We will prove that the graphs of examples 1) and 2) are the unique triangulated $(n-1)$-minimal graphs having some nonhated edge.

We conjecture that example 3) gives exactly the remaining triangulated $(n-1)$-minimal graphs of diameter 3.

We note that, if e is any nonhated edge of a triangulated graph, then it is a bridge. Otherwise e is contained in a cycle of G and, if this cycle is chosen of minimum length, it is chordless and hence, since G is a triangulated graph, it is a triangle.

Theorem 4.2. *Let G be any graph having a nonhated edge e. Then G is triangulated $(n-1)$-minimal iff it is either a star graph or it has a vertex u which is adjacent to every other but a vertex w, $N(w)$ is the trivial graph and $N(u)$ is the union of $N(w)$ with some stars.*

Proof. We only prove the nontrivial direction.

As remarked above, $e = \{v, z\}$ is a bridge; let $G - e$ be the union of the graphs H and K, where $v \in H$ and $z \in K$. We distinguish two cases.

1) v and z are not end-vertices.

First of all we note that all the remaining vertices are adjacent to v or z. Let $x \in H - v$, $y \in K - z$. Obviously v and z, as well as x and y, have no common neighbor. Thus we obtain, by Proposition 4.1, that x adj v and y adj z.

Then we claim that H and K cannot have both order greater than 2. Indeed, if x_1, $x_2 \in H - v$ and y_1, $y_2 \in K - z$, then x_1 and y_1, as well as x_2 and y_2, have no common neighbor; thus, by Prop. 4.1, x_1 adj x_2 and y_1 adj y_2. This, together with the first remark, gives that both H and K are complete graphs. Thus $\lambda(G - e) = n - 1$, which is in contradiction with the assumption G $(n-1)$-minimal. This proves our claim, so we can think that K has order 2.

Finally, we note that v has degree $n-2$ and achieve the proof in the actual case by applying Th. 3.3.

2) z, say, is an end-vertex.

In this case K reduces to the trivial graph on z. In H there are two vertices p and q having no common neighbor, otherwise $\lambda(G - e) = n - 1$, which is in contradiction with the assumption G $(n-1)$-minimal. Now, applying Prop. 4.1, we see that there is only one possibility: $q = v$, p adj v and pq is a nonhated edge, hence it is a bridge.

If p is not an end-vertex, then we go back to case 1); so let p be an end-vertex. In this case we claim that v is adjacent to every other vertex of G. Otherwise, if t

is a vertex of G which is not adjacent to v, we note that t and p, as well as v and z, have no common neighbor, and it is easily seen that this leads to a contradiction with Prop. 4.1. Thus G has to be a star graph. □

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