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ON THE TRACE THEORY FOR FUNCTIONS IN SOBOLEV SPACES WITH MIXED L_p-NORM

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INTRODUCTION

In this paper we prove a theorem on the trace on $\partial\Omega \times (0,T)$ for functions in the Sobolev space $W^{2,1}_{p,q}(\Omega_T) := \{f \mid \partial^\alpha_x f, \partial_t f \text{ (distr. sense)} \in L_q(0,T,L_p(\Omega))\,\forall|\alpha| \leqslant 2\}$ with $1 < p \leqslant q < \infty$; here $\Omega_T := \Omega \times (0,T)$ and $\Omega \subset \mathbb{R}^n$ with compact sufficiently smooth boundary. Our results, which seem to be sharp, are applicable to the Dirichlet- and Neumann problem for the heat equation and Navier-Stokes equations with *inhomogeneous* boundary conditions. The corresponding problems with homogeneous boundary conditions have been studied in $L_q\big(0,T,L_p(\Omega)\big)$-spaces with q different from p by various authors: compare v. Wahl [7] for parabolic equations and Iwashita [4], v. Wahl [8] for the Navier-Stokes system. Our results, stated in Theorem 1, generalize the classical trace theory developed for $q = p$ only (see Ladyshenskaya [6], chapter II, Lemma 3.4.; Il'in and Solonnikov [3]); an elaboration of part of their work can also be found in Weidemaier [9].

We use the method of integral representation introduced by the Russian school (cf. Appendix A) and some weighted inequalities of Hardy-type (cf. Appendix B).

Let us fix our notation: Γ is the boundary of Ω and $\Gamma_T := \Gamma \times (0,T)$. Moreover $Q^{n+1}(0,T^{\underline{\kappa}}) := \prod_{i=1}^{n+1}(0,T^{\kappa_i})$ for $\underline{\kappa} := (\kappa_1,\dots,\kappa_{n+1})$, $Q^{n-1}(\alpha) := (-\alpha,\alpha)^{n-1}$, $Q^n_+(\alpha,\beta) := Q^{n-1}(\alpha) \times (0,\beta)$ for $\alpha,\beta > 0$. The typical point in $Q^n_+(\alpha,\beta) \times (0,T)$ is denoted (x,t). The prime characterizes $(n-1)$-dimensional quantities: thus we write $x \in \mathbb{R}^n$ as $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$. The t-coordinate is sometimes also referred to as the $(n+1)$-th coordinate. The superscript $\check{}$ always indicates the deletion of a coordinate, for example

$$\overset{i}{\check{y}} = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{n+1}) \quad \text{and} \quad \overset{n,n+1}{\check{Q}}{}^{n+1}(0, T^{\underline{\kappa}}) := \prod_{\substack{i=1 \\ i \notin \{n,n+1\}}}^{n+1} (0, T^{\kappa_i}).$$

The natural norm in $L_q(0, T, L_p(\Omega))$ is denoted by $\|\cdot\|_{p,q,\Omega_T}$. We use the notation c^* to emphazise the non-dependence of the constant c on the quantity T.

Main result

For the convenience of the reader we shortly introduce our notation used in the description of the boundary of Ω and some function spaces on it.

For $\Omega \subset \mathbb{R}^n$ with compact boundary "$\Omega \in C^{1,1}$" is defined as in the book by Kufner [5], 6.2.2; this in particular implies that there exist finitely many open subsets $U_i \subset \mathbb{R}^n$ $(i = 1, \dots, M)$ and invertible mappings $\Psi_i \in C^{1,1}(\overline{Q^n_+(\alpha, \beta)}, \mathbb{R}^n)$ such that

$$\Gamma \cap U_i = \Psi_i(Q^{n-1}(\alpha) \times \{0\}), \quad \bigcup_{i=1}^{M} (\Gamma \cap U_i) = \Gamma$$
$$\Omega \cap U_i = \Psi_i(Q^{n-1}(\alpha) \times (0, \beta));$$

let us remark that Ψ_i equals $A_i^{-1} \circ Q_i$ in the notation of [5], 6.2.9, where $Q_i(x', x_n) := (x', a(x') + x_n)$ with a certain $a(\cdot) \in C^{1,1}(\overline{Q^{n-1}(\alpha)})$ and A_i is linear and invertible. From the explicit form of Q_i it is easy to see that Q_i^{-1} is also $C^{1,1}$. Moreover there exists an open subset $U_0 \subset \mathbb{R}^n$ such that

$$\overline{U}_0 \subset \Omega, \quad \bigcup_{i=0}^{M} (\Omega \cap U_i) = \Omega.$$

Ψ_i^* defined by $\Psi_i^* g(x, t) := g(\Psi_i(x), t)$ is the pullback induced by Ψ_i in the spatial variables. We denote by $(\varphi_i)_i$ a partition of unity on $\overline{\Omega}$ with $\varphi_i \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} \varphi_i \subset U_i$ for $i = 0, \dots, M$.

The spaces $L_p(\Gamma)$ $(1 \leqslant p < \infty)$ are defined as in [5], 6.3.2: a function u defined a.e. on Γ belongs to $L_p(\Gamma)$ iff $u \circ \Psi_i(\cdot, 0) \in L_p(Q^{n-1}(\alpha))$ for each $i = 1, \dots, M$; in this case

$$\|u\|^p_{p,\Gamma} := \sum_{i=1}^{M} \|u \circ \Psi_i(\cdot, 0)\|^p_{p, Q^{n-1}(\alpha)}.$$

The spaces $\dot{W}^s_p(\Gamma)$, $s > 0$, are defined similarly (see [5], 6.7.2 and 6.8.6). Finally we define

$$X^{\alpha,\beta}_{p,q}(\Gamma_T) := L_q(0, T, W^\alpha_p(\Gamma)) \cap \{g \mid |g|_{\mathcal{L}^{0,\beta}_q(\Gamma_T)} < \infty\} \quad \text{for } \alpha > 0,\ \beta \in (0, 1)$$

with

$$\|\cdot\|_{X^{\alpha,\beta}_{p,q}(\Gamma_T)} := \|\cdot\|_{L_q(0,T,W^{\alpha}_p(\Gamma))} + |\cdot|_{\mathcal{L}^{0,\beta}_q(\Gamma_T)},$$

$$|g|^q_{\mathcal{L}^{0,\beta}_q(\Gamma_T)} := \int_0^T h^{-(1+q\beta)} \|\Delta_{n+1,h}\, g\|^q_{L_q(0,T-h,L_p(\Gamma))} \mathrm{d}h$$

with $\Delta_{n+1,h}\, g(\xi,t) := g(\xi,t+h) - g(\xi,t)$ for $\xi \in \Gamma$.

Now we are ready to formulate our main result.

Theorem 1. *Assume that $\Omega \subset \mathbb{R}^n$ has compact boundary and belongs to the class $C^{1,1}$; let $1 < p \leqslant q < \infty$ and $s(m) = 2 - m - 1/p$.*

(i) *Then for each $k = 1, \ldots, n$ and $m = 0, 1$ there is a unique linear continuous map $\gamma_{k,m} : W^{2,1}_{p,q}(\Omega_T) \longrightarrow X^{s(m),s(m)/2}_{p,q}(\Gamma_T)$ such that $\gamma_{k,m} f = \partial^m_k f\big|_{\Gamma_T}$ for $f \in D := W^{2,1}_{p,q}(\Omega_T) \cap \{f \mid f(\cdot,t) \in C^1(\overline{\Omega})\ \forall t \in (0,T)\}$.*

(ii) *Moreover the norm of each $\gamma_{k,m}$ is independent of T.*

Remark 2. The space $X^{s,s/2}_{p,q}(\Gamma_T)$ coincides for $q = p$ with $W^{s,s/2}_p(\Gamma_T)$ in Ladyshenskaya [6].

P r o o f of Theorem 1. The estimate for the spatial regularity follows from the well-known trace theorem $W^{2-m}_p(\Omega) \ni u \longmapsto u\big|_\Gamma \in W^{2-m-1/p}_p(\Gamma)$ (cf. Kufner [5], 6.10.3) together with an easy scaling argument in t. In the sequel we shall concentrate on the proof for the time-regularity of the trace: since D defined above is dense in $W^{2,1}_{p,q}(\Omega_T)$ and $X^{s(m),s(m)/2}_{p,q}(\Gamma_T)$ is a complete space (two facts for which the (routine) proofs will be given later in Lemma 3 and Lemma 4), it is sufficient to consider $f \in D$. Moreover, since $f = \sum_{i=0}^M f \cdot \varphi_i$ (the φ_i are the functions of the partition of unity introduced above) and since $\Gamma \cap \operatorname{supp} \varphi_0 = \emptyset$, it is sufficient to consider $f_i := f \cdot \varphi_i$ $(i = 1, \ldots, M)$. Furthermore we are going to reduce the proof to a situation in half-space by flattening the boundary: for u with support contained in U_i we have (see [5], 6.3.9 Lemma)

$$\|u\|_{p,\Gamma} \leqslant c^* \cdot \big\|u\big(\Psi_i(\cdot,0)\big)\big\|_{p,Q^{n-1}(\alpha)};$$

applying the last inequality with $u(\cdot) = \Delta_{n+1,h}\partial^m_k f_i(\cdot,t)$ we see that it is sufficient to prove

$$\text{(1)} \qquad \big|\big(\Psi^*_i(\partial^m_k f_i)\big)\big|_{x_n=0}\big|_{\mathcal{L}^{0,s(m)/2}_q(Q^{n-1}(\alpha)\times(0,T))} \leqslant c^* \cdot \|f_i\|_{W^{2,1}_{p,q}(\Omega_T)},$$

(where $|\cdot|_{\mathcal{L}^{0,\beta}_q(Q^{n-1}(\alpha)\times(0,T))}$ is defined, of course, in the same way as $|\cdot|_{\mathcal{L}^{0,\beta}_q(\Gamma_T)}$, but with Γ replaced with $Q^{n-1}(\alpha)$ everywhere). We further claim that the last inequality follows from

$$\text{(2)} \qquad \big|\big(\partial^m_{x_j}(\Psi^*_i f_i)\big)\big|_{x_n=0}\big|_{\mathcal{L}^{0,s(m)/2}_q(Q^{n-1}(\alpha)\times(0,T))} \leqslant c^* \cdot \|\Psi^*_i f_i\|_{W^{2,1}_{p,q}(Q^n_+(\alpha,\beta)\times(0,T))}$$

($j = 1, \ldots, n$). For the proof of this claim we note that by the chain rule for weak derivatives (cf. [5], proof of 5.7.3) and the $C^{1,1}$-regularity of Ψ_i^{-1} the function $\Psi_i^*(\partial_k f_i)$ is a linear combination of spatial derivatives of $\Psi_i^* f_i$ with L_∞-coefficients (which do not depend on t). In order to pass from the r.h. side of (2) to the r.h. side of (1), we remark that Ψ_i^* induces an isomorphism $W_{p,q}^{2,1}((U_i \cap \Omega)_{T'}) \to W_{p,q}^{2,1}(Q_+^n(\alpha, \beta)_T)$ (use again the chain rule, the $C^{1,1}$-regularity of Ψ_i and Ψ_i^{-1}, the transformation rule for integrals and the fact that the Jacobians of Ψ_i and Ψ_i^{-1} are in L_∞).

A last technical remark: for later use of the integral representation in Appendix A it is useful to consider $\Psi_i^* f_i$ in (2) as being defined on $\mathbb{R}_+^n \times (0, 2T)$. This is possible, since extending $\Psi_i^* f_i$ by zero in its spatial variables and reflecting it (cf. Adams [1], p. 83) in its t-variable yield a linear extension operator E_T, which is continuous with respect to the $W_{p,q}^{2,1}$-norms and whose operator norm is bounded uniformly in T. Thus, denoting $E_T(\Psi_i^* f_i)$ by f again, it is enough to prove

$$(3)\qquad |(\partial_j^m f)|_{x_n=0}|_{\mathcal{L}_q^{0,s(m)/2}(Q^{n-1}(\alpha)\times(0,T))} \leqslant c^* \cdot \|f\|_{W_{p,q}^{2,1}(\mathbb{R}_+^n \times (0,2T))}.$$

In the sequel we shall prove (3). By density it is clearly no restriction to assume that $f \in C^2(\overline{\mathbb{R}_+^n} \times [0, 2T])$ additionally. We start from representation (A.1) for $\partial_j^m f$: splitting $\int_0^T (\ldots)\mathrm{d}v = \int_0^h (\ldots)\mathrm{d}v + \int_h^T (\ldots)\mathrm{d}v$ in the sum in the second line in (A.1) we get

$$\partial_j^m f(\cdot) = H_1(\cdot) + \sum_{i=1}^{n+1} \tilde{B}_i \{ H_2^{(i)}(\cdot) + H_3^{(i)}(\cdot) \} \quad \text{for } m = 0, 1,$$

where

$$(4)\qquad H_1(\cdot) := \frac{A}{T^r} \int \ldots \int_{Q^{n+1}(0,T^{\underline{\kappa}})} f(\cdot + y)\Pi(y, T)\mathrm{d}y,$$

$$H_2^{(i)}(\cdot) := \int_0^h v^{-(1+r)} \int \ldots \int_{Q^{n+1}(0,v^{\underline{\kappa}})} \partial_i^{l_i} f(\cdot + y) \cdot K_i(y, v)\mathrm{d}y\mathrm{d}v,$$

$$H_3^{(i)}(\cdot) := \int_h^T v^{-(1+r)} \int \ldots \int_{Q^{n+1}(0,v^{\underline{\kappa}})} \partial_i^{l_i} f(\cdot + y) \cdot K_i(y, v)\mathrm{d}y\mathrm{d}v.$$

We choose $\underline{l} := (2, \ldots, 2, 1) \in \mathbb{N}^{n+1}$ and $\underline{\kappa} := (\frac{1}{2}, \ldots, \frac{1}{2}, 1) \in \mathbb{R}^{n+1}$. Abbreviating $(\gamma H_1)(x', t) := H_1(x', 0, t)$, we find

$$(5)\qquad \|\Delta_{n+1,h}(\gamma H_1)\|_{p,q,Q^{n-1}(\alpha)\times(0,T-h)} \leqslant h \cdot \|\partial_t(\gamma H_1)\|_{p,q,Q^{n-1}(\alpha)\times(0,T)}$$

(use $|\Delta_{n+1,h}\, g(\tau)| \leqslant \int_0^h |g'(\tau+s)|\mathrm{d}s$ and Minkowski's integral inequality, cf. Wheeden and Zygmund [10], p. 143); now

$$|\partial_t(\gamma H_1)(x',t)| \leqslant \frac{|A|}{T^r} \cdot \|\Pi(\cdot,T)\|_{\infty,Q^{n+1}(0,T^{\underline{\kappa}})} \cdot |Q^{n+1}(0,T^{\underline{\kappa}})|^{1/p'} \cdot \left\|\partial_t f((x',0,t)+\cdot)\right\|_{p,Q^{n+1}(0,T^{\underline{\kappa}})}$$

by (4) and Hölder's inequality; hence

$$\leqslant c^* \cdot T^{-|\underline{\kappa}|\cdot(1-1/p')-m\cdot\kappa_j} \cdot \left\|\partial_t f((x',0,t)+\cdot)\right\|_{p,Q^{n+1}(0,T^{\underline{\kappa}})}$$

by kernel-estimate (A.2). Thus

$$\|\partial_t(\gamma H_1)(\cdot,t)\|_{p,Q^{n-1}(\alpha)} \leqslant c^* \cdot T^{-|\underline{\kappa}|\cdot(1-1/p')-m\cdot\kappa_j} \cdot \left| \overset{n,n+1}{\check{Q}}{}^{n+1}(0,T^{\underline{\kappa}})\right|^{1/p} \times\Big(\int_0^{T^{\kappa_{n+1}}} \int_0^{T^{\kappa_n}} \|\partial_t f(\cdot,y_n,t+y_{n+1})\|^p_{p,\mathbb{R}^{n-1}}\mathrm{d}y_n\mathrm{d}y_{n+1}\Big)^{1/p}$$

and consequently, since $|\overset{n,n+1}{\check{Q}}{}^{n+1}(0,T^{\underline{\kappa}})| = T^{|\underline{\kappa}|-3/2}$ and $\kappa_{n+1} = 1$,

$$\Big(\int_0^T \|\partial_t(\gamma H_1)(\cdot,t)\|^q_{p,Q^{n-1}(\alpha)}\mathrm{d}t\Big)^{1/q} \leqslant c^* \cdot T^{-m\cdot\kappa_j-3/2p} \times\Big(\int_0^T \Big(\int_0^T \int_0^{T^{\kappa_n}} \|\partial_t f(\cdot,y_n,t+y_{n+1})\|^p_{p,\mathbb{R}^{n-1}}\mathrm{d}y_n\mathrm{d}y_{n+1}\Big)^{q/p}\mathrm{d}t\Big)^{1/q}.$$

By Minkowski's integral inequality the last integral does not exceed

$$\Big(\int_0^T \Big(\int_0^T \Big(\int_0^{T^{\kappa_n}} \|\partial_t f(\cdot,y_n,t+y_{n+1})\|^p_{p,\mathbb{R}^{n-1}}\mathrm{d}y_n\Big)^{q/p}\mathrm{d}t\Big)^{p/q}\mathrm{d}y_{n+1}\Big)^{1/p},$$

which is majorized by

$$T^{1/p}\Big(\int_0^{2T} \Big(\int_0^{T^{\kappa_n}} \|\partial_t f(\cdot,y_n,\tau)\|^p_{p,\mathbb{R}^{n-1}}\mathrm{d}y_n\Big)^{q/p}\mathrm{d}\tau\Big)^{1/q}$$

after integrating out the y_{n+1}-variable. These estimates imply

$$\text{r.h. side in (5)} \leqslant c^* \cdot h \cdot T^{-(m\cdot\kappa_j+\frac{1}{2p})} \cdot \|\partial_t f\|_{p,q,\mathbb{R}^{n-1}\times(0,T^{\kappa_n})\times(0,2T)};$$

thus, abbreviating $\varrho := \frac{1}{2}(2-m-\frac{1}{p})$, we see that

$$|\gamma H_1|_{\mathcal{L}^{0,\varrho}_q(Q^{n-1}(\alpha)\times(0,T))} = \Big(\int_0^T h^{-(1+q\varrho)}\|\Delta_{n+1,h}(\gamma H_1)\|^q_{p,q,Q^{n-1}(\alpha)\times(0,T-h)}\mathrm{d}h\Big)^{1/q}$$
$$\leqslant c^* \cdot T^{-(m\cdot\kappa_j+\frac{1}{2p})} \cdot \Big(\int_0^T h^{-1+q(1-\varrho)}\mathrm{d}h\Big)^{1/q}\|\partial_t f\|_{p,q,\mathbb{R}^n_+\times(0,2T)};$$

now $1-\varrho=\frac{1}{2}(m+\frac{1}{p})$ and the T factors in the last inequality cancel (since $\kappa_j=\frac{1}{2}$), as desired.

Let us turn our attention to $H_2^{(i)}$: trivially, for $h\leqslant T$,

(6) $$\|\Delta_{n+1,h}(\gamma H_2^{(i)})\|_{p,q,Q^{n-1}(\alpha)\times(0,T-h)}\leqslant 2\cdot\|\gamma H_2^{(i)}\|_{p,q,Q^{n-1}(\alpha)\times(0,T)};$$

furthermore, using kernel estimate (A.3) (with $s=0$), we get

(7)
$$|\gamma H_2^{(i)}(x',t)|\leqslant c^*\cdot\int_0^h v^{-(1+|\underline{\kappa}|+\varepsilon\kappa_n)+\frac{1}{2}(2-m)}\int\limits_{Q^{n+1}(0,v^{\underline{\kappa}})}\!\!\dots\!\int y_n^\varepsilon\cdot\left|\partial_i^{l_i}f((x',0,t)+y)\right|\mathrm{d}y\mathrm{d}v;$$

we now represent the integrand as

$$\left\{v^{-\frac{1}{p'}(1+|\underline{\kappa}|)+\frac{1}{2}(\varrho-\varepsilon\cdot\kappa_n)}\right\}\cdot\left\{v^{-\frac{1}{p}(1+|\underline{\kappa}|-\frac{1}{2})+\frac{1}{2}(\varrho-\varepsilon\kappa_n)}\cdot y_n^\varepsilon\cdot\left|\partial_i^{l_i}f((x'0,t)+y)\right|\right\}$$

(note that $\frac{1}{2}(2-m)=\varrho+1/2p$); we choose $\varepsilon\in(0,\varrho/\kappa_n)$; Hölder's inequality (with p', p) in y-v space then yields

(8) $$\text{r.h. side in (7)}\leqslant c^*\cdot\left(\int_0^h v^{-1+\frac{p'}{2}(\varrho-\varepsilon\cdot\kappa_n)}\mathrm{d}v\right)^{1/p'}\cdot I^{1/p}$$

with

$$I:=\int_0^h\int\limits_{Q^{n+1}(0,v^{\underline{\kappa}})}\!\!\dots\!\int v^{-(1+|\underline{\kappa}|-\frac{1}{2})+\frac{p}{2}(\varrho-\varepsilon\cdot\kappa_n)}\cdot y_n^{\varepsilon\cdot p}\cdot\left|\partial_i^{l_i}f((x',0,t)+y)\right|^p\mathrm{d}y\mathrm{d}v,$$

where in the first integral we took into account that $|Q^{n+1}(0,v^{\underline{\kappa}})|=v^{|\underline{\kappa}|}$; the first integral clearly is proportional to $h^{\frac{1}{2}(\varrho-\varepsilon\cdot\kappa_n)}$. Thus, by (7) and (8),

(9) $$\|\gamma H_2^{(i)}(\cdot,t)\|_{p,Q^{n-1}(\alpha)}\leqslant c^*\cdot h^{\frac{1}{2}(\varrho-\varepsilon\cdot\kappa_n)}\cdot\tilde{I}^{1/p}$$

with

$$\tilde{I}:=\int_0^h v^{-(1+|\underline{\kappa}|-\frac{1}{2})+\frac{p}{2}(\varrho-\varepsilon\cdot\kappa_n)}|\overset{n,n+1}{\check{Q}}{}^{n+1}(0,v^{\underline{\kappa}})|\times$$
$$\times\int_0^{v^{\kappa_{n+1}}}\int_0^{v^{\kappa_n}}y_n^{\varepsilon\cdot p}\cdot\|\partial_i^{l_i}f(\cdot,y_n,t+y_{n+1})\|_{p,\mathbb{R}^{n-1}}^p\mathrm{d}y_n\mathrm{d}y_{n+1}\mathrm{d}v;$$

abbreviating $F_i(y_n,\tau):=y_n^{\varepsilon\cdot p}\cdot\|\partial_i^{l_i}f(\cdot,y_n,\tau)\|_{p,\mathbb{R}^{n-1}}^p$, (9) implies

(10) $$\left(\int_0^T\|\gamma H_2^{(i)}(\cdot,t)\|_{p,Q^{n-1}(\alpha)}^q\mathrm{d}t\right)^{1/q}\leqslant c^*\cdot h^{\frac{1}{2}(\varrho-\varepsilon\cdot\kappa_n)}\times$$
$$\times\left(\int_0^T\left(\int_0^h\int_0^{v^{\kappa_{n+1}}}v^{-2+\frac{p}{2}(\varrho-\varepsilon\cdot\kappa_n)}\int_0^{v^{\kappa_n}}F_i(y_n,t+y_{n+1})\mathrm{d}y_n\mathrm{d}y_{n+1}\mathrm{d}v\right)^{q/p}\mathrm{d}t\right)^{1/q}.$$

By Minkowski's integral inequality the last integral does not exceed

$$\left(\int_0^h \int_0^{v^{\kappa_{n+1}}} \left(\int_0^T \left(v^{-2+\frac{p}{2}(\varrho-\varepsilon\cdot\kappa_n)} \int_0^{v^{\kappa_n}} F_i(y_n, t+y_{n+1})\mathrm{d}y_n\right)^{q/p} \mathrm{d}t\right)^{p/q} \mathrm{d}y_{n+1}\mathrm{d}v\right)^{1/p},$$

which is majorized by

$$\left(\int_0^h v^{-1+\frac{p}{2}(\varrho-\varepsilon\cdot\kappa_n)} \left(\int_0^{T+v} \left(\int_0^{v^{\kappa_n}} F_i(y_n,\tau)\mathrm{d}y_n\right)^{q/p} \mathrm{d}\tau\right)^{p/q} \mathrm{d}v\right)^{1/p}$$

and thus also by

$$c^* \cdot h^{\frac{1}{2}(\varrho-\varepsilon\cdot\kappa_n)} \cdot \left(\int_0^{T+h} \left(\int_0^{h^{\kappa_n}} y_n^{\varepsilon\cdot p} \cdot \|\partial_i^{l_i} f(\cdot, y_n, \tau)\|^p_{q,\mathbf{R}^{n-1}} \mathrm{d}y_n\right)^{q/p} \mathrm{d}\tau\right)^{1/q};$$

here we integrated out the y_{n+1} and v variables successively (recall that $\kappa_{n+1} = 1$). By (6), (10) and the last estimates

(11)

$$\begin{aligned}
|\gamma H_2^{(i)}|^q_{\mathcal{L}_q^{0,\varrho}(Q^{n-1}(\alpha)\times(0,T))} &= \int_0^T h^{-(1+q\varrho)} \|\Delta_{n+1,h}(\gamma H_2^{(i)})\|^q_{p,q,Q^{n-1}(\alpha)\times(0,T-h)}\mathrm{d}h \\
&\leqslant c^* \cdot \int_0^T h^{-(1+q\varepsilon\kappa_n)} \int_0^{T+h} \left(\int_0^{h^{\kappa_n}} y_n^{\varepsilon\cdot p} \cdot \|\partial_i^{l_i} f(\cdot, y_n,\tau)\|^p_{p,\mathbf{R}^{n-1}}\mathrm{d}y_n\right)^{q/p} \mathrm{d}\tau\mathrm{d}h \\
&\leqslant c^* \cdot \int_0^{2T} \int_0^T h^{-(1+q\varepsilon\kappa_n)} \left(\int_0^{h^{\kappa_n}} y_n^{\varepsilon\cdot p} \cdot \|\partial_i^{l_i} f(\cdot, y_n,\tau)\|^p_{p,\mathbf{R}^{n-1}}\mathrm{d}y_n\right)^{q/p} \mathrm{d}h\mathrm{d}\tau,
\end{aligned}$$

the last step by Fubini's theorem and since $h \leqslant T$. By the Hardy-type inequality in Appendix B, Lemma B.1(i) (applied with $r = q/p \geqslant 1 = s$, $\gamma = \kappa_n$ and ε replaced with $\varepsilon \cdot p$; note that then indeed $\varepsilon \cdot p \cdot \gamma \cdot r = q \cdot \varepsilon \cdot \kappa_n$) we get for the inner integral in the last line

$$\begin{aligned}
\int_0^T h^{-(1+q\varepsilon\kappa_n)} &\left(\int_0^{h^{\kappa_n}} y_n^{\varepsilon\cdot p} \cdot \|\partial_i^{l_i} f(\cdot, y_n,\tau)\|^p_{p,\mathbf{R}^{n-1}}\mathrm{d}y_n\right)^{q/p} \mathrm{d}h \\
&\leqslant c^* \cdot \left(\int_0^{T^{\kappa_n}} \|\partial_i^{l_i} f(\cdot, y_n,\tau)\|^p_{p,\mathbf{R}^{n-1}}\mathrm{d}y_n\right)^{q/p};
\end{aligned}$$

using this estimate in the last line in (11) we get the desired result for $H_2^{(i)}$.

Finally, let us turn to $H_3^{(i)}$; we again use (5) and observe that the correct expression for $\partial_t(\gamma H_3^{(i)})$ is obtained just by replacing K_i (in the definition of $H_3^{(i)}$) by $\partial_{y_{n+1}} K_i$

(integrate by parts); after estimating $\partial_{y_{n+1}} K_i$ according to (A.3) we arrive at

$$(12)\qquad |\partial_t(\gamma H_3^{(i)})(x',t)| \leqslant c^* \cdot \int_h^T v^{-(1+|\underline{\kappa}|+\varepsilon\cdot\kappa_n)-\frac{m}{2}} \int\limits_{Q^{n+1}(0,v^{\underline{\kappa}})}\!\!\cdots\!\!\int y_n^{\varepsilon} \cdot \left|\partial_i^{l_i} f\big((x',0,t)+y\big)\right| \mathrm{d}y\mathrm{d}v$$

(see (7); here the v-exponent is smaller by one, since $\partial_{y_{n+1}} K_i$ entails (in (A.3)) the additional factor v^{-1}); in the last integral we write the integrand in the form (note that $-m/2 = \frac{1}{2}p + \varrho - 1$)

$$\left\{v^{-\frac{1}{p'}(1+|\underline{\kappa}|)-(1-\varrho-\delta)}\right\} \cdot \left\{v^{-\frac{1}{p}(1+|\underline{\kappa}|-\frac{1}{2})-(\varepsilon\kappa_n+\delta)} \cdot y_n^{\varepsilon} \cdot |\partial_i^{l_i} f(\ldots)|\right\},$$

where we introduced $\delta \in (0, 1-\varrho) \cap (0, 1/q)$. Now we apply Hölder's inequality (with p', p) in y-v space and get

$$\text{r.h. side in (12)} \leqslant c^* \cdot \Big(\int_h^T v^{-1-p'\cdot(1-\varrho-\delta)}\mathrm{d}v\Big)^{1/p'} \cdot J^{1/p}$$

with

$$J := \int_h^T v^{-(1+|\underline{\kappa}|-\frac{1}{2})-p\cdot(\varepsilon\cdot\kappa_n+\delta)} \int\limits_{Q^{n+1}(0,v^{\underline{\kappa}})}\!\!\cdots\!\!\int y_n^{\varepsilon\cdot p} \cdot \left|\partial_i^{l_i} f\big((x',0,t)+y\big)\right|^p \mathrm{d}y\mathrm{d}v;$$

from this we get (see (10))

$$(13)\qquad \Big(\int_0^T \|\partial_t(\gamma H_3^{(i)})(\cdot,t)\|^q_{p,Q^{n-1}(\alpha)}\mathrm{d}t\Big)^{1/q} \leqslant c^* \cdot h^{-(1-\varrho-\delta)} \times$$
$$\times\Big(\int_0^T \Big(\int_h^T \int_0^{v^{\kappa_{n+1}}} v^{-2-p\cdot(\varepsilon\cdot\kappa_n+\delta)} \int_0^{v^{\kappa_n}} F_i(y_n, t+y_{n+1})\mathrm{d}y_n\mathrm{d}y_{n+1}\mathrm{d}v\Big)^{q/p}\mathrm{d}t\Big)^{1/q}.$$

After applying Minkowski's integral inequality and integrating out the y_{n+1} variable (as after (10)) we see that the last integral does not exceed

$$\Big(\int_h^T v^{-1-p\cdot(\varepsilon\cdot\kappa_n+\delta)}\Big(\int_0^{T+v}\Big(\int_0^{v^{\kappa_n}} F_i(y_n,\tau)\mathrm{d}y_n\Big)^{q/p}\mathrm{d}\tau\Big)^{p/q}\mathrm{d}v\Big)^{1/p}$$
$$=: \Big(\int_h^T g(v)\mathrm{d}v\Big)^{1/p};$$

from (5), (13) and the last estimate we get

$$|\gamma H_3^{(i)}|^q_{\mathcal{L}_q^{0,\varrho}(Q^{n-1}(\alpha)\times(0,T))} = \int_0^T h^{-(1+q\varrho)}\|\Delta_{n+1,h}(\gamma H_3^{(i)})\|^q_{p,q,Q^{n-1}(\alpha)\times(0,T-h)}\mathrm{d}h$$
$$\leqslant c^* \cdot \int_0^T h^{-1+q\delta}\Big(\int_h^T g(v)\mathrm{d}v\Big)^{q/p}\mathrm{d}v;$$

now we apply Lemma B.1′(ii) with $r = s := q/p$ and $a \cdot r := 1 - q\delta < 1$ and get (since the total exponent of the weight on the r.h. side in this Lemma equals $s \cdot (-a + 1/s' + 1/r)$, which equals $q\delta - 1 + q/p$ in our case)

$$
\begin{aligned}
&\leqslant c^* \cdot \int_0^T v^{-1+q(\delta+1/p)} g(v)^{q/p} \mathrm{d}v \\
&= c^* \cdot \int_0^T v^{-(1+q\varepsilon\kappa_n)} \int_0^{T+v} \Big(\int_0^{v^{\kappa_n}} y_n^{\varepsilon \cdot p} \cdot \|\partial_i^{l_i} f(\cdot, y_n, \tau)\|_{p,\mathbb{R}^{n-1}}^p \mathrm{d}y_n \Big)^{q/p} \mathrm{d}\tau \mathrm{d}v,
\end{aligned}
$$

where we inserted the definitions of g and F_i. The last line is identical with the second line in (11), so that the desired result for $H_3^{(i)}$ follows.

Thus (3) and with it the Theorem are proved. □

We still have to prove two auxiliary results:

Lemma 3. *Let the assumptions of Theorem 1 be fulfilled. Then D is dense in $W_{p,q}^{2,1}(\Omega_T)$.*

P r o o f. Take $f \in W_{p,q}^{2,1}(\Omega_T)$; clearly $f(\cdot, t) \in W_p^2(\Omega)$ for a.e. $t \in (0, T)$, say on $(0,T) \setminus E$, $|E| = 0$. Redefining f by $f(\cdot, t) \equiv 0$ for $t \in E$, we may assume $f(\cdot, t) \in W_p^2(\Omega)$ for all $t \in (0, T)$.

The approximation problem can be localized by considering $f\varphi_i$, φ_i from the partition of unity. Next we will flatten the boundary: fix $i \in \{1, \ldots, M\}$ and denote $u := \Psi_i^*(f\varphi_i)$. Then $u \in \mathcal{P}$, which means the following: a function g defined on $Q_+^n(\alpha, \beta) \times (0, T)$ is called *spatially properly supported* and we write $g \in \mathcal{P}$, if there exists $\varepsilon > 0$ such that $\operatorname{supp} g(\cdot, t) \subset Q^{n-1}(\alpha - \varepsilon) \times [0, \beta - \varepsilon]$ for a.e. $t \in (0, T)$. Since Ψ_i^* induces an isomorphism $W_{p,q}^{2,1}\big((U_i \cap \Omega)_T\big) \to W_{p,q}^{2,1}\big(Q_+^n(\alpha, \beta)_T\big)$ and since for a $\Phi \in W_{p,q}^{2,1}\big(Q_+^n(\alpha, \beta)_T\big) \cap \mathcal{P}$ we may regard $(\Psi_i^*)^{-1}\Phi$ as an element of $W_{p,q}^{2,1}(\Omega_T)$ (by zero continuation), it is sufficient to solve the approximation problem in $W_{p,q}^{2,1}\big(Q_+^n(\alpha, \beta)_T\big)$ and in such a way that the approximating functions belong to $\mathcal{P}$ also. Since u_δ with $u_\delta(x', x_n, t) := u(x', x_n + \delta, t)$ tends to u in $W_{p,q}^{2,1}\big(Q_+^n(\alpha, \beta)_T\big)$ for $\delta \downarrow 0$ and u_δ has the same properties as u (for δ small), it is sufficient to approximate u_δ. To achieve this, set

$$
u_k := \varrho_{1/k} * \big((u_\delta)^0 \theta\big),
$$

where $\varrho_{1/k}$ is the usual smooth mollifier with $\|\varrho_{1/k}\|_{1,\mathbb{R}^n} = 1$ and $\operatorname{supp} \varrho_{1/k} \subset B_{1/k}(0)$, $\theta = \theta(x)$ is a smooth function with $\theta \equiv 1$ on $\overline{\mathbb{R}_+^n}$ and $\operatorname{supp} \theta \subset \mathbb{R}^{n-1} \times (-\delta/2, \infty)$ and "$*$" denotes convolution in x and "^{0}" denotes extension by zero (in x)

to the whole space. By standard arguments we then have for all t

$$u_k(\cdot,t) \to ((u_\delta)^0\theta)(\cdot,t) \text{ in } W_p^2(\mathbb{R}^n) \quad (k\to\infty),$$
$$\|u_k(\cdot,t)\|_{W_p^2(\mathbb{R}^n)} \leqslant c^* \cdot \left\|((u_\delta)^0\theta)(\cdot,t)\right\|_{W_p^2(\mathbb{R}^n)}$$
$$\leqslant c^*\cdot\|(u_\delta)^0(\cdot,t)\|_{W_p^2(\mathbb{R}^{n-1}\times(-\delta/2,\infty))} \leqslant c^*\cdot\|u(\cdot,t)\|_{W_p^2(Q_+^n(\alpha,\beta))}$$

for δ small; this implies by Lebesgue's theorem

$$u_k\big|_{Q_+^n(\alpha,\beta)_T} \to u_\delta \text{ in } L_q\Big(0,T,W_p^2(Q_+^n(\alpha,\beta))\Big).$$

What remains to be shown is

$$\partial_t u_k\big|_{Q_+^n(\alpha,\beta)_T} \to \partial_t u_\delta \text{ in } L_q\Big(0,T,L_p(Q_+^n(\alpha,\beta))\Big);$$

this follows as above, if we show that

$$\partial_t u_k = \varrho_{1/k} * \Big(((\partial_t u)_\delta)^0\theta\Big) \text{ in } \mathcal{D}'(\mathbb{R}^{n+1}\times(0,T));$$

the last line follows easily, if we show that

$$\partial_t((u_\delta)^0\theta) = ((\partial_t u)_\delta)^0\theta \text{ in } \mathcal{D}'(\mathbb{R}^{n+1}\times(0,T)); \tag{14}$$

to prove (14) take $\varphi\in C_0^\infty(\mathbb{R}^{n+1}\times(0,T))$; then we have

$$\int_0^T \int\!\!\cdots\!\!\int_{\mathbb{R}^{n+1}} (u_\delta)^0\theta\partial_t\varphi$$
$$= \int_0^T\int_{-\delta/2}^{\beta-\delta}\int\!\!\cdots\!\!\int_{Q^{n-1}(\alpha)} (u_\delta\theta\partial_t\varphi)(x',x_n,t)\,\mathrm{d}x'\mathrm{d}x_n\mathrm{d}t$$
$$= \int_0^T\int_{\delta/2}^{\beta}\int\!\!\cdots\!\!\int_{Q^{n-1}(\alpha)} u(x',x_n,t)\theta(x',x_n-\delta)\partial_t\varphi(x',x_n-\delta,t)\,\mathrm{d}x'\mathrm{d}x_n\mathrm{d}t$$
$$= \int_0^T\int_0^\beta\int\!\!\cdots\!\!\int_{Q^{n-1}(\alpha)} \ldots\mathrm{d}x'\mathrm{d}x_n\mathrm{d}t \quad \text{(since } \theta \text{ cuts off in } x_n)$$
$$= \int_0^T\int_0^\beta\int\!\!\cdots\!\!\int_{Q^{n-1}(\alpha)} \eta(x',x_n)u(x',x_n,t)\theta(x',x_n-\delta)\partial_t\varphi(x',x_n-\delta,t)\,\mathrm{d}x'\mathrm{d}x_n\mathrm{d}t,$$

where η is a smooth cut-off function with $\eta \equiv 1$ on $\bigcup_{t\in(0,T)}$ supp $u(\cdot,t)$ and $\eta \in \mathcal{P}$; the last line can be rephrased as

$$\int_0^T \int_0^\beta \underset{Q^{n-1}(\alpha)}{\int \dots \int} u(x', x_n, t)\partial_t \tilde{\varphi}(x', x_n, t)\mathrm{d}x'\mathrm{d}x_n \mathrm{d}t,$$

where $\tilde{\varphi}(x', x_n, t) := \eta(x', x_n)\theta(x', x_n - \delta)\varphi(x' x_n - \delta, t)$ belongs to $C_0^\infty(Q_+^n(\alpha,\beta) \times (0,T))$. Now we may shift the ∂_t from $\tilde{\varphi}$ to u and reverse the above chain of reasoning to end up with

$$-\int_0^T \underset{\mathbb{R}^{n+1}}{\int \dots \int} ((\partial_t u)_\delta)^0 \theta\varphi.$$

□

Lemma 4. *Let the assumptions of Theorem 1 be fulfilled. Let $\alpha \in (0,2)$ and $\beta \in (0,1)$. Then $X_{p,q}^{\alpha,\beta}(\Gamma_T)$ is complete.*

P r o o f. Let (g_k) be a Cauchy sequence in $X_{p,q}^{\alpha,\beta}(\Gamma_T)$; then (g_k) is also a Cauchy sequence in $L_q(0,T,W_p^\alpha(\Gamma))$ and by the completeness of this latter space we find a $g \in L_q(0,T,W_p^\alpha(\Gamma))$ such that $g_k \to g$ in $L_q(0,T,W_p^\alpha(\Gamma))$. This implies $\|\Delta_{n+1,h}(g_k - g_j)\|_{L_q(0,T-h,L_p(\Gamma))} \to \|\Delta_{n+1,h}(g - g_j)\|_{L_q(0,T-h,L_p(\Gamma))}$ for $k \to \infty$, so that by Fatou's Lemma we may conclude $|g - g_j|_{\mathcal{L}_q^{0,\beta}(\Gamma_T)} \to 0$ for $j \to \infty$. The proof is complete.

□

Appendix A

Here we give the details about the integral representation used earlier: for a smooth f we have (cf. Il'in and Solonnikov [3], p. 70, (6) with $m_i = 0$, $k_i = l_i$)

$$\partial^{\underline{l}} f(x,t) = \frac{A}{T^r} \underset{Q^{n+1}(0,T^{\underline{\kappa}})}{\int \dots \int} f((x,t)+y)\,\Pi(y,T)\mathrm{d}y$$

$$+\sum_{i=1}^{n+1} B_i \int_0^T v^{-(1+r)} \underset{Q^{n+1}(0,v^{\underline{\kappa}})}{\int \dots \int} f((x,t)+y)\,\Pi_i(\dot{y},v)\partial_i^{l_i}\psi_i(y_i,v)\mathrm{d}y\mathrm{d}v$$

for $\nu_j \leqslant l_j - 1$, where (cf. [3], pp. 69–70)

$$\Pi(y,T) := \prod_{j=1}^{n+1} \partial_j^{l_j} \chi_j(y_j, T),$$

$$\chi_j(y_j, T) := y_j^{l_j - \nu_j - 1} \int_{y_j}^{T^{\kappa_j}} (T^{\kappa_j} - s)^{\mu_j} s^{\lambda_j} \mathrm{d}s,$$

$$\Pi_i(\overset{i}{y}, v) := \prod_{\substack{j=1\\ j\neq i}}^{n+1} \partial_j^{l_j} \chi_j(y_j, v),$$

$$\psi_i(y_i, v) := y_i^{l_i + \lambda_i - \nu_i} \cdot (v^{\kappa_i} - y_i)^{\mu_i}$$

with certain parameters $l_j, \mu_j, \lambda_j \in \mathbb{N}$ and certain $A, B_i \in \mathbb{R}$; here $\underline{\kappa} = (\kappa_1, \dots, \kappa_{n+1}) \in \mathbb{R}^{n+1}$ and $r := |\underline{\kappa}| + \underline{\kappa} \cdot (\underline{\lambda} + \underline{\mu})$, where $\underline{\lambda} := (\lambda_1, \dots, \lambda_{n+1})$ etc. We choose the parameters μ_i, λ_i so large that $\partial_i^k \psi_i(y_i, v)$ vanishes for $k = 1, \dots, l_i$ at $y_i = 0$ and $y_i = v^{\kappa_i}$. Hence, integrating by parts and introducing $K_i(y, v) := \Pi_i(\overset{i}{y}, v)\psi_i(y_i, v)$, $0 \leqslant y_i \leqslant v^{\kappa_i}$, we have show that

$$\partial^{\underline{\nu}} f(x,t) = \frac{A}{T^r} \int \cdots \int_{Q^{n+1}(0,T^{\underline{\kappa}})} f\big((x,t)+y\big)\Pi(y,T)\mathrm{d}y + \sum_{i=1}^{n+1} \tilde{B}_i \int_0^T v^{-(1+r)} \int \cdots \int_{Q^{n+1}(0,v^{\underline{\kappa}})} \partial_i^{l_i} f\big((x,t)+y\big) K_i(y,v)\mathrm{d}y\mathrm{d}v. \tag{A.1}$$

(The kernels Π, K_i in this representation clearly depend on $\underline{\nu}$, $\underline{\kappa}$, $\underline{\lambda}$, $\underline{\mu}$, $\underline{l}$, but this dependence is suppressed in our notation.) They satisfy (uniformly in $y \in Q^{n+1}(0, v^{\underline{\kappa}})$)

$$|\partial_y^{\underline{\alpha}} \Pi(y,v)| \leqslant c \cdot v^{r - |\underline{\kappa}| - \underline{\kappa}\cdot(\underline{\nu}+\underline{\alpha})} \quad \forall |\underline{\alpha}| \leqslant 2, \tag{A.2}$$

$$|\partial_{y_{n+1}}^s K_i(y,v)| \leqslant c \cdot y_n^{\varepsilon} \cdot v^{r + \kappa_i l_i - |\underline{\kappa}| - \underline{\kappa}\cdot\underline{\nu} - \varepsilon\kappa_n - s\kappa_{n+1}}, \quad \big(0 \leqslant s \leqslant 1,\ 1 \leqslant i \leqslant n+1,\ \varepsilon \in (0, \varepsilon_0)\big). \tag{A.3}$$

For the proof of these inequalities, we first note that $\partial_j^{l_j + \alpha_j} \chi_j(y_j, v)$ is a linear combination of terms of the form $(v^{\kappa_j} - y_j)^{\varrho_1} y_j^{\varrho_2}$ with $\varrho_1 + \varrho_2 = \mu_j + \lambda_j - \nu_j - \alpha_j$, $\varrho_2 > 0$ (for λ_j large) and consequently

$$|\partial_j^{l_j + \alpha_j} \chi_j(y_j, v)| \leqslant c \cdot y_j^{\varepsilon} \cdot v^{-\kappa_j(\varepsilon + \alpha_j)} \cdot v^{\kappa_j(\mu_j + \lambda_j - \nu_j)} \quad (0 \leqslant y_j \leqslant v^{\kappa_j})$$

for $\varepsilon \in (0, \varrho_2)$; this implies (for $k = 1, \dots, n-1$)

$$|\partial_{n+1}^s \Pi_k(\overset{k}{y}, v)| \leqslant c \cdot y_n^{\varepsilon} \cdot v^{-\kappa_n\varepsilon - \kappa_{n+1}\cdot s} \cdot v^{\underline{\kappa}\cdot\underline{\delta} - \kappa_k\cdot\delta_k},$$

$$|\partial_{n+1}^s \Pi_n(\overset{n}{y}, v)| \leqslant c \cdot v^{-\kappa_{n+1}\cdot s} \cdot v^{\underline{\kappa}\cdot\underline{\delta} - \kappa_n\delta_n},$$

$$|\Pi_{n+1}(\overset{n+1}{y}, v)| \leqslant c \cdot y_n^{\varepsilon} \cdot v^{-\kappa_n\cdot\varepsilon} \cdot v^{\underline{\kappa}\cdot\underline{\delta} - \kappa_{n+1}\cdot\delta_{n+1}},$$

where $\underline{\delta} := \underline{\mu} + \underline{\lambda} - \underline{\nu}$. The definition of ψ_i easily implies

$$|\psi_k(y_k, v)| \leqslant v^{\kappa_k \cdot (l_k + \delta_k)},$$
$$|\psi_n(y_n, v)| \leqslant y_n^{\varepsilon} \cdot v^{-\kappa_n \cdot \varepsilon} \cdot v^{\kappa_n \cdot (l_n + \delta_n)},$$
$$|\partial_{n+1}^{s} \psi_{n+1}(y_{n+1}, v)| \leqslant c \cdot v^{-s \cdot \kappa_{n+1}} \cdot v^{\kappa_{n+1} \cdot (l_{n+1} + \delta_{n+1})};$$

since $K_i(y, v) = \Pi_i(\overset{\prime}{y}, v)\psi_i(y_i, v)$, these formulas yield (A.3). For (A.2) compare Il'in and Solonnikov [3], p. 72.

Appendix B

We state some basic inequalities.

Lemma B.1. *Suppose that* $1 \leqslant s \leqslant r < \infty$, $f \in L_s(0, T^\gamma)$, $0 < \varepsilon, \gamma < \infty$, $0 < T \leqslant \infty$. *Then*

(i) $\left\| x^{-1/r - \varepsilon\gamma} \cdot \int_0^{x^\gamma} y^{\varepsilon - 1/s'} f(y) dy \right\|_{L_r(0,T,dx)} \leqslant c(\ldots) \|f\|_{L_s(0,T^\gamma)}$,

(ii) $\left\| x^{-1/r + \varepsilon\gamma} \cdot \int_{x^\gamma}^{T^\gamma} y^{-\varepsilon - 1/s'} f(y) dy \right\|_{L_r(0,T,dx)} \leqslant c(\ldots) \|f\|_{L_s(0,T^\gamma)}$,

where $c(\ldots) = c(\varepsilon, \gamma, r, s) = \gamma^{-1/r} \left(\frac{\mu}{\varepsilon}\right)^\mu$, $\mu = 1 - \frac{1}{s} + \frac{1}{r}$.

P r o o f. Compare Besov [2], 2.15, p. 28. □

Putting $s = r = 1$ in (i) and reformulating (ii) (for $\gamma = 1$) in an equivalent way, we get a version which is sometimes handier for our purposes:

Lemma B.1′. *Let the assumptions of the preceding Lemma hold. Then*

(i) $\int_0^T x^{-1-\varepsilon} \int_0^x y^\varepsilon \cdot f(y) dy dx \leqslant \varepsilon^{-1} \int_0^T f(y) dy$ *for all* $f \in L_1(0, T)$, $f \geqslant 0$;

(ii) *If* $a \cdot r < 1$, *then*

$$\left\| x^{-a} \int_x^T f(y) dy \right\|_{L_r(0,T,dx)} \leqslant c(a, r, s) \cdot \|y^{-a + 1/s' + 1/r} f(y)\|_{L_s(0,T,dy)},$$

for all f *with r.h. side finite.*

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