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RIEMANNIAN REGULAR σ-MANIFOLDS

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Symmetric spaces and their generalizations play an important role in modern differential geometry and its applications, [4], [5]. In this paper we introduce and study the so-called Riemannian regular σ-manifolds which generalize on the one hand the spaces with reflections [6] and on the other hand the Riemannian regular s-manifolds [4]. We want to point out that the term "subsymmetry" was first used in [8]. The main point of the present paper is to show that any Riemannian regular σ-manifold is a fibre bundle over the base space $N = G/H$, with a standard fibre Λ and a structure group G, which is associated with the principal fibre bundle $G(G/H, H)$. The manifold N is a regular s-manifold. When M is compact then N is a Riemannian regular s-manifold.

All manifolds and mappings are supposed to belong to the class C^∞, $\mathcal{X}(M)$ denotes the algebra of vector fields on M. TM denotes the tangent bundle, I the identity operator.

1. RIEMANNIAN LOCALLY REGULAR σ-MANIFOLDS

Definition 1.1. We shall call a connected Riemannian manifold (M, g) with a family of local isometries $\{s_x : x \in M\}$ a Riemannian locally regular σ-manifold (R.l.r. σ-m.), if

1) $s_x(x) = x$, 2) the tensor fields S: $S_x = (s)_{x*x}$ is smooth and invariant under any subsymmetry s_x, 3) there exists a connection $\overline{\nabla}$ on M invariant under any s_x, such that $\overline{\nabla} S = \overline{\nabla} g = 0$.

As $S_x = (s_{x*x})$, it is evident that

$$g(SX, SY) = g(X, Y), \qquad X, Y \in \mathcal{X}(M). \tag{1.1}$$

If a tensor field S is O-deformable, then the existence of a connection $\overline{\nabla}$ ($\overline{\nabla} S = \overline{\nabla} g = 0$) follows from (1.1), [1]. Let the closure $G = \mathrm{CL}(\{s_x\})$ of the group generated by the set $\{s_x : x \in M\}$ in the full isometry group $I(M)$ be a transitive Lie group of transformations.

Then M is a Riemannian homogeneous space with the canonical connection $\overline{\nabla}$. S is G-invariant (S is invariant under every s_x) and it follows that $\overline{\nabla} S = \overline{\nabla} g = 0$, [3].

Definition 1.2. We shall call a connected Riemannian manifold (M, g) with a family of local isometries $\{s_x : x \in M\}$ a Riemannian locally regular σ-manifold of order k (R.l.r. σ-m.o.k), if

1) $s_x(x) = x$,

2) the tensor field S determined by the formula $S_x = (s_{x*x})$ is smooth, invariant under any s_x and satisfies the condition $S^k = I$.

Let M be a R.l.r. σ-m. (R.l.r. σ-m.o.k) and suppose all the symmetries are determined globally. Then we shall call M a Riemannian regular σ-manifold (R.r. σ-m. and R.r. σ-m.o.k, respectively).

The following theorem shows that any R.l.r. σ-m.o.k is a R.l.r. σ-m.

Theorem 1.1. *Let M be R.l.r. σ-m.o.k, $S^k = I$, let ∇ be a Riemannian connection of g. Then the connection*

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{k} \sum_{j=1}^{k-1} \nabla_X (S^j) S^{k-j} Y \tag{1.2}$$

$$= \frac{1}{k} \sum_{j=0}^{k-1} S^j \nabla_X S^{k-j} Y, \quad X, Y \in \mathscr{X}(M),$$

is determined on M, $\overline{\nabla} S = \overline{\nabla} g = 0$, and $\overline{\nabla}$ is invariant under every s_x.

P r o o f. $\overline{\nabla}$ is obviously a connection. We have

$$\overline{\nabla}_X (S) Y = \frac{1}{k} \sum_{j=0}^{k-1} (S^j \nabla_X S^{k-j+1} Y - S^{j+1} \nabla_X S^{k-j} Y)$$

$$= \frac{1}{k} (\nabla_X S^{k+1} Y - S^k \nabla_X S Y) = 0,$$

$$g(\overline{\nabla}_X Y, Z) + g(Y, \overline{\nabla}_X Z) = \frac{1}{k} \sum_{j=0}^{k-1} [g(S^j \nabla_X S^{k-j} Y, Z) + g(Y, S^j \nabla_X S^{k-j} Z)]$$

$$= \frac{1}{k} \sum_{j=0}^{k-1} [g(\nabla_X S^{k-j} Y, S^{k-j} Z) + g(S^{k-j} Y, \nabla_X S^{k-j} Z)]$$

$$= \frac{1}{k} \sum_{j=0}^{k-1} X g(S^{k-j} Y, S^{k-j} Z) = X g(Y, Z),$$

that is $\overline{\nabla} g = 0$. As ∇ and S are invariant under every s_x, it follows from (1.2) that $\overline{\nabla}$ is also invariant under every s_x. □

The condition $\overline{\nabla} S = 0$ on R.l.r. σ-m. M implies that S has on M a constant Jordan normal form. An almost product structure can be defined on $M: T(M) = T^1(M) \oplus T^2(M)$, where T^1 is a distribution corresponding to the eigenvalue 1, $T^2 = T^{1\perp}$.

In the case when $T^1 = \{0\}$, M is a Riemannian locally regular s-manifold [4]. Further on we assume $T^1 \neq \{0\}$.

Theorem 1.2. *Let M be a R.l.r. σ-m. Then the distribution T^1 is integrable and its maximal integral manifolds are totally geodesic submanifolds with respect to ∇.*

Proof. From the fact that connections ∇, $\overline{\nabla}$ are invariant it follows that the tensor field $h = \nabla - \overline{\nabla}$ is also invariant under every s_x. Since h is invariant and $s_x = (s_{x*x})$, it follows that $h(SX, SY) = Sh(X, Y)$, X, $Y \in \mathfrak{X}(M)$. Let X, $Y \in T^1$, then $Sh(X, Y) = h(SX, SY) = h(X, Y)$ and $h(X, Y) = \nabla_X Y - \overline{\nabla}_X Y \in T^1$.

Since $\overline{\nabla} S = 0$, T^1 is invariant under $\overline{\nabla}$ and we get

$$\overline{\nabla}_X Y \in T^1, \quad \nabla_X Y = \overline{\nabla}_X Y + h(X, Y) \in T^1, \quad [X, Y] = \nabla_X Y - \nabla_Y X \in T^1,$$

T^1 is autoparallel under ∇ and it follows that its maximal integral sumbmanifolds are totally geodesic. □

The distribution T^1 defines the foliation $\tilde{\Lambda} = \{\Lambda_x : x \in M\}$. The fibres of $\tilde{\Lambda}$ will be called the mirrors.

The canonical connection is unique for any Riemannian locally regular s-manifold [4]. For R.l.r. σ-m. we have

Proposition 1.3. *Let $\overline{\nabla}$, $\overline{\nabla}'$ be canonical connections from Definition* 1.1 *and $X \in T^2$. Then $\overline{\nabla}_X = \overline{\nabla}'_X$ on M.*

Proof. S has no fixed vectors except the null vector in T^2, hence $(I - S)$ is an isomorphism on T^2 and $(I - S)X \neq 0$, $X \in T^2$, $X \neq 0$. Let $X \in T^2$, $Y \in \mathfrak{X}(M)$, let $\overline{\nabla}$, $\overline{\nabla}'$ be canonical connections from Definition 1.1, $E = \overline{\nabla} - \overline{\nabla}'$. Then

$$E_X Y = E_{(I-S)X_1} SY_1 = E_{X_1} SY_1 - E_{SX_1} SY_1 - SE_{X_1} Y_1 - SE_{X_1} Y_1 = 0$$

and $\overline{\nabla}_X = \overline{\nabla}'_X$ ($X = (I - S)X_1$, $Y = SY_1$, $SE_{X_1} Y_1 = E_{X_1} SY_1$ because $\overline{\nabla}(S) = \overline{\nabla}'(S) = 0$, $SE_{X_1} Y_1 = E_{SX_1} SY_1$ because E is invariant under every s_x). □

In this section we assume that M is a R.r.σ-m.

Lemma 2.1 [2]. *Let ϱ and ψ be isometries on (M, g), $\varrho(x) = \psi(x)$, $\varrho_*(x) = \psi_*(x)$ for some $x \in M$. Then $\varrho = \psi$ on M.*

Lemma 2.2. *All the subsymmetries s_x are affine transformations with respect to $\overline{\nabla}$.*

Proof obviously follows from Definition 1.1.

Proposition 2.3. *Let M be a R.r.σ-m. and s_x a subsymmetry on M. Then we have $s_x\big|_{\Lambda_x} = \mathrm{id}\big|_{\Lambda_x}$ and if $x_1 \in \Lambda_X$, then $s_x = s_{x_1}$ on M.*

Proof. Since s_x and S commute, T^1 and Λ are invariant under s_x and it follows that $s_x(\Lambda_x) = \Lambda_x$. For the restriction $s_x\big|_{\Lambda_x}$ we have $s_x(x) = x$, $s_{x*x} = I$. According to Lemma 2.1, $s_x = \mathrm{id}$ on Λ_x. Let $x_1 \in \Lambda_x$, then $s_{x_1}\big|_{\Lambda_x} = \mathrm{id}$ and $s_{x_1}(x) = s_x(x) = x$. Consider $v \in T_x(M)$ and a curve τ_t connecting x and x_1. Denote the parallel transport with respect to the connection $\overline{\nabla}$ by $\bar{\tau}_t$. According to Lemma 2.2, all subsymmetries commute with the parallel transport; the parallel transport commutes with S, because $\overline{\nabla} S = 0$. Thus $\bar{\tau}_t(s_{x_1*x}(v)) = s_{x_1*x_1}(\bar{\tau}_t(v)) = S\bar{\tau}_t(v) = \bar{\tau}_t(Sv)$ and we get $s_{x_1*x} = s_{x*x} = S$. According to Lemma 2.1 $s_{x_1} = s_x$ on M. □

Theorem 2.4. *Let M be R.r.σ-m., $N = \{\Lambda_x : x \in M\}$, $\pi : M \to N : x \mapsto \Lambda_x$. Then N is a smooth manifold and π is a differentiable submersion.*

Proof. According to [7] it is sufficient to show that the foliation is regular. Let $U(x)$ be a convex neighbourhood of x in which there exists a foliated chart of the foliation $\tilde{\Lambda}$, [9], and let $x_1 \in U(x)$. Suppose that $\overline{\Lambda}_{x_1}$, $\overline{\Lambda}_{x_2}$ are connected components of $\Lambda_{x_1} \cap U(x)$ which do not coincide $(x_2 \in U(x))$. Then there exists a unique minimizing geodesic $\gamma(t)$ in $U(x)$, where $t \in [t_1, t_2]$, $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$. The isometry s_x transforms γ into a geodesic $\gamma' \subset U(x)$ and γ' is a minimizing geodesic [2]. Proposition 2.3 yields that $s_{x_1}(\Lambda_{x_1}) = \Lambda_{x_1}$ and $s_{x_1}(x_1) = x_1$, $s_{x_1}(x_2) = x_2$. Since the minimizing geodesic which connects x_1 and x_2 is unique we have $\gamma' = \gamma$. Thus $s_{x_1}(\gamma) = \gamma$ and $s_{x_1*x_1}(\dot{\gamma}) = S_{x_1}(\dot{\gamma}) = \dot{\gamma}$ and hence $\dot{\gamma}_{x_1} \in T^1_{x_1}$.

According to Theorem 1.4, Λ_{x_1} is a totally geodesic submanifold of M, so $\gamma \subset \Lambda_{x_1}$. Because $\overline{\Lambda}_{x_1}$, $\overline{\Lambda}_{x_2}$ are arewise connected in $U(x)$, they coincide. The contradiction obtained proves the theorem. □

3. Riemannian regular σ-manifold as a fibre bundle

Let $I(m)$ be the full isometry group of R.r. σ-m. M equipped with the compact open topology and let $G = \mathrm{CL}(\{s_x\})$ be the closure in $I(M)$ of the group generated by the set $\{s_x : x \in M\}$. Then G is a Lie group of transformations.

Lemma 3.1. *The foliation $\tilde{\Lambda}$ is invariant under all transformations of the group G, that is, G transforms mirrors into mirrors.*

Proof. Consider a sequence $\{g_n\} \to g \in G$ where $g_n \in G$. As S is invariant under subsymmetries, S is also invariant under each g_n. But then $g_* \cdot S = S \cdot g_*$. As the tensor field S is invariant under the group G, T^1 is also invariant under G. It follows that G transforms mirrors of the foliation $\tilde{\Lambda}$ into mirrors. □

Lemma 3.2 [4]. *If $G \subset I(M)$ is a closed subgroup then all G-orbits are closed in M.*

Let us define the action of the group G on the manifold N: $G \times N \to N : (g, y) \mapsto \pi(g \cdot x)$, where $y = \pi(x)$. From Lemma 3.1 we see that this definition is correct. The action is obviously differentiable.

Theorem 3.3. *Let M be a R.r. σ-m., and N the corresponding manifold of mirrors. Then the group G is a transitive Lie group of transformations of the manifold N.*

Proof. Let $x_0 \in M$, let $U(x_0)$ be a convex neighbourhood of x_0 with respect to ∇, which is a foliated chart of the foliation $\tilde{\Lambda}$. Suppose that x is an arbitrary point in $U(x_0)$, $x \notin \Lambda_{x_0}$, r is a distance from x_0 to the G-orbit $G(x)$ of the point $x : r = \inf_{g \in G} d(x_0, g(x))$. Since $G(x)$ is closed, one can find $z \in G(x)$ such that $r = d(x_0, z)$. Let us suppose that $z \notin \Lambda_{x_0}$. Then there is a geodesic segment of the length r joining x_0 and z. Let w be a point of this segment between x_0 and z. Then $\dot{\gamma}_w \notin T^1$ because otherwise, according to Theorem 1.2, the whole segment would lie in Λ_w and $z \in \Lambda_w = \Lambda_{x_0}$. Thus $s_w(z) \neq z$, $s_w(z) \in G(x)$.

Hence all the points x_0, z, w, $s(w)$ lie in $U(x)$. Using the triangle inequality we get

$$\begin{aligned} d(x_0, s_w(z)) < d(x_0, w) + d(w, s_w(z)) &= d(x_0, w) + d(s_w(w)s_w(z)) \\ &= d(x_0, w) + d(w, z) = d(x_0, z) = r. \end{aligned}$$

The contradiction obtained shows that $z \in \Lambda_{x_0}$. Thus, for any mirror $y = \Lambda_x$, $y \in \pi(U(x_0))$, one can find an element of the group G transforming y into $y_0 = \Lambda_{x_0}$, and for any $y_1, y_2 \in \pi(U(x_0))$ there exists a transformation $g \in G$ such that $y_2 = g(y_1)$.

Covering a segment of the curve between two arbitrary points of N by a finite number of neighborhoods like $\pi(U(x_0))$ we conclude that the group is a transitive Lie group of transformations of N. □

Corollary 3.4. *All fibres of the foliation $\tilde{\Lambda}$ are diffeomorphic to the standard fibre $\Lambda = \Lambda_0$, where $o \in M$ is a fixed point.*

It is well known that the component of identity of a Lie group acting transitively on the manifold N is also transitive on N, so later on we will assume the group G to be connected.

Corollary 3.5. *Let $o \in M$ and let H be the isotropy subgroup of $\Lambda_0 \in N$. The mapping $G/H \to N : gH \mapsto \Lambda_{g(0)}$ is a diffeomorphism of the manifolds G/H and N.*

Let $G(G/H, H)$ be a principal fibre bundle with the base G/H and the structure group H. Since H acts on the manifold $\Lambda = \Lambda_0$ to the left, it is possible to consider $G \times {}_H\Lambda$, which is the fibre bundle over the base space G/H with the standard fibre Λ and the structure group H associated with the principal fibre bundle.

Let $g \otimes x$ be the equivalence class containing (g, x), where $(gh, x) \sim (g, hx)$, $h \in H$.

Theorem 3.6. *Let M be a R.r.σ-m. The mappings $\Phi\colon G \times {}_H\Lambda \to M : g \otimes x \mapsto g(x)$ and $G/H \to N : gH \mapsto \Lambda_{g(0)}$ are diffeomorphisms. The following diagram is commutative:*

$$\begin{array}{ccc} G \times {}_H\Lambda & \longrightarrow & M \\ \downarrow & & \downarrow \\ G/H & \longrightarrow & N \end{array} \tag{3.1}$$

P r o o f. Φ is obviously a correctly defined, differentiable mapping, Φ is surjective because G is transitive on N. Let us check the injectivity of Φ. Let $g_1(x_1) = g_2(x_2)$, then

$$g_1^{-1}g_2 = h \in H \qquad \text{and} \qquad g_1 \otimes x_1 = g_1 h \otimes h^{-1}x_1 = g_2 \otimes x_2.$$

The mapping $G \times \Lambda \to M : (g, x) \mapsto g(x)$ is a submersion and the following diagram is commutative:

$$\begin{array}{ccc} G \times \Lambda & \longrightarrow & M \\ & \searrow \quad \swarrow & \\ & G \times {}_H\Lambda & \end{array}$$

Thus Φ is a diffeomorphism and the diagram (3.1) is evidently commutative. □

Let $o \in M$ be again a fixed point, $y_0 = \Lambda_0 \in N$. According to Proposition 2.3 every subsymmetry s_x defines a diffeomorphism s_y of the manifold N, where $y \in \pi(x)$. It is clear that $s_y(y) = y$ and $s_{y*y} = \bar{S}$, where the Jordan normal form $\bar{S}$ coincides with the normal form of the tensor field S restricted to T^2. It is also evident that $\bar{S}$ is invariant under the group G acting transitively on N.

Lemma 4.1. *Let $g(\Lambda_0) = \Lambda_x$, where $x = g(o) \in M$. Then $s_x = g \cdot s_0 \cdot g^{-1}$ on M, $g \in G$.*

P r o o f. $s_x(x) = x$ and $(g \cdot s_0 \cdot g^{-1})(x) = x$. Then $s_{x*x} = S_x$ and $(g \cdot s_0 \cdot g^{-1})_{*x} = g_{*0} \cdot s_{0*0} \cdot g_{*x}^{-1} = g_{*0} \cdot S_0 \cdot g_{*x}^{-1} = S_x$, because S is G-invariant. According to Lemma 2.1, s_x coincides with $g \cdot s_0 \cdot g^{-1}$ on M. □

Proposition 4.2. *Let M be a R.r. σ-m. and let N be a manifold of mirrors. Then $\mu \colon N \times N \to N : (y_1, y_2) \mapsto s_{y_1}(y_2)$ is a real analytic mapping.*

P r o o f. $N \cong G/H$ has the structure of a real analytic manifold such that the action of G on N and the projection $p \colon G \to G/H$ are analytic [2]. One can find a neighbourhood $W \subset N$ of a point y_0 for which there exists an analytic section ν: $W \to G$ of the fibre bundle $p \colon G \to G/H$. According to Lemma 4.1, $s_y = \pi(s_x) = \pi(g \cdot s_0 \cdot g^{-1}) = g \cdot s_{y_0} \cdot g^{-1}$. Therefore, for any $y \in W$, $s_y = \nu(y) \cdot s_{y_0} \cdot \left(\nu(y)\right)^{-1}$, $s_{y_0} \in G$ is analytic. Thus, the mapping $(y_1, y_2) \mapsto s_{y_1}(y_2)$ is analytic on $W \times N$ and, in fact, on $M \times M$. □

Definition 4.1 [4]. A regular s-manifold is a manifold N with a multiplication $\mu \colon N \times N \to N$ such that the mappings $s_y : N \to N$, $y \in N$ given by $s_y(z) = \mu(y, z)$ satisfy the following axioms:

1) $s_y(y) = y$,

2) each s_y is a diffeomorphism,

3) $s_y \cdot s_z = s_w \cdot s_y$, where $w = s_y(z)$,

4) for each $y \in N$, $s_{y*y} : T_y(N) \to T_y(N)$ has no fixed vectors except the null vector.

Theorem 4.3. *Let M R.r. σ-m. and N its manifold of mirrors. Then N is a regular s-manifold.*

P r o o f. According to Proposition 4.2, μ is differentiable, the axioms 1) and 2) are evident, 4) follows from the fact that $S|_{T^2}$ has no fixed vectors except the null one. Consider the axiom 3). Let x, u, $v \in M$, $\pi(x) = y$, $\pi(u) = z$, $\pi(v) = w$. Let us prove that $s_x \cdot s_u = s_v \cdot s_x$. We have

$$(s_x \cdot s_u)(u) = (s_v \cdot s_x)(u) = v,$$
$$(s_x \cdot s_u)_{*u} = s_{x*u} \cdot s_{u*u} = s_{x*u} \cdot S_u = S_v \cdot s_{x*u} = s_{v*v} \cdot s_{x*u} = (s_v \cdot s_x)_{*u}.$$

According to Lemma 2.1 we have $s_x \cdot s_u = s_v \cdot s_x$. Projecting this equality onto N, we obtain that $s_y \cdot s_z = s_w \cdot s_y$, where $w = s_y(z)$. □

Theorem 4.4. *Let a R.r. σ-m. M be compact. Then its manifold of mirrors N is a Riemannian regular s-manifold.*

Proof. Since the group $I(M)$ of all isometries of M is compact, the group G is also compact. Assume $<,>^*$ is an arbitrary Riemannian metric on N, $X, Y \in T_y(N)$. The elements of the group G are isometries with respect to the following metric $<, >$ on N:

$$\langle X, Y\rangle = \int\limits_{g\in G} \langle g_*X, g_*Y\rangle^* .$$

The rest follows from Theorem 4.3. □

Remark 4.5. If H is not compact then G/H can not be a Riemannian regular s-manifold because according to [3], the isotropy subgroup of a homogeneous Riemannian space must be compact.

5. The main example of a Riemannian regular σ-manifold of order k

Let (N, g^2) be a Riemannian regular homogeneous s-manifold of order k [4], then $N \cong G/H$ where $G_0^\sigma \subset H \subset G^\sigma$, $G^\sigma = \{g \in G \colon \sigma(g) = g\}$, G_0^σ is the component of the identity of G^σ, σ is the automorphism of the group G ($\sigma^k = \mathrm{id}$). (Here G is a connected group of isometries which acts transitively on N). Let $G(G/H, H)$ be a principal fibre bundle with the base G/H and the structure group H. Let (Λ, g^1) be the Riemannian manifold and let H act on Λ to the left. We consider the fibre bundle $G \times_H \Lambda$ which is associated with $G(G/H, H)$, and again denote by $g \otimes x$ the equivalence class containing (g, x), where $(gh, x) \sim (g, hx)$, $h \in H$.

Now we will state the main theorem of this section.

5.1. *$M \cong G \times_H \Lambda$ is a R.r. σ-m.o.k.*

The proof will be given step by step in the next paragraphs.

Lemma 5.2 [5]. *The formulas*

$$pH \cdot qH = p^\sigma (p^\sigma)^{-1} \cdot q^\sigma \cdot H, \quad p^\sigma = \sigma(p), \quad q^\sigma = \sigma(q), \quad p, q \in G$$

define a regular multiplication on N.

Lemma 5.3. *The formula*

$$(p \otimes u) \cdot (q \otimes v) = p(p^\sigma)^{-1} q^\sigma \otimes v$$

defines a regular multiplication on $M \cong G \times_H \Lambda$.

The projection $\pi\colon G \times_H \Lambda \to G/H$ is a homomorphism of spaces with multiplications.

The proof is analogous to that considered in [6] when $\sigma^2 = \mathrm{id}$.

We have a family of symmetries $\{s_y\colon y \in N\}$ on N, $s_y(z) = y \cdot z$, and a tensor field $\bar{S}_y = s_{y*y}$ which is invariant under all s_y. It is clear that $\bar{S}^k = I$. The family of subsymmetries $\{s_x\colon x \in M\}$, $s_x(z) = x \cdot z$, and the tensor field $S_x = s_{x*x}$ are defined on M. S is invariant under all s_x from regularity condition. Since π is a homomorphism of spaces with multiplications, we have

$$\pi \cdot s_x = s_{\pi(x)}, \quad \pi_x \cdot S = \bar{S}. \tag{5.1}$$

Lemma 5.4. *Let Λ_x be the fibre which contains $x \in M$. Then $s_x = \mathrm{id}$ on Λ_x and if $x_1 \in \Lambda_x$ then $s_x = s_{x_1}$.*

P r o o f. Let $x = p \otimes u$, $z = q \otimes v \in \Lambda_x$, then $p = qH$ because $\pi(x) = \pi(z)$, $x \cdot z = (p \otimes u) \cdot (q \otimes v) = (q \otimes hu) \cdot (q \otimes v) = q(q^\sigma)^{-1} \cdot q^\sigma \otimes v = q \otimes v$. If $x_1 = p_1 \otimes u_1 \in \Lambda_x$, then $p_1 = ph$ because $\pi(x) = \pi(x_1)$ and $x_1 = p_1 \otimes u_1 = p \otimes hu_1$, $x_1 \cdot \bar{z} = (p \otimes hu_1) \cdot (\bar{q} \otimes \bar{v}) = p(p^\sigma)^{-1} \bar{q}^\sigma \otimes v = x \cdot \bar{z}$, $\forall \bar{z} \in M$. □

The foliation $\tilde{\Lambda} = \{\Lambda_x\colon x \in M\}$ defines the distribution T^1 on M. According to Lemma 5.4 $S|_{T^1} = I$ and since $\bar{S}$ has no fixed vectors except the null vector, the eigenspace of S_x corresponding to the eigenvalue 1 coincides with T_x^1. Let T_x^2 be the direct sum of all eigenspaces of S_x except T_x^1. From (5.1) we get $S^k = I$, and $\pi_*\colon T_x^2 \to T_{\pi(x)}(N)$ is an isomorphism. The structure of the almost product $T(M) = T^1 \oplus T^2$ is defined on M. The action of the group G on the homogeneous space $N \cong G/H$ induces the action of G on $M \cong G \times_H \Lambda\colon (q, p \otimes u) \mapsto q \cdot p \otimes u$ and we have

$$\pi(q \cdot x) = q \cdot \pi(x), \quad p, q \in G, \quad x \in M.$$

Lemma 5.5. *The tensor field S is invariant under all elements of G on M.*

P r o o f. We shall show that $(q \cdot s_x)(z) = (s_{g(x)} q)(z)$, $q \in G$, x, $z \in M$. Indeed, $q \cdot (x \cdot z) = q \cdot p(p^\sigma)^{-1} \cdot r^\sigma \otimes v$, $(qp \otimes u) \cdot (qr \otimes v) = (qp) \cdot (q^\sigma p^\sigma)^{-1} \cdot q^\sigma \cdot r^\sigma \otimes v = q \cdot p \cdot (p^\sigma)^{-1} \cdot r^\sigma \otimes v$ where $x = p \otimes u$, $z = r \otimes v$. Considering the tangent mappings we get $g_* \cdot S_x = S_{g(x)} \cdot g_* x$. □

According to Lemma 5.5 the distributions T^1, T^2 are invariant under G, hence the foliation $\tilde{\Lambda}$ is also G-invariant.

Define the following Riemannian metric on the distribution T^2:

$$g_x^2(X,Y) = g_{\pi(x)}^2(\pi_* X, \pi_* Y), \quad X, Y \in T_x^2.$$

Then $g^2(p_* X, p_* Y) = g^2(\pi_* \cdot p_* X, \pi_* \cdot p_* Y) = g^2(p_* \cdot \pi_* X, p_* \cdot \pi_* Y) = g^2(X,Y)$, where X, $Y \in T^2$, $p \in G$. Thus the elements of the group G are isometries on T^2. Let $o \in M$ be a fixed point and $\Lambda_0 = \Lambda$.

Define a Riemannian metric on the distribution T^1 as follows:

$$g_x^1(X,Y) = g^1(p_* X, p_* Y), \quad p \in G, \quad p(x) \in \Lambda, \quad X, Y \in T^1.$$

The element p exists because G is a transitive Lie group of transformations of N. Let $g \in G$, $g(x) \in \Lambda$ then Λ is invariant under $h = p \cdot g^{-1}$ and $h \in H$. Since H acts on Λ as an isometry group, we get $g^1(g_* X, g_* Y) = g^1(h_* g_* X, h_* g_* Y) = g^1(p_* X, p_* Y)$, X, $Y \in T^1$.

It follows that the metric g^1 is well-defined on T^1. It is clear that the elements of the group G are isometries on T^1.

Define a Riemannian metric on M as follows: $g\big|_{T^1} = g^1$, $g\big|_{T^2} = g^2$, T^1, T^2 are orthogonal in the metric g. From the above we see that G is an isometry group with respect to g. The transformation s_x is identified with an element of G and s_x is an isometry, too.

Hence Theorem 5.1 follows.

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